General solutions of covariant superstring equations of motion

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The equations of motion arising from the Green-Schwarz Lagrangian for the superstring are solved for both commuting and anticommuting variables. The form of the solution depends on the number of independent Grassmann parameters; we are able to give the most general solution in some cases, but not all. The method of solution is to use an octonionic formalism for ten-dimensional vectors and spinors, and the solution is given in terms of a number of octonion parameters.

I. INTRODUCTION

The covariant formulation of superstring theory of Green and Schwarz[1-3] has not been fully explored at the classical level. A complete knowledge of the classical solutions of the equations of motion arising from the Green-Schwarz Lagrangian can be expected to be helpful in quantizing the theory;[4] it will enable one to elucidate the structure of the gauge symmetries and hence to give an explicit description of the phase space of the superstring.

In this paper we look for the general solution of the equations of motion of the Green-Schwarz superstring in its critical dimension $D=10$. We find that there are different forms of solution, depending on the number of independent Grassmann parameters in the fermionic variables of the superstring, and on the assumptions one makes about the dependence of the bosonic variables on Grassmann parameters. Thus the general solution would contain a number of different cases. We have made the simplifying assumption that both bosonic and fermionic variables have their lowest possible Grassmann degree. There are still a number of different cases to be considered; we have found the general solution in one of these cases, and a wide class of solutions in all the others.

Our solutions constitute a considerably wider class than those found by previous authors.[3-7] They require a choice of gauge on the world sheet, since we assume a parametrization in which the metric is constant, but they are fully covariant in space-time coordinates.

In Sec. II we discuss the equations of motion and put them in the form in which we will solve them. The solution uses an octonionic formalism[8,9] which we expound in Sec. III. In Sec. IV we use this to find general solutions under the assumption that the bosonic variables do not depend on Grassmann parameters (have no "soul," in DeWitt's terminology.[10])

II. THE EQUATIONS OF MOTION

The variables in the Green-Schwarz formulation of the superstring are the world-sheet metric $g^{\alpha\beta}$ ($\alpha,\beta=1,2$), the space-time position, X^μ ($\mu=0,\ldots,9$) of the world sheet, and two anticommuting Majorana-Weyl spinors θ^A ($A=1,2$). We note that Majorana spinors ψ have a symmetry property in their covariant bilinears, namely,

$$\bar\psi_1\gamma^\mu\psi_2=\pm\bar\psi_2\gamma^\mu\psi_1\ , \tag{1}$$

where the + sign applies to commuting spinors and the − sign to anticommuting ones; and that Majorana-Weyl spinors in ten dimensions satisfy a famous identity which reads

$$\gamma^\mu\psi\bar\psi\gamma_\mu\psi=0 \tag{2}$$

for commuting spinors ψ, and

$$\epsilon_{ijk}\gamma^\mu\psi_i\bar\psi_j\gamma_\mu\psi_k=0 \tag{3}$$

for anticommuting spinors ψ_1,ψ_2,ψ_3. As we will see in the next section, this identity appears naturally in the octonionic formalism.

The variables $g^{\alpha\beta}$, X^μ, and θ^A are all functions of the world-sheet parameters $\sigma^\alpha=(\sigma,\tau)$, and the action is

$$S=\int d\sigma\, d\tau[-\tfrac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\Pi^\mu_\alpha\Pi_{\mu\beta}-\epsilon^{\alpha\beta}\Pi^\mu_\alpha(v^1_{\mu\beta}-v^2_{\mu\beta})+\epsilon^{\alpha\beta}v^{1\mu}_\alpha v^2_{\mu\beta}]\ , \tag{4}$$

where

$$\Pi^\mu_\alpha=\partial_\alpha X^\mu-v^{1\mu}_\alpha-v^{2\mu}_\alpha\ , \tag{5}$$

$$v^{A\mu}_\alpha=\bar\theta^A\gamma^\mu\partial_\alpha\theta^A \quad \text{no sum over } A\ . \tag{6}$$

To discuss the equations of motion it is convenient to introduce on the world sheet a zweibein $\{u^\alpha_+,u^\alpha_-\}$ satisfying

$$u_+^\alpha u_-^\beta + u_-^\alpha u_+^\beta = g^{\alpha\beta} \tag{7a}$$

from which it follows that

$$u_+^\alpha u_-^\beta - u_+^\beta u_-^\alpha = \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} , \tag{7b}$$

$$u_+^\alpha u_{+\alpha} = 0 = u_-^\alpha u_{-\alpha} , \tag{7c}$$

$$u_+^\alpha u_{-\alpha} = 1 , \tag{7d}$$

and

$$\sqrt{-g} = (\epsilon_{\alpha\beta} u_+^\alpha u_+^\beta)^{-1} . \tag{7e}$$

Then any world-sheet vector V^α can be written as

$$V^\alpha = V_- u_+^\alpha + V_+ u_-^\alpha \tag{8}$$

with $V_\pm = V^\alpha u_{\pm\alpha}$.

Now the equation of motion obtained from the action (4) by varying $g^{\alpha\beta}$ can be written

$$\Pi_+^\mu \Pi_{+\mu} = 0 = \Pi_-^\mu \Pi_{-\mu} \tag{9}$$

and that obtained by varying X^μ is

$$\partial_\alpha[\sqrt{-g}\,(-\partial^\alpha X^\mu + 2v_+^{1\mu} u_-^\alpha + 2v_-^{2\mu} u_+^\alpha)] = 0 . \tag{10}$$

The equations for θ^1 and θ^2 can be reduced, using (10) and the anticommuting Fierz identity (3), to

$$\Pi_-^\mu \gamma_\mu \partial_+ \theta^1 = 0 , \tag{11a}$$

$$\Pi_+^\mu \gamma_\mu \partial_- \theta^2 = 0 . \tag{11b}$$

[Note the significant role played by the fact that the θ's are anticommuting variables. If they were commuting their equations of motion would both be consequences of the X^μ equation (10).]

We now use the reparametrization invariance of the action (4) and change to world-sheet coordinates $\sigma^\pm = \sigma \pm \tau$ in which the components of the metric $g^{\alpha\beta}$ are constant; we take them to be

$$g^{++} = 0 = g^{--}, \quad g^{+-} = 1 . \tag{12}$$

Then the zweibein can be taken to be

$$u_+^\alpha = (1,0), \quad u_-^\alpha = (0,1) \quad (\alpha = +,-) ; \tag{13}$$

this removes a potential ambiguity in the symbol $\partial_\pm$, for it gives

$$\partial_\pm = u_\pm^\alpha \partial_\alpha = \frac{\partial}{\partial \sigma^\pm} . \tag{14}$$

Now (10) becomes

$$\partial_+ \partial_- X^\mu = \partial_- v_+^{1\mu} + \partial_+ v_-^{2\mu} . \tag{15}$$

From the definition (5) of Π_α^μ we obtain

$$\partial_+ \partial_- X^\mu = \partial_+ \Pi_-^\mu + \partial_+ v_-^{1\mu} + \partial_+ v_-^{2\mu} \tag{16a}$$

$$= \partial_- \Pi_+^\mu + \partial_- v_+^{1\mu} + \partial_- v_+^{2\mu} . \tag{16b}$$

Eliminating $\partial_+ \partial_- X$ from Eqs. (15) and (16) yields

$$\partial_+ \Pi_- = \partial_- v_+^1 - \partial_+ v_-^1 , \tag{17a}$$

$$\partial_- \Pi_+ = \partial_+ v_-^2 - \partial_- v_+^2 . \tag{17b}$$

Using the definition (6) of $v_\alpha^{A\mu}$ and the symmetry property (1) for anticommuting spinors, these become

$$\partial_+ \Pi_-^\mu = 2\partial_- \bar{\theta}^1 \gamma^\mu \partial_+ \theta^1 , \tag{18a}$$

$$\partial_- \Pi_+^\mu = 2\partial_+ \bar{\theta}^2 \gamma^\mu \partial_- \theta^2 . \tag{18b}$$

Equations (9), (11), and (18) are the equations to be solved for $\Pi_\pm^\mu$ and θ^A. They fall into two similar pairs, with Π_- and θ^1 decoupled from Π_+ and θ^2; we need only consider one pair, and will therefore drop the indices and consider the equations

$$\Pi^\mu \Pi_\mu = 0 , \tag{19}$$

$$\Pi^\mu \gamma_\mu \partial_+ \theta = 0 , \tag{20}$$

$$\partial_+ \Pi^\mu = 2\partial_- \bar{\theta} \gamma^\mu \partial_+ \theta . \tag{21}$$

We shall see that (19) is a consequence of (20).

Our procedure will be to take a lightlike vector Π^μ, for which there exists a convenient parametrization,[6,7] find the general solution of (20) for $\partial_+\theta$, then find the general solution of (21) for $\partial_-\theta$, and finally impose the integrability condition which would make it possible to find θ given $\partial_+\theta$ and $\partial_-\theta$. For commuting variables this program can be carried out in full generality. Grassmann variables, however, admit more complicated types of solutions and we must analyze the Grassmann structure of the variables.

The commuting variable Π and the anticommuting variable θ belong, respectively, to the even and odd part of a Grassmann algebra, and can, in principle, contain terms of any degree. Equating terms of each degree in Eqs. (20) and (21) would then give two towers of equations. For simplicity we will assume that both Π and θ contain only terms of lowest degree; i.e., the components of Π are pure numbers while those of θ are basic anticommuting elements of the Grassmann algebra (vectors in V if the Grassmann algebra is the exterior algebra of a vector space V). Then (20) remains one equation, and (21) splits into two equations:

$$\partial_+ \Pi = 0 , \tag{22}$$

$$\partial_- \bar{\theta} \gamma^\mu \partial_+ \theta = 0 . \tag{23}$$

Thus our assumption requires II (i.e., Π_-) to be a function of $(\sigma - \tau)$ only, as is generally assumed. In principle, however, Π_- could have a soul (terms of higher Grassmann degree) which could be a function of both $\sigma - \tau$ and $\sigma + \tau$.

The solution of (23) depends on the number of independent Grassmann elements among the components of $\partial_+\theta$. This number is at most eight, since for a given Π Eq. (20) reduces to 8 the number of independent spinor components in $\partial_+\theta$. Let it be g; then we can write

$$\partial_+ \theta = \alpha_1 \psi_1 + \cdots + \alpha_g \psi_g , \tag{24}$$

where $\alpha_1, \ldots, \alpha_g$ are anticommuting scalars and $\psi_1, \ldots, \psi_g$ are commuting Majorana-Weyl spinors. Now put

$$\partial_- \theta = \alpha_1 \phi_1 + \cdots + \alpha_h \phi_h \ , \tag{25}$$

where $g \le h \le 8$, $\alpha_{g+1}, \ldots, \alpha_h$ are independent anticommuting scalars and the ϕ_i are commuting spinors. Then, because of the independence of the α_i and the $\alpha_i \alpha_j$ with $i<j$, Eqs. (20) and (23) give

$$\Pi_\mu \gamma^\mu \psi_i = 0 \quad \text{for } 1 \le i \le g \ , \tag{26}$$

$$\bar{\phi}_i \gamma^\mu \psi_j = \bar{\phi}_j \gamma^\mu \psi_i \quad \text{for } 1 \le i < j \le g \ , \tag{27}$$

$$\bar{\phi}_j \gamma^\mu \psi_i = 0 \quad \text{for } 1 \le i \le g < j \ . \tag{28}$$

Thus the fermionic equation (23) can be reduced to a number of bosonic equations (26)–(28). The remaining equations (19) and (20) also involve essentially commuting variables, since θ occurs only linearly in them. To solve these equations we use the octonionic formalism for ten-dimensional vectors and spinors which is summarized in the next section.

III. OCTONIONIC FORMALISM (REFS. 8, 9, 11, AND 12)

A vector X^μ in real $(9+1)$-dimensional Minkowski space is represented by a 2×2 Hermitian octonionic matrix

$$X = \begin{bmatrix} x^+ & x \\ \bar{x} & x^- \end{bmatrix} , \tag{29}$$

when $x^\pm = X^0 \pm X^9$ and $x = X^8 + X^1 e_1 + \cdots + X^7 e_7$ is an octonion representing the transverse components of X; $\bar{x} = X^8 - X^1 e_1 - \cdots - X^7 e_7$ is its octonion conjugate. Then the Minkowski square of the ten-vector X, with signature $(-+\cdots+)$, is

$$X^\mu X_\mu = -x^+ x^- + |x|^2 = -\det X \ . \tag{30}$$

In particular a lightlike ten-vector can be constructed from a column $\xi = \binom{p}{q}$ with two octonion components p,q by forming the singular Hermitian matrix

$$X = \xi \xi^\dagger \ , \tag{31}$$

where the dagger denotes matrix transpose together with octonion conjugation.

The Minkowski inner product of two ten-vectors X^μ, Y^μ, corresponding to Hermitian 2×2 matrices X and Y, can be written as

$$X^\mu Y_\mu = \tfrac{1}{2} \mathrm{Re}[\mathrm{tr}(X Y^t)] \ , \tag{32}$$

where the index-lowering operation t is accomplished in matrix form by

$$Y^t = J Y^T J, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \tag{33}$$

Majorana spinors ψ in ten dimensions have 32 real components and will be regarded as being made up of two 16-component objects, $\psi = \binom{\xi}{\eta}$, where ξ and η each have two octonion components. A suitable representation of the Dirac matrices is defined by

$$X_\mu \gamma^\mu = \begin{bmatrix} 0 & X \\ X^t & 0 \end{bmatrix} \tag{34a}$$

(in block form), where X is the 2×2 matrix corresponding to the ten-vector X^μ. Note that this gives

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{34b}$$

Here octonionic matrices are to act on octonionic columns by the obvious combination of matrix multiplication and octonion multiplication. Regarded as operators on octonionic columns, these γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\,\mathrm{diag}(-1, +1, \ldots, +1) \ . \tag{35}$$

Their product is

$$\gamma^{11} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{36}$$

so the objects $\binom{\xi}{0}$ and $\binom{0}{\eta}$ are the chiral projections of ψ. Each of these Majorana-Weyl spinors has two nonzero octonion components.

Identifying the usual 32-real-component Majorana spinor with its four-octonion component form, the correspondence between the expressions for the scalar and vector bilinears formed from two spinors ψ_1 and ψ_2 is

$$\begin{aligned} &\bar{\psi}_1 \psi_2 \to \mathrm{Re}(\psi_1^\dagger \gamma^0 \psi_2) \ , \\ &\bar{\psi}_1 \gamma^\mu \psi_2 \to \mathrm{Re}[\psi_1^\dagger \gamma^0 (\gamma^\mu \psi_2)] \ , \end{aligned} \tag{37}$$

where on the left γ^μ is a 32×32 component real matrix, while on the right-hand side the matrix products are performed using octonionic multiplication, and Re denotes the real part of an octonion. Since ψ_1 and ψ_2 are Majorana spinors, these bilinears have the symmetry properties of Eq. (1).

The 2×2 matrix representing the ten-vector $V^\mu = \bar{\psi}_1 \gamma^\mu \psi_2$ can be expressed in terms of the chiral components of ψ_1 and ψ_2 as

$$V = -(\xi_1 \xi_2^\dagger \pm \xi_2 \xi_1^\dagger) + (\eta_1 \eta_2^\dagger \pm \eta_2 \eta_1^\dagger)^t \tag{38}$$

if

$$\psi_i = \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix}$$

with the superscript t defined as in (33), and with the signs chosen as appropriate for commuting or anticommuting spinors.

In this formalism the Fierz identity (2) appears as a simple consequence of the properties of the octonions as a division algebra.[6,13] Suppose $\psi = \binom{\xi}{0}$; then according to (38) the ten-vector $\bar{\psi}\gamma^\mu \psi$ corresponds to the 2×2 matrix $V = -2\xi\xi^\dagger$ and so, by (34),

$$(\bar{\psi}\gamma_\mu \psi)\gamma^\mu \psi = -2 \begin{bmatrix} 0 \\ (\xi\xi^\dagger)^t \xi \end{bmatrix} . \tag{39}$$

Now if $\xi = \binom{p}{q}$ we have

$$(\xi\xi^\dagger)^t = \begin{bmatrix} -|q|^2 & p\bar{q} \\ q\bar{p} & -|p|^2 \end{bmatrix} \tag{40}$$

and then $(\xi\xi^\dagger)'\xi=0$ follows from the alternative law for the octonions.[14]

Since (2) is true for all commuting Majorana-Weyl spinors ψ, it amounts to an identity on products of γ matrices which is symmetric in three of its matrix indices. This identity yields (3) for anticommuting Majorana-Weyl spinors.

IV. SOULLESS SOLUTIONS

We now use the octonionic formalism to solve Eqs. (19)–(21). Let Π denote the 2×2 matrix corresponding to the ten-vector Π^μ, and write the Majorana-Weyl spinor θ as

$$\theta=\begin{bmatrix}\zeta\\0\end{bmatrix}, \tag{41}$$

where ζ has two octonion components. Then the equations to be solved are

$$\det\Pi=0\ , \tag{42}$$

$$\Pi'\partial_+\zeta=0\ , \tag{43}$$

$$\partial_+\Pi=0\ , \tag{44}$$

$$2(\partial_-\zeta)(\partial_+\zeta)^\dagger-2(\partial_+\zeta)(\partial_-\zeta)^\dagger=0\ . \tag{45}$$

As we have seen, Eq. (42) can be solved by taking

$$\Pi=\xi\xi^\dagger \tag{46}$$

for some two-component spinor ξ, i.e., $\Pi^\mu=\bar{\psi}\gamma^\mu\psi$ for some Majorana-Weyl spinor ψ; conversely, if $\det\Pi=0$ there is a two-component spinor ξ such that $\Pi=\xi\xi^\dagger$. Let $\xi=\binom{p}{q}$. This is a redundant parametrization of the lightlike vector Π; if ξ' is another possible choice, it must be of the form

$$\xi'=\begin{bmatrix}(pq^{-1})q'\\q'\end{bmatrix}\quad\text{with } |q'|=|q| \text{ if } q\neq 0$$

or (47)

$$\xi'=\begin{bmatrix}p'\\0\end{bmatrix}\quad\text{with } |p'|=|p| \text{ if } q=0\ .$$

Thus each lightlike vector corresponds to a 7-sphere of spinors ξ. (If we normalize Π by requiring $\Pi^0=\frac{1}{2}\mathrm{tr}\Pi=1$, then Π is restricted to an 8-sphere in the light cone and ξ is restricted to the 15-sphere $\xi^\dagger\xi=1$; the map $\xi\to\Pi$ is the Hopf map from the 15-sphere to the 8-sphere, whose fibers are 7-spheres.[15])

As is pointed out in Refs. 6 and 7, the supersymmetric identity (2) implies that if $\Pi^\mu=\bar{\psi}\gamma^\mu\psi$, then the equation of motion (20) is solved by $\partial_+\theta=\psi$; in our formalism, if Π is given by (46), then (43) is solved by $\partial_+\zeta=\xi$. The general solution of (43) in these circumstances is

$$\partial_+\zeta=\begin{bmatrix}(pq^{-1})\sigma\\\sigma\end{bmatrix}, \tag{48}$$

i.e., apart from a real multiple $\partial_+\zeta$ belongs to the same fiber as ξ under the Hopf map. If ζ were a commuting variable, it would follow that ξ could be replaced by $\partial_+\zeta$; in fact, for commuting variables we have

$$\Pi'\xi=0\leftrightarrow\Pi=r\xi\xi^\dagger\ , \tag{49}$$

where r is real [showing that the first equation of motion, Eq. (19), is a consequence of the second, Eq. (20)]. This would give a nonredundant parametrization of Π and $\partial_+\zeta$ by means of 17 real parameters (r,ξ). However, the Grassmann nature of $\partial_+\zeta$ means that we must retain the distinction between the parameters describing Π and those describing $\partial_+\zeta$. Since (p,q) occur only in the combination pq^{-1}, there are nine parameters (r,pq^{-1}) for Π and eight (σ) for $\partial_+\zeta$. [We keep the two symbols (p,q) and use the combination pq^{-1} to emphasize that these refer to the octonionic projective line, with a point at infinity included, so that the parameter space is an 8-sphere.]

On expanding $\partial_\pm\theta$ in terms of independent Grassmann elements, as in Sec. II, we are led to consider commuting spinors

$$\psi_i=\begin{bmatrix}\chi_i\\0\end{bmatrix},\quad \phi_j=\begin{bmatrix}\omega_j\\0\end{bmatrix}\qquad (i=1,\ldots,g;j=1,\ldots,h) \tag{50}$$

satisfying (26)–(28), which become

$$\Pi\chi_i=0\ , \tag{51}$$

$$\chi_i\omega_j^\dagger+\omega_j\chi_i^\dagger=\chi_j\omega_i^\dagger+\omega_i\chi_j^\dagger\quad(i<j\leq g)\ , \tag{52}$$

$$\chi_i\omega_j^\dagger+\omega_j\chi_i^\dagger=0\quad(j>g)\ . \tag{53}$$

With $\Pi=\xi\xi^\dagger$ and $\xi=\binom{p}{q}$, as before, the general solution of (51) is

$$\chi_i=\begin{bmatrix}(pq^{-1})s_i\\s_i\end{bmatrix}, \tag{54}$$

where s_i is any octonion. Now the general solution of (53) is

$$\omega_j=\begin{bmatrix}(pq^{-1})t_j\\t_j\end{bmatrix}\quad\text{with } \langle s_i,t_j\rangle=0\ , \tag{55}$$

where the angular brackets denote the Euclidean inner product between octonions. This can be seen as follows.

If χ_i is given, Eq. (53) is a linear equation for ω_j. Putting $\omega_j=\binom{x}{y}$ and writing out (53) as a matrix equation, using (54), shows that $\langle y,s_i\rangle=0$ and x is uniquely determined by y. Thus the space of ω_j satisfying (53) is seven dimensional. Now let $\chi(\tau)$ be a curve of spinors, labeled by a real parameter τ with $\chi(0)=\chi_i$, which belong to the same fiber as χ_i under the Hopf map: i.e.,

$$\chi(\tau)\chi(\tau)^\dagger=\chi_i\chi_i^\dagger\ . \tag{56}$$

Then, according to (47), $\chi(\tau)$ is of the form

$$\chi(\tau)=\begin{bmatrix}(pq^{-1})s(\tau)\\s(\tau)\end{bmatrix}\quad\text{with } |s(\tau)|=|s_i|\ . \tag{57}$$

Differentiating with respect to τ and putting $\tau=0$ shows that $\omega_j = \xi'(0)$ is a solution of (53). From (57) we see that this is of the form (55); since $t_j = s'(0)$ lies in the tangent space to the 7-sphere at $s(0) = s_i$, this is a seven-dimensional space of solutions.

Thus each ω_j with $j > g$ is specified by an octonion t_j which is orthogonal to $s_i, \ldots, s_g$. Since the s_i are independent, we can choose $8-g$ independent t_j to give $\omega_{g+1}, \ldots, \omega_8$.

Now we turn to Eq. (52). This equation is specifically due to the anticommuting nature of θ, and we do not have the most general solution. Clearly $\omega_i = \chi_i$ is one solution. More generally, if we suppose that ω_i, like χ_i, belongs to the same Hopf fiber as ξ, i.e., that

$$\omega_i = \begin{bmatrix} (pq^{-1})t_i \\ t_i \end{bmatrix} , \tag{58}$$

then (52) reduces to

$$\langle s_i, t_j \rangle = \langle t_i, s_j \rangle . \tag{59}$$

We can assume that $\langle s_i, s_j \rangle = \delta_{ij}$; then the general solution of (59) is

$$t_i = \sum_{j=1}^{g} a_{ij} s_j + t_i' , \tag{60}$$

where (a_{ij}) is a symmetric $g \times g$ matrix and t_i' is orthogonal to $s_1, \ldots, s_g$. The part of ω_i containing t_i' can be incorporated in ω_{g+1} and so we can assume that $t_i' = 0$.

We have now arrived at forms for $\partial_\pm \theta$ which we can describe as follows. We will say that two Majorana-Weyl spinors ψ_1, ψ_2 "belong to the same Hopf fiber" if

$$\psi_i = \begin{bmatrix} \xi_i \\ 0 \end{bmatrix} , \quad \xi_i = \begin{bmatrix} p_i \\ q_i \end{bmatrix} \quad \text{with } p_1 q_1^{-1} = p_2 q_2^{-1} . \tag{61}$$

We will call the octonion q_i the "Hopf parameter" of ψ_i. The solutions are

$$\Pi^\mu = \bar{\psi}_0 \gamma^\mu \psi_0 , \tag{62}$$

$$\partial_+ \theta = \sum_{i=1}^{g} \alpha_i \psi_i , \tag{63}$$

$$\partial_- \theta = \sum_{i=1}^{g} \alpha_i a_{ij} \psi_j + \sum_{i=g+1}^{8} \alpha_i \psi_i , \tag{64}$$

where $\alpha_1, \ldots, \alpha_8$ are anticommuting scalars, $\psi_0, \ldots, \psi_8$ are commuting Majorana-Weyl spinors belonging to the same Hopf fiber, the Hopf parameters of $\psi_1, \ldots, \psi_8$ being orthogonal unit octonions, and $a_{ij} = a_{ji}$.

From (63) we obtain an integrability condition which must be satisfied by α_i, ψ_i, and a_{ij}: namely,

$$\sum_{i=1}^{g} \partial_-(\alpha_i \psi_i) = \sum_{i=1}^{g} \partial_+(\alpha_i a_{ij} \psi_j) + \sum_{i=g+1}^{8} \partial_+(\alpha_i \psi_i) . \tag{65}$$

The solution (62)–(64) is probably the most general for $g \geq 3$, but we have not proved that $\psi_1, \ldots, \psi_g$ must belong to the same Hopf fiber as ψ_0. This is certainly not so if $g = 1$, when the general solution is

$$\partial_+ \theta = \alpha_1 \psi_1 , \tag{66}$$

$$\partial_- \theta = \alpha_1 \phi + \alpha_2 \psi_2 + \cdots + \alpha_8 \psi_8 , \tag{67}$$

where the ψ_i are as before but ϕ is arbitrary.

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