Modular invariance and stochastic quantization

Carlos R. Ordóñez and Mark A. Rubin
Theory Group, Rockefeller University, New York, New York 10021

Daniel Zwanziger
Physics Department, New York University, New York, New York 10003

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In Polyakov path integrals and covariant closed-string field theory, integration over Teichmüller parameters must be restricted by hand to a single modular region. This problem has an analog in Yang-Mills gauge theory—namely, the Gribov problem, which can be resolved by the method of stochastic gauge fixing. This method is here employed to quantize a simple modular-invariant system: the Polyakov point particle. In the limit of a large gauge-fixing force, it is shown that suitable choices for the functional form of the gauge-fixing force can lead to a restriction of Teichmüller integration to a single modular region. Modifications which arise when applying stochastic quantization to a system in which the volume of the orbits of the gauge group depends on a dynamical variable, such as a Teichmüller parameter, are pointed out, and the extension to Polyakov strings and covariant closed-string field theory is discussed.

I. INTRODUCTION

A. Modular transformations and string field theory

Modular transformations are discrete gauge transformations of the fields—spacetime coordinates and world-sheet metric—in the Polyakov path integral for the first-quantized string. As gauge transformations, they leave the path integrand invariant. Therefore, in evaluating the path integral, one must be sure to avoid "overcounting," that is, to avoid including in the path integral field configurations related by modular transformations. However, since the modular transformations are *discrete* symmetries, the usual Faddeev-Popov gauge-fixing technique is of no help in avoiding this particular source of overcounting, since that technique is tailored to *continuous* gauge symmetries. Of course, it is also important to avoid overcounting due to the other, continuous gauge symmetries of the Polyakov path integral. For this purpose the Faddeev-Popov technique is fine. If one first uses the Faddeev-Popov technique to avoid the multiple counting of field configurations related by these continuous gauge symmetries, one finds that the path integral is reduced from an integral over the infinite-dimensional space of field configurations to an integral over a finite-dimensional space labeled by coordinates which are termed "Teichmüller parameters." At any rate, gauge fixing with respect to *continuous* gauge symmetries can be accomplished in as systematic a fashion as in Yang-Mills theory.

However, we must still deal with the *discontinuous* modular symmetries. In terms of the reduced configuration space, i.e., the space of Teichmüller parameters, the effect of the modular transformations is to carve up this space into subregions, referred to as modular regions, such that the field configurations in any *one* modular region are, in fact, *all* possible physically distinct field configurations (that is, field configurations not related by either continuous or discrete gauge transformations). Any modular region is a gauge copy of any other modular region; specifically, one is mapped into another by a modular transformation. So, after we have gauge fixed with respect to continuous gauge symmetries using the Faddeev-Popov technique, gauge fixing with respect to the discrete modular symmetries can be accomplished by restricting the range over which the Teichmüller parameters vary so as to cover only a single modular region. This restriction to the modular region must be performed by hand, and differently for each world-sheet topology; there is no systematic algorithm, analogous to the Faddeev-Popov technique, for implementing it.

At the first-quantized level, this need for *ad hoc* restriction to the modular region is inconvenient; when we come to covariant closed-string field theory, however, the existence of modular symmetry presents more serious problems. The tree-level two-point function ("two-string" function, really) of a second-quantized theory must coincide with the propagator of the corresponding first-quantized theory. There exist covariant closed-string field theories which satisfy this requirement. However, when one connects up these propagators (which, for the closed string, are path integrals over world surfaces with tubular topology), according to the Feynman rules of the field theory, to form more complicated world surfaces, one finds that the resulting amplitudes suffer from overcounting. These (incorrect) field-theoretic amplitudes have the form of first-quantized path integrals in which the range of integration of Teichmüller parameters has (incorrectly) *not* been restricted to a single modular region. This comes about because the more complicated world surfaces constructed from the propagator tend to have *more* modular symmetry than that of the propagator. (In fact, the path integral for the propagator lacks modular symmetry completely.)

One approach to the construction of a correct covariant closed-string field theory is the following. To start, find an alternative method of *first* quantization in which, upon gauge fixing, the range of Teichmüller integration is "automatically" restricted to a single modular region, or, equivalently, in which the Teichmüller-space integrand automatically includes a weighting factor which assigns weights to different modular regions in such a way that each physically distinct field configuration is assigned a total weight of unity. A second-quantized theory based upon this modified first quantization, if it can be developed, might then be hoped to yield amplitudes which do not suffer from the modular-overcounting problem.

B. Gribov copies and stochastic gauge fixing

As an "alternative first-quantization method," we choose stochastic quantization. [Treatments of the string which involve stochastic quantization, but which do not deal with the question of modular invariance, may be found in Ref. 1 (stochastic first quantization) and Ref. 2 (stochastic second quantization).] The reason for this choice is the observation that stochastic quantization can provide a handle on otherwise difficult global questions. In particular, the modular copies in the first-quantized string theory are, in some respects, analogous to the Gribov copies which occur in Yang-Mills theory. In the latter case, the gauge slice obtained after fixing, Faddeev-Popov style, all continuous gauge continuous gauge symmetries, is *infinite* dimensional. This "infinite-dimensional Teichmüller space" consists of regions, analogous to the modular regions of the string's Teichmüller space, which are related to each other by "large" gauge transformations analogous to the string's modular transformations. There are differences: in the Yang-Mills case, these "large" transformations are continuous rather than discrete. (See the discussion in Sec. IV D and Fig. 1.) Also, it turns out that, if one limits oneself to *perturbative* computations in the Yang-Mills theory, the existence of Gribov regions on the gauge slice is irrelevant, since perturbation theory only probes a small region of configuration space near some background-field configuration. The first-quantized string path integral, on the other hand, can and must be evaluated exactly, so the existence of the Gribov-modular copies must be taken into account.

The relevant point, for our purposes, is that one may incorporate into stochastic quantization a procedure, known as stochastic gauge fixing, which not only accomplishes the task of the usual Faddeev-Popov technique, but also takes into account the problem of Gribov copies. We summarize here the key ideas; for a detailed discussion, see Ref. 3.

In stochastic quantization, one begins with the Euclidean path integral. Consider, for example, a theory of n fields $\phi_i(x), i=1,\ldots,n$, defined on a Euclidean-signature manifold whose coordinates we will denote for

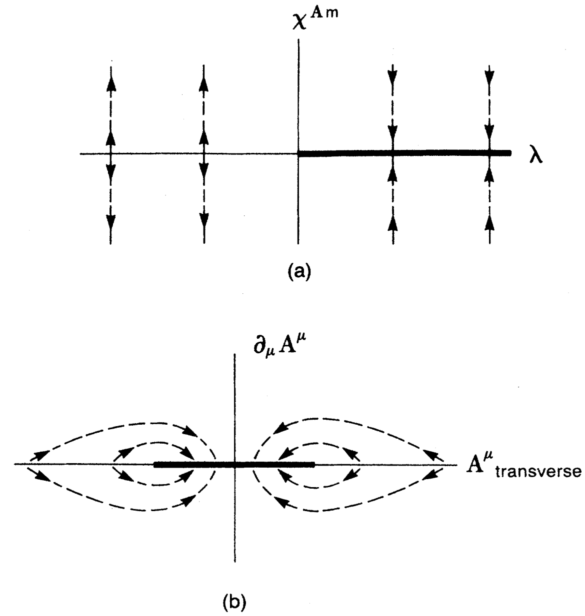

FIG. 1. Schematic representations of integral curves of continuous gauge transformations generated by the gauge-fixing force K_{gf}. (a) Polyakov point particle. Each curve moves either toward or away from the submanifold $\chi^{Am}=0$ $[e(\tau)=\lambda]$. Points with $\lambda>0$ (darker region) are connected to points with $\lambda<0$ by discrete gauge transformations only. For suitable choice of K_{gf} the flow is toward the region with $\lambda>0$ and away from the region with $\lambda<0$ (see Sec. IV). (b) Yang-Mills theory in Lorentz gauge. Each curve moves either toward or away from the submanifold $\partial_\mu A^{\mu=0}$. Points in the interior of the first Gribov horizon (darker region) are connected to points in the exterior by continuous gauge transformations. For suitable choice of K_{gf} the flow is toward the interior of the first Gribov horizon and away from the exterior (see Ref. 3).

now simply by x. If the classical action functional is $S_{\text{cl}}[\phi_1,\ldots,\phi_n]\equiv S_{\text{cl}}[\phi]$, the vacuum matrix element of a product of N fields is

$$\langle \phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)\rangle = \frac{1}{Z}\int \prod_{i=1}^{n} D\phi_i \phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)e^{-S_{\text{cl}}[\phi]}\,, \tag{1.1}$$

where Z is the vacuum-to-vacuum amplitude:

$$Z=\int \prod_{i=1}^{n} d\phi_i e^{-S_{\text{cl}}[\phi]}\,. \tag{1.2}$$

Since we are dealing with the Euclidean theory, we can regard (1.1) as an average in the usual sense of probability theory:

$$\langle \phi_{i_1}(x_1)\cdots\phi_{1_N}(x_N)\rangle = \int \prod_{i=1}^{n} D\phi_i \phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)P[\phi]\,, \tag{1.3}$$

where the "probability density" $P[\phi]$ is given by

$$P[\phi]=\frac{e^{-S_{\rm cl}[\phi]}}{Z} \tag{1.4}$$

and is manifestly normalized; from (1.2) and (1.4),

$$\int \prod_i D\phi_i P[\phi]=1 \; . \tag{1.5}$$

In stochastic quantization, the probability density $P[\phi]$ is viewed as the equilibrium limit of a diffusion process which takes place in the space of fields $\phi_i(x)$ and which obeys a diffusion equation, the Fokker-Planck equation:

$$\frac{\partial P_t[\phi]}{\partial t}=-\int dx \sum_{i=1}^{n} \frac{\delta}{\delta\phi_i(x)}\left[\left[-\frac{\delta}{\delta\phi_i(x)}+K_{{\rm cl},i}(x)\right]P[\phi]\right] \; . \tag{1.6}$$

[In actual practice we will employ a "generally covariant" form of (1.6); see Sec. II.] The variable t in (1.6) is referred to as the "fictitious time," since it is only the equilibrium value of $P_t[\phi]$, the value to which $P_t[\phi]$ relaxes at large t, which is to be used in Eq. (1.3) for computing physically meaningful averages:

$$\lim_{t\to\infty}\frac{\partial P_t[\phi]}{\partial t}=0 \; , \tag{1.7}$$

$$\lim_{t\to\infty} P_t[\phi]=P[\phi] \; . \tag{1.8}$$

Using (1.7) and (1.8), we see that $P[\phi]$, as given in (1.4), is a solution to the Fokker-Planck equation (1.6) provided that $K_{{\rm cl},i}(x)$, the so-called "classical drift force," is given by

$$K_{{\rm cl},i}(x)=-\frac{\delta S_{\rm cl}[\phi]}{\delta\phi_i(x)} \; . \tag{1.9}$$

If $K_{{\rm cl},i}(x)$ has this form, any initial distribution of probability $P_{t_0}[\phi], t_0<\infty$, will be "pushed" through configuration space so that it approaches the form (1.4) for large values of the fictitious time t.

The Fokker-Planck equation (1.6) is an Eulerian representation of the diffusion process; we sit at a given location ϕ in configuration space and watch the probability fluid $P_t[\phi]$ flow past. The corresponding Lagrangian description, in which we follow "a bit of field" ϕ on its travels, is given by the Langevin equation

$$\frac{\partial\phi_i}{\partial t}=K_{{\rm cl},i}(\lambda)+\eta_i(x) \; , \tag{1.10}$$

where $\eta_i(x)$ is a stochastic random variable.

Now, let the theory under consideration be a gauge theory. We can then add to $K_{{\rm cl},i}(x)$ a "gauge-fixing force" $\alpha K_{{\rm gf},i}(x)$ (α is a positive real parameter which we include for later convenience), where $K_{{\rm gf},i}(x)$ can be any function of ϕ, provided it points only in directions in field-configuration space corresponding to gauge transformations.[3,4] (The precise geometrical meaning of this statement will be given in Sec. IV.) Different choices for $K_{{\rm gf},i}(x)$ will lead to different equilibrium $P[\phi]$'s; but, since $\alpha K_{{\rm gf},i}(x)$ only causes ϕ to undergo a gauge transformation, these different P's will give identical results when used to evaluate gauge-invariant quantities. In the limit of the infinite the gauge-fixing force $\alpha\to\infty$, the $t\to\infty$ the probability distribution may be forced onto the submanifold where $K_{{\rm gf},i}(x)=0$ (e.g., in QED this might be the submanifold $\partial_\mu A^\mu=0$).

However, if the form of the gauge-fixing force is properly chosen, it causes the probability to concentrate, not on the entire gauge slice, but, rather, *within a single Gribov region on that gauge slice.* This is thus the type of "alternative quantization" we seek.

In this paper we apply stochastic gauge fixing to a simplified model, the Polyakov point particle, which possesses an analog of the modular invariance of the Polyakov string. We find that the method can yield the desired restriction of Teichmüller integration to a single modular region. (For suggested solutions to the modular problem along completely different lines, see Refs. 5 and 15.)

II. STOCHASTIC QUANTIZATION FOR REPARAMETRIZATION-INVARIANT THEORIES

A. The superspace-covariant Fokker-Planck equation

We remarked earlier that the expression (1.6) for the Fokker-Planck equation was not completely general. Specifically, the Fokker-Planck equation (1.6) is a continuity equation for the flow of probability density in "superspace," i.e., in the configuration space of the theory. It states that the rate of change of the probability density at a point in superspace is given by the negative of the divergence of a superspace probability current J_i:

$$\frac{\partial P_t[\phi]}{\partial t}=-\int dx\frac{\delta}{\delta\phi_i(x)}J_i(x) \; , \tag{2.1}$$

where

$$J_i(x)=\left[-\frac{\delta}{\delta\phi_i(x)}+K_i(x)\right]P[\phi] \; , \tag{2.2}$$

and where

$$K_i(x)=K_{{\rm cl},i}(x)+\alpha K_{{\rm gf},i}(x) \tag{2.3}$$

is the total drift force, the sum of classical and gauge-fixing contributions.

Equation (2.1) is only a correct equation of continuity if the ϕ_i's are Cartesian coordinates in a flat configuration space. To deal with the notions of flatness and curvature, we will introduce a metric structure in this space, and work with the modification of (2.1) and (2.2) which is generally covariant under changes of coordinates in superspace (configuration space):

$$\frac{\partial P_t}{\partial t}=-\frac{1}{\sqrt{\det G}}\int dx\frac{\delta}{\delta\phi^i(x)}[\sqrt{\det G}\;J^i(x)] \; , \tag{2.4}$$

where

$$J^i(x)=\left[-\int d\bar{x}\, G^{ij}(x,\bar{x})\frac{\delta}{\delta\phi^j(\bar{x})}+K^i(x)\right]P[\phi] \, . \tag{2.5}$$

Here we have introduced the superspace metric $G_{ij}(x,\bar{x})$, with inverse $G^{ij}(x,\bar{x})$ and determinant G. We have also placed the indices on superspace coordinates $\phi^i(x)$ and the contravariant vector $J^i(x)$ in their conventional "upstairs" position. So, for example, since the drift force is a tangent vector, as seen in the Langevin equation (1.10),

$$K^i(x)=K^i_{\mathrm{cl}}(x)+\alpha K^i_{\mathrm{gf}}(x) \, , \tag{2.6}$$

where

$$K^i_{\mathrm{cl}}(x)=-\int d\bar{x}\, G^{ij}(x,\bar{x})\frac{\delta S_{\mathrm{cl}}[\phi]}{\delta\phi^i(\bar{x})} \, . \tag{2.7}$$

Also, superspace integrals (path integrals) must be weighted by the volume element in superspace:

$$\langle \phi^{i_1}(x_1)\cdots\phi^{i_N}(x_N)\rangle = \frac{1}{Z}\int \prod_{i=1}^{n} D\phi_i \sqrt{\det G}\; \phi^{i_1}(x_1)\cdots\phi^{i_N}(X_N)P[\phi] \, . \tag{2.8}$$

There is one further, and extremely important, modification to the formalism that must be made in the case of reparametrization-invariant theories. The path integral in a gauge theory is not simply (1.1), or even its covariantized version. We really wish to perform the path integration only over gauge-inequivalent field configurations; when we integrate over *all* field configurations, we are doing the same as if we had performed the path integration over gauge-inequivalent configurations, but then weighted the contribution to the path integral of each configuration by the number of field configurations related to it by gauge transformations. To correct for this, we should divide the path integrand by $V_g[\phi]$, the volume of the gauge group:

$$\langle \phi^{i_1}(x_1)\cdots\phi^{i_N}(x_N)\rangle = \frac{1}{Z}\int \prod_{i=1}^{n} D\phi_i \sqrt{\det G}\; \phi^{i_1}(x_1)\cdots\phi^{i_N}(x_N)\frac{e^{-S_{\mathrm{cl}}[\phi]}}{V_g[\phi]} \, . \tag{2.9}$$

In the usual Faddeev-Popov approach to evaluating (2.9), one separates out of the path integral over the fields an infinite factor proportional to V_g, which cancels the factor of V_g in the denominator. Since the stochastic gauge-fixing force prevents this infinite factor from appearing, one might think that the inclusion of V_g in the denominator of (2.9) is unnecessary. However, as we will see in Polyakov point-particle or string theory, V_g is equal to an infinite constant—i.e., an infinity which is independent of the fields $\phi^i(x)$— times a finite "scale factor" $\tilde{V}_g$, which *may be* a functional of the ϕ's (through its dependence on the Teichmüller parameters) and thus cannot be ignored.[6] (In Yang-Mills theory $\tilde{V}_g$ is independent of the fields.) To ensure that the Fokker-Planck equation gives, in the $t\to\infty$ limit, a probability density $P[\phi]$ suitable for use in (2.9), we can proceed in one of two ways, which are (at least in the $t\to\infty$ limit) manifestly equivalent to each other.

We could retain the form (2.7) for the classical drift force $K^i_{\mathrm{cl}}(x)$ as minus the derivative of the usual classical action S_{cl}. If we use that form of $K^i_{\mathrm{cl}}(x)$ in the Fokker-Planck equation (2.4), the equilibrium (i.e., $\partial P_t/\partial t=0$) solution for P is the $P\propto e^{-S_{\mathrm{cl}}}$ as in (1.4). Then, in using P to compute matrix elements, we must, by hand, divide the path integrand by $\tilde{V}_g$.

Or, we can (and will) take an alternative form for the classical drift force:

$$\tilde{K}^i_{\mathrm{cl}}(x)=-\int d\bar{x}\, G^{ij}(x,\bar{x})\frac{\delta\tilde{S}_{\mathrm{cl}}[\phi]}{\delta\phi^i(\bar{x})} \, , \tag{2.10}$$

where $\bar{S}_{\mathrm{cl}}$ is the "effective classical action"

$$\tilde{S}_{\mathrm{cl}}[\phi]=S_{\mathrm{cl}}[\phi]+\ln\tilde{V}_g \, . \tag{2.11}$$

Using this in the Fokker-Planck equation (2.4) will give an equilibrium P proportional to $e^{-S_{\mathrm{cl}}}/\tilde{V}_g$, which, if used directly in (2.8), will give the desired path integral (2.9).

To proceed to solve the Fokker-Planck equation for $P[\phi]$, we must first specify the classical action S_{cl} as an explicit functional on a configuration space with coordinates $\phi^i(x)$. This we do in Sec. III.

B. Global restoring property of the gauge-fixing force

To provide a global gauge fixing, a requirement which the gauge-fixing force should satisfy, at least within a *continuously connected sector* of the orbit space, is the rather weak condition that it have the "restoring property" at large distances in superspace; that is, that it keep the probability from diffusing to infinity along the gauge orbits. This will be assured by constructing a force K_{gf} which satisfies globally, with respect to a suitably chosen functional I, the inequality $K_{\mathrm{gf}}\cdot\nabla I\le 0$. In dynamical theory, I is called a Liapunov function for the force field K_{gf}. The inequality implies that I decreases monotonically along the integral curves of K_{gf}. This assures the restoring property, at least within the sector, and so for this purpose the inequality is an adequate substitute for the condition $K_{\mathrm{gf}}=-\nabla I$ which is unavailable for nonconservative forces. If the gauge-fixing force is taken to be of the form αK_{gf}, where α is a gauge parameter, then in the limit $\alpha\to\infty$ the probability becomes concentrated on the absolute minimum which is achieved by the Liapunov functional I on each sector of the orbit. (The important issue of how the probability is distributed between the different sectors will be addressed in Sec. IV.)

Consider the diffusion equation

$$\partial P/\partial t=\nabla_e\cdot[(G^{-1}\nabla_e-K)P] \tag{2.12}$$

for the probability distribution $P=P[e,t)$, which is a functional of the einbein $e=e(\tau)$, and a function of the fictitious time t. [By decorating with appropriate indices and increasing the number of arguments of e, the discussion holds for a veilbein or metric tensor in a space of arbitrary dimension. Also, we temporarily ignore the path variable $x(\tau)$.] Here τ is a curve parameter, ∇_e is the functional derivative $\delta/\delta e(\tau)$, the center dot represents the divergence calculated with the superspace metric G, and G^{-1} is the inverse metric in superspace, which converts a one-form in superspace such as $\delta P/\delta e(\tau)$ into a tangent vector according to

$$(G^{-1}\delta P/\delta e)(\tau)=\int d\tau' G^{-1}(\tau,\tau')\delta P/\delta e(\tau') , \quad (2.13)$$

as, for example, in (2.7). The drift force K is made up of two pieces:

$$K=K_{\rm cl}+K_{\rm gf} , \quad (2.14)$$

a classical drift force which is conservative,

$$K_{\rm cl}(\tau)=-G^{-1}\delta S_{\rm cl}/\delta e)(\tau) , \quad (2.15)$$

and a "gauge-fixing force"

$$K_{\rm gf}(\tau)=(P_e v)(\tau) \quad (2.16)$$

which is not.[3] Here $v(\tau)$ is an arbitrary function of τ, so that $\epsilon v(\tau)$ represents an arbitrary infinitesimal reparametrization, and $\epsilon P_e v(\tau)$ is the corresponding change which this reparametrization induces in the einbein $e(\tau)$: namely,

$$\epsilon P_e v=\epsilon v\partial e/\partial\tau+\epsilon e\partial v/\partial\tau . \quad (2.17)$$

[In Eq. (3.19) $P_e v$ is written $L_v e$, where L_v is the Lie derivative of e with respect to v; however, the notation $P_e v$, which emphasizes the linear dependence on v, is more convenient here.]

How has v to be chosen in order for $K_{\rm gf}$ to have the restoring property globally, which keeps the probability distribution from drifting (or worse, being pushed) to infinity along gauge orbits? This is necessary to ensure that $P[e]=P[e,\infty)$ actually does exist. We will give v in terms of the Liapunov function I:

$$v=-G(\lambda)P_e^\dagger(\delta I[e]/\delta e)(\tau) , \quad (2.18)$$

where $P_e^\dagger=-P_e$, and where

$$\lambda=\int_0^1 d\tau\, e(\tau) . \quad (2.19)$$

We emphasize that we are here dealing with the restoring property within a *single connected sector*; in particular we have taken that sector to be continuously connected to the identity. (See Sec. IV concerning the possible choices in this regard.)

Under the action of $K_{\rm gf}$ alone–i.e., with the classical force $K_{\rm cl}$ and the noise force being "turned off" and v as in Eq. (2.18)—the Langevin equation for e becomes

$$\partial e/\partial t=K_{\rm gf}=P_e v \quad (2.20)$$

and $I[e]$ decreases monotonically, for we have

$$\begin{aligned}\partial I[e]/\partial t&=\int d\tau\,\delta I[e]/\delta e(\tau)(P_e v)(\tau)\\ &=-\int d\tau(P_e^\dagger\delta I[e]/\delta e)(\tau)(P_e^\dagger\delta I[e]/\delta e)(\tau)\\ &\le 0 .\end{aligned}$$

Note that no knowledge of the orbit space was required to achieve this global property. A reasonable choice for the Liapunov function is

$$I[e]=\int_0^1 d\tau\, e(\tau)[e(\tau)-\lambda]^2/e^2(\tau) . \quad (2.21)$$

With v as in (2.18), $K_{\rm gf}=P_e v$ is globally a restoring force, since $\partial I[e]/\partial t\le 0$ means that the motion always approaches the point $e(\tau)=\lambda$ in e space. The restoring property holds if v is replaced by αv, where α is a positive number (or even a positive kernel). By letting α approach infinity, so the gauge-fixing force gets arbitrarily large, the equilibrium distribution $P[e]$ gets concentrated on the absolute minimum of $I[e]$ on each gauge orbit.

III. THE POLYAKOV POINT PARTICLE

A. Action and configuration space

The Euclidean action for the Polyakov point particle is[7,11]

$$S_{\rm cl}=\tfrac{1}{2}\int_0^1 d\tau|g|^{1/2}\left[g^{ab}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\mu}{\partial\tau}+M^2\right] . \quad (3.1)$$

In this expression, τ is the parameter labeling the particle's world line. $g(\tau)$ is the determinant of the world-line metric tensor $g_{ab}(\tau)$, and $g^{ab}(\tau)$ is its inverse. The indices a,b are tensor indices on the world line; since the world line is one dimensional, they only take on a single value, and thus really only serve to denote tensor character. Finally, $x^\mu(\tau)$, for $\mu=1,\ldots,D$, are the coordinates of the particle word line in Euclidean spacetime. The integrand in (3.1) is clearly a scalar density on the world line of the particle. So, $S_{\rm cl}$ is invariant under all transformations of world-line coordinates, the orientation-preserving ones continuously connected to the identity, as well as the orientation-reversing ones. These world-line-coordinate transformations are the gauge transformations of the system.

We now introduce the einbein $e_a{}^\alpha(\tau)$, a one-form field on the world line satisfying

$$e_a{}^\alpha e_b{}^\beta\delta_{\alpha\beta}=g_{ab} . \quad (3.2)$$

As with a and b, the indices α and β each take on only a single value. In what follows we will sometimes dispense with these single-valued indices; but it is often useful to retain them to be sure that our expressions in fact have the correct tensor character, and so that we can make use of our familiarity with higher-dimensional tensor manipulations.[8] We will sometimes introduce an "index" on the world-line coordinate:

$$\tau^a=\tau . \quad (3.3)$$

The inverse einbein $e^a{}_\alpha(\tau)$ is a vector field on the world line satisfying

$$e^a{}_\alpha e_a{}^\beta=\delta_\alpha^\beta . \quad (3.4)$$

Denote by $e(\tau)$ the "determinant" of the einbein,

$$e=\det e_a{}^\alpha . \quad (3.5)$$

Then

$$e_a{}^\alpha = e \ , \tag{3.6a}$$

$$e^a{}_\alpha = e^{-1} \ , \tag{3.6b}$$

$$e^2 = g_{ab} = g \ . \tag{3.6c}$$

Using the above, $S_{\rm cl}$ in (3.1) becomes

$$S_{\rm cl} = \tfrac{1}{2}\int_0^1 d\tau \left[\frac{1}{|e|}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\mu}{\partial \tau} + |e| M^2 \right] \ . \tag{3.7}$$

B. Modular invariance

The Fokker-Planck equation describes diffusion in superspace; i.e., the configuration space of the theory. In the case at hand, that space is the direct product of two infinite-dimensional spaces; the e superspace with coordinates $e_a{}^\mu(\tau)$ and the x superspace with coordinates $x^\mu(\tau)$. Not all configurations $e_a{}^\mu(\tau), x^\mu(\tau)$ are physically distinct, in that configurations can be transformed into other configurations by gauge transformations. Let us replace the coordinates $e_a{}^\alpha(\tau)$ by a different set, such that some describe gauge-invariant degrees of freedom, and others label "pure gauge" aspects.

Define λ as in (2.19). Then we can write

$$e_a{}^\alpha(\tau) = \lambda \chi_a{}^\alpha(\tau) \ . \tag{3.8}$$

Note that λ is invariant under transformations of the world line coordinate τ which are continuously connected to the identity, and hence do not reverse the ends; but that it changes sign under "large" coordinate transformations, i.e., those that include a reversal of the ends, such as

$$\tau \to \tau' = 1 - \tau \ . \tag{3.9}$$

For orientation-preserving coordinate transformations, λ and $\chi_a{}^\alpha(\tau)$ are, respectively, the gauge-invariant and gauge-variant coordinates on the space of einbeins, related to the original coordinates $e_a{}^\mu(\tau)$ by (2.19) and (3.8).

Classically the einbein has the important property $e(\tau) \neq 0$ for all τ, because $e^2(\tau) = g(\tau)$ and the metric is strictly positive on a Riemannian manifold. Hence $e(\tau)$, which is continuous, never changes sign, so $\lambda \neq 0$. Moreover, λ has the sign of $e(\tau)$, so $\chi(\tau) > 0$. Because $\lambda = 0$ is excluded, the range of λ breaks up into two disjoint regions, $\lambda > 0$ and $\lambda < 0$, which are mapped into each other under (3.9). The effect of this mapping on the einbein is

$$e'(\tau') = -e(\tau) \tag{3.10}$$

by the tensor transformation law for einbeins. The configuration space thus falls into two disjoint regions, $e(\tau) > 0$ and $e(\tau) < 0$, characterized by the sign of λ. The changes of $e_a^\alpha(\tau)$ under world line coordinate transformations may be viewed as "motions in e superspace." Given any $e_a{}^\alpha(\tau)$, one can perform a change of world-line coordinates which transforms $e_a{}^\alpha(\tau)$ to the constant λ related to $e_a{}^\mu(\tau)$ by (3.8):

$$e_{\bar a}{}^\alpha(\bar\tau(\tau)) = \frac{\partial \tau}{\partial \bar\tau} e_a{}^\alpha(\tau) = \lambda \ , \tag{3.11}$$

where $\bar\tau(\tau)$ is such that

$$\frac{\partial \tau}{\partial \bar\tau} = \frac{\lambda}{e} = \chi \tag{3.12}$$

and is always defined for nonsingular e's. In terms of the e-superspace coordinates λ and $\chi(\tau)$, the transformation (3.11) maps the point $(\lambda, \chi(\tau))$ to the point $(\lambda, 1)$.

Let us evaluate the action (3.7) at the point in superspace where $e = \lambda$ (i.e., using the world-line coordinate system where $e = \lambda$):

$$S_{\rm cl} = \tfrac{1}{2}\int_0^1 d\tau \left[\frac{1}{|\lambda|}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\mu}{\partial \tau} + |\lambda| M^2 \right] \ . \tag{3.13}$$

However, since $S_{\rm cl}$ and $|\lambda|$ are invariant under all world-line coordinate transformations, the expression (3.13) is valid at *all* points in superspace. That is, at the point $(\lambda, \chi(\tau))$, $S_{\rm cl}$ has the form (3.13).

Note that the eigenvalues $\mu_n = (\pi n/\lambda)^2$ of the Laplacian operator defined by

$$-\frac{1}{e(\tau)}\frac{\partial}{\partial \tau}\frac{1}{e(\tau)}\frac{\partial}{\partial \tau}\psi_n(\tau) = \mu_n \psi_n(\tau)$$

are reparametrization invariant. The corresponding eigenfunctions are

$$\psi_n(\tau) = \sqrt{2}\sin(n\pi\tau^*) \ ,$$

where τ^* is the coordinate defined by

$$d\tau^* = \chi(\tau) d\tau \ . \tag{3.14}$$

The components $x_n{}^{\mu *}$ of $x^\mu(\tau)$ in the basis

$$x^\mu(\tau) = x_0^{\mu *} + \sum_{n>0} x_n^\mu \sqrt{2}\sin n\pi\tau^* \ , \tag{3.15}$$

$$x_0^\mu(\tau^*) = x_0^\mu(0) + [x^\mu(1) - x^\mu(0)]\tau^* \ , \tag{3.16}$$

are also coordinate invariant. Here τ^* is regarded as a function of τ. We take the x_n^* and $e(\tau)$ as global coordinates; so, as we will see in Sec. IV C, the gauge-fixing force, which is tangent to the gauge orbit, has no component in the x^* direction. Henceforth we will use the x_n^* coordinates and drop the $*$, except where specifically indicated otherwise. Note that we are using the eigenfunction expansion appropriate to the open interval with fixed end points. This will be used for the propagator in Sec. V. For the loop, treated in Sec. IV, we will use periodic eigenfunctions.

From (3.13) we see that $\chi(\tau)$ really *is* a "pure gauge" variable, in that $S_{\rm cl}$ is completely independent of it. $S_{\rm cl}$ does depend on the value of λ, but is invariant under

$$\lambda \to -\lambda \ . \tag{3.17}$$

These are the "modular transformations," i.e., the large gauge transformations of the Polyakov point particle. (The fact that the point particle possesses this simple analog of the string's modular invariance has been pointed out by Govaerts.[9]) Note that χ is only transformed by the parts of the gauge group which *are* continuously connected to the identity, i.e., those which are genuine world-line diffeomorphisms. (Every world-line coordinate transformation is the product of a diffeomorphism with either the identity or the inversion.)

C. Active viewpoint for superspace transformations

Active diffeomorphisms on a manifold are implemented by Lie dragging along the integral curves of a vector fields on the manifold.[8] Under an infinitesimal active diffeomorphism,

$$e_a{}^{\alpha}(\tau)\rightarrow e_a{}^{\alpha}(\tau)+\epsilon(L_{\xi}e^{\alpha})_a \ , \tag{3.18}$$

where the action of the Lie derivative with respect to the world-line vector field $\xi^a(\tau)$ on $e_a{}^{\mu}(\tau)$ is given by

$$(L_{\xi}e^{\alpha})_a=\xi^b\frac{\partial}{\partial\tau^b}e_a{}^{\alpha}+e_a{}^{\alpha}\frac{\partial}{\partial\tau^b}\xi^b \ , \tag{3.19}$$

or, dropping indices,

$$L_{\xi}e=\frac{\partial}{\partial\tau}(e\xi) \ . \tag{3.20}$$

Because $x^{\mu}(\tau)$ is a world-line scalar, the action of an infinitesimal diffeomorphism on $x^{\mu}(\tau)$ is simply

$$x^{\mu}(\tau)\rightarrow x^{\mu}(\tau)+\epsilon L_{\xi}x^{\mu}(\tau) \ , \tag{3.21}$$

where

$$L_{\xi}x^{\mu}=\xi\frac{\partial x^{\mu}}{\partial\tau} \ . \tag{3.22}$$

Integrating (3.18) over τ and using (3.19) and (3.20), we see that infinitesimal active diffeomorphisms have the following effect on λ:

$$\lambda\rightarrow\lambda+\epsilon(e\xi|_{\tau=1}-e\xi|_{\tau=0}) \ . \tag{3.23}$$

We also see from (3.21) and (3.22) that active diffeomorphisms will tend to move the initial and final locations of the particle, $x^{\mu}(0)$ and $x^{\mu}(1)$, unless ξ vanishes at $\tau=0,1$. But $x^{\mu}(0)$ and $x^{\mu}(1)$ must be fixed if we are computing the transition amplitude (propagator from one spacetime point to another), as we will below. We thus restrict all diffeomorphism-generating vector fields $\xi^a(\tau)$ to satisfy

$$\xi^a(0)=\xi^a(1)=0 \ . \tag{3.24}$$

Then, from (3.20) and (3.23),

$$L_{\xi}\lambda=0 \ , \tag{3.25}$$

$$\lambda\rightarrow\lambda \ , \tag{3.26}$$

while use of (3.8), (3.20), and the Leibniz rule for Lie derivatives gives

$$L_{\xi}\chi=\frac{\partial}{\partial\tau}(\chi\xi) \ . \tag{3.27}$$

D. The superspace metric

The superspace metric G is a symmetric, bilinear, positive-semidefinite map from tangent vectors on superspace to the real numbers. We can view the tangent vectors' components, in the $(e_a{}^{\alpha}(\tau),x^{\mu}(\tau))$ coordinate basis, as "small deformation" of the coordinates, which we write as $\delta e_a{}^{\alpha}(\tau),\delta x^{\mu}(\tau)$. If we demand that G be a *reparametrization-invariant* mapping (i.e., the real numbers it gives as output are scalars under all changes of world-line coordinates) and that G is "ultralocal," i.e., involves no derivatives with respect to superspace coordinates, then G is block diagonal and, up to overall constants multiplying each block, acts in the following manner:

$$(\delta e_1,G\delta e_2)=\int_0^1 d\tau\int_0^1 d\bar{\tau}\delta e_1(\tau)G(\tau,\bar{\tau})\delta e_2(\bar{\tau}) \ , \tag{3.28a}$$

$$(\delta x_1,G\delta x_2)=\int_0^1 d\tau\int_0^1 d\bar{\tau}\delta x_1{}^{\mu}(\tau)G_{\mu\nu}(\tau,\bar{\tau})deltsx_2{}^{\nu}(\bar{\tau}) \ , \tag{3.28b}$$

$$(\delta e,G\delta x)=\int_0^1 d\tau\int_0^1 d\bar{\tau}\delta e(\tau)G(\tau,\bar{\tau})_{\nu}\delta x^{\nu}(\bar{\tau}) \ , \tag{3.28c}$$

$$(\delta x,G\delta e)=\int_0^1 d\tau\int_0^1 d\bar{\tau}\bar{\delta}\delta x^{\mu}(\tau)G_{\mu}(\tau,\bar{\tau})detae(\bar{\tau}) \ , \tag{3.28d}$$

where

$$G(\tau,\bar{\tau})=|e|^{-1}\delta(\tau-\bar{\tau}) \ , \tag{3.29a}$$

$$G_{\mu\nu}(\tau,\bar{\tau})=|e|\delta_{\mu\nu}\delta(\tau-\bar{\tau}) \ , \tag{3.29b}$$

$$G(\tau,\bar{\tau})_{\nu}=G_{\mu}(\tau,\bar{\tau})=0 \ . \tag{3.29c}$$

In principle, we could now proceed to solve the Fokker-Planck equation, either by using the supermetric components (3.29a)–(3.29c) in the $(e(\tau),x^{\mu}(\tau))$ basis, or by transferring to the $(\lambda,\chi(\tau),x^{\mu}(\tau))$ basis. However, it simplifies matters greatly to work with a set of superspace coordinates labeled by discrete indices, rather than the continuous index τ; and the specific simplifying choice of discretely indexed coordinates depends on the specific computation we are doing. We will start with the simplest possible one, the vacuum amplitude: i.e., a closed loop.

IV. THE LOOP

A. The superspace metric

The configuration space here is the set of spacetime embeddings and einbeins for a world line with the topology of a circle; that is, all functions on the world line must be periodic in τ with unit period. So, for a set of discrete coordinates in the e superspace, we choose the Fourier components of $\chi(\tau)$:

$$\chi(\tau)=\chi^0+\sqrt{2}\sum_{m=1}^{\infty}\chi^{Cm}\cos(2\pi m\tau)+\sqrt{2}\sum_{m=1}^{\infty}\chi^{Sm}\sin(2\pi m\tau) \ . \tag{4.1}$$

The e-superspace coordinates $\{\lambda,\chi^0,\chi^{Cm},\chi^{Sm},m=1,2,\ldots\}$ are not independent, since $\chi(\tau)$ is subject to the constraint

$$\int_0^1 d\tau\chi(\tau)=1 \ , \tag{4.2}$$

as follows from (2.19) and (3.8). In terms of the discrete coordinates, this implies

$$\chi^0 = 1 \ . \tag{4.3}$$

So,

$$\chi(\tau) = 1 + \sum_{A=C,S} \sum_{m=1}^{\infty} \chi^{Am} h_{Am}(\tau) \ , \tag{4.4a}$$

$$e(\tau) = \lambda + \sum_{A=C,S} \sum_{m=1}^{\infty} \chi^{Am} \lambda h_{Am}(\tau) \ , \tag{4.4b}$$

where

$$h_{Cm} = \sqrt{2}\cos(2\pi m\tau), \quad m = 1,2,\ldots \ , \tag{4.4c}$$

$$h_{Sm} = \sqrt{2}\sin(2\pi m\tau), \quad m = 1,2,\ldots \ . \tag{4.4d}$$

In the x superspace we use the Fourier components of $x^{\mu}(\tau)$ as coordinates:

$$x^{\mu}(\tau) = \sum_{m=0}^{\infty} x^{\mu Cm} h_{Cm}(\tau) + \sum_{m=1}^{\infty} \chi^{\mu Sm} h_{Sm} \ , \tag{4.5}$$

with

$$h_{C0}(\tau) = 1 \ . \tag{4.6}$$

A set of unconstrained coordinates on superspace is thus

$$\{Z^i\} \equiv \{\lambda, \chi^{Am}, x^{\mu Am}\} \ . \tag{4.7}$$

The coordinates (4.7) determine a holonomic basis for tensors in superspace. We already know the components of the superspace metric in the coordinates $\{e(\tau), x^{\mu}(\tau)\}$ [Eqs. (3.29a)–(3.29c)] and the explicit relation between the two sets of coordinates [Eqs. (4.4b) and (4.5)] so we can compute the components of the superspace metric in the coordinate basis (4.7) using the usual rule for changing tensor components from one coordinate basis to another. We find

$$G_{\lambda\lambda} = |\lambda|^{-1} \ , \tag{4.8a}$$

$$G_{Am,Bn} = |\lambda| \phi_{Am,Bn} \ , \tag{4.8b}$$

$$G_{\mu Am,\nu Bn} = |\lambda| \delta_{\mu\nu} \delta_{AB} \delta_{mn} \ , \tag{4.8c}$$

where

$$\phi_{Am,Bn} \equiv \int_0^1 d\tau \frac{h_{Am}(\tau) h_{Bn}(\tau)}{|\chi(\tau)|} \ , \tag{4.8d}$$

$$\phi_{Am,Bn} = \delta_{AB} \delta_{mn} \ \text{ if } \chi^{Am} = 0 \ \ \forall_{A,m} \ . \tag{4.8e}$$

The contravariant components are thus

$$G^{\lambda\lambda} = |\lambda| \ , \tag{4.9a}$$

$$G^{Am,Bn} = |\lambda|^{-1} \phi^{Am,Bn} \ , \tag{4.9b}$$

$$G^{\mu Am,\nu Bn} = |\lambda|^{-1} \delta_{\mu\nu} \delta_{AB} \delta_{mn} \ , \tag{4.9c}$$

where $\phi^{Am,Bn}$ is the matrix inverse to $\phi_{Am,Bn}$. All other components vanish. The expression (4.8a) for the Teichmüller piece of the supermetric, as well as the statement that the gauge-Teichmüller piece $G_{\lambda,\, Am}$ vanishes is actually valid only in the region of superspace where

$$\chi(\tau) > 0 \quad \forall \tau \ . \tag{4.10}$$

[Indeed, as already mentioned, any χ arising from a nonsingular einbein must satisfy (4.10).] Since the solution we ultimately find for P will have support only in an (infinitesimal) region near $\chi(\tau) = 1 \ \ \forall \tau$, this leads to no inconsistency.

From (4.8a)–(4.8c), the determinant of the (covariant) metric is

$$\det G = G_{\lambda\lambda} \left[\prod_{A=C,S} \prod_{m=1}^{\infty} G_{Am,\,Am} \right] \prod_{\mu=1}^{D} \left[G_{\mu C0,\mu C0} \left[\prod_{A=C,S} \prod_{m=1}^{\infty} G_{\mu Am,\mu Am} \right] \right] \ , \tag{4.11}$$

where D is the dimensionality of Euclidean spacetime. Evaluating the infinite products by the formula, derivable using ζ-function regularization,[10,11]

$$\prod_{m=1}^{\infty} a m^{b} = a^{-1/2} (2\pi)^{b/2} \tag{4.12}$$

we obtain

$$\det G = |\lambda|^{-2} \phi \ , \tag{4.13a}$$

where

$$\phi \equiv \det \phi_{Am,Bn} \ . \tag{4.13b}$$

(In evaluating (4.11) we have made use of (4.12) with $b=0$; that is,

$$\prod_{m=1}^{\infty} a = a^{-1/2} \ . \tag{4.12'}$$

Clearly, the answer will depend on how we group the factors in the infinite product; as indicated in (4.11), we have separated out the zero mode and applied (4.12′) individually to the nonzero cosine and sine modes. If we had used a slightly different set of superspace coordinates (see, e.g., Ref. 11), b would not have been zero, and the grouping of factors would have been unique. Here, we employ this grouping throughout [see Eqs. (4.19) and (4.20)]. It would be nice to demonstrate in a rigorous fashion that this grouping is mandatory as well when $b=0$, without having to appeal to other superspace coordinate systems; but we have not done so.) Thus, in (2.9),

$$\int \prod_i D\phi_i \sqrt{\det G} \to \int_{-\infty}^{\infty} d\lambda |\lambda|^{-1} \prod_{A=C,S} \prod_{m=1}^{\infty} \left[\int_{-\infty}^{\infty} d\chi^{Am} \phi(\chi) \prod_{\mu=1}^{D} dx^{\mu Am} \right] \ . \tag{4.14}$$

B. The classical drift force

In terms of the Fourier expansion (4.5) of $x^\mu(\tau), S_{\rm cl}$ in (3.13) becomes

$$S_{\rm cl}=\sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2 m^2}{|\lambda|}(x^{\mu Am})^2+\frac{|\lambda|}{2}M^2 .$$

To calculate the effective action $\tilde{S}_{\rm cl}$, as described in Sec. II A, we need V_g, the volume of the group of gauge transformations. As discussed in Sec. II A we only need to know V_g up to a field-independent factor; the discrete gauge transformations give just such a field-independent factor, so we need only compute the volume of the diffeomorphisms. Diffeomorphisms are parametrized by world-line vector fields $\xi^a(\tau)$, so we may take as a discrete set of coordinates in the space of diffeomorphisms the Fourier components of $\xi^a(\tau)$. In the notation of (4.4c), (4.4d), and (4.6),

$$\xi^a(\tau)=\sum_{m=0}^{\infty}\xi^{Cm}h_{Cm}(\tau)+\sum_{m=1}^{\infty}\xi^{Sm}h_{Sm}(\tau) . \tag{4.15}$$

Thus,

$$V_g=\int\prod_{m=0}^{\infty}d\xi^{Cm}\prod_{n=1}^{\infty}d\xi^{Sn}\sqrt{\det H} , \tag{4.16}$$

where $H_{ab}(\tau,\tau')$ is a suitable metric tensor on the space of ξ's, i.e., a bilinear symmetric map from small deformations of the ξ's to positive real numbers:

$$(\delta\xi_1,H\delta\xi_2)=\int_0^1 d\tau\int_0^1 d\bar{\tau}\,\delta\xi_1^a(\tau)H_{ab}(\tau,\bar{\tau})\delta\xi_2^b(\bar{\tau}) .$$

Demanding reparametrization invariance determines H up to an overall constant:

$$H_{ab}(\tau,\tau')=|e(\tau)|e_a{}^{\alpha}(\tau)e_b{}^{\beta}(\tau')\delta(\tau-\tau') , \tag{4.17a}$$

or, dropping indices,

$$\begin{aligned}H(\tau,\tau')&=|e(\tau)|^3\delta(\tau-\tau')\\&=|\lambda|^3|\chi(\tau)|^3\delta(\tau-\tau') .\end{aligned} \tag{4.17b}$$

In the ξ^{Am} basis,

$$\begin{aligned}H_{Am,Bn}&=\int_0^1 d\tau\int_0^1 d\bar{\tau}\frac{\partial\xi^a(\tau)}{\partial\xi^{Am}}\frac{\partial\xi^b(\bar{\tau})}{\partial\xi^{Bn}}H_{ab}(\tau,\bar{\tau})\\&=|\lambda|^3\delta_{AB}\delta_{mn}\int_0^1 d\tau[h_{Am}(\tau)]^2|\chi(\tau)|^3 .\end{aligned} \tag{4.18}$$

Therefore,

$$\det H=H_{C0,C0}\prod_{A=C,S}\prod_{m=1}^{\infty}H_{Am,Am}=|\lambda|^3\int_0^1 d\tau|\chi|^3\prod_{A=C,S}\prod_{m=1}^{\infty}\lambda^3\int_0^1 d\tau(h_{Am})^2|\chi|^3 . \tag{4.19}$$

Applying (4.12) to the infinite products in (4.19),

$$\det H=|\lambda|^3(|\lambda|^{-3/2})^2\left[\int_0^1 d\tau|\chi|^3\right]\left[\prod_{A=C,S}\prod_{m=1}^{\infty}\int_0^1 d\tau(h_{Am})^2|\chi|^3\right] . \tag{4.20}$$

So $\det H$ is independent of λ,

$$\det H=\det H(\chi) . \tag{4.21}$$

Using this in (4.16),

$$V_g=\int\prod_{m=0}^{\infty}d\xi^{Cm}\prod_{n=1}^{\infty}d\xi^{Sn}\sqrt{\det H(\chi)} . \tag{4.22}$$

So V_g, and hence $\tilde{V}_g$ (see Sec. II A), is independent of λ.

Any $\chi(\tau)$ can be obtained by starting from some arbitrarily chosen fixed $\chi_0(\tau)$ and Lie dragging by a suitable vector field ξ^a; e.g., for χ close to χ_0,

$$\chi=(1+L_\xi)\chi_0 \tag{4.23}$$

[see (3.27)]. So the integration in (4.22) is over the entire space of χ's and it turns out, for the case of the loop, that V_g is independent, not only of λ and $x^\mu(\tau)$, but also of $\chi(\tau)$ and can therefore be ignored.

Using (2.10), (4.9a)–(4.9c), and the above expression for $S_{\rm cl}$ in terms of the Fourier components of $x^\mu(\tau)$, we find the components of the classical drift force to be

$$K_{\rm cl}^{\lambda}=\sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2m^2}{\lambda}(x^{\mu Am})^2-\frac{\lambda}{2}M^2 , \tag{4.24a}$$

$$K_{\rm cl}^{Am}=0 , \tag{4.24b}$$

$$K^{\mu Am}=-\frac{4\pi^2m^2}{\lambda^2}x^{\mu Am} . \tag{4.24c}$$

As expected, the components of the classical drift force in the pure-gauge χ^{Am} directions vanish, since the classical action is gauge invariant.

C. The gauge-fixing force

For the gauge-fixing force we must choose a field of vectors in superspace which is everywhere tangent to some set of integral curves generated by continuous gauge transformations. Any such set of curves in superspace corresponds to a vector field $\xi^a(\tau)$ on the world line. Let ϵ be the curve parameter. Then, near any point e in superspace, the equation of the curve (to first order in ϵ) is

$$e(\epsilon)=e+\epsilon L_\xi e . \tag{4.25}$$

The tangent vector (derivative operator)[8] to this curve is given by

$$\frac{\partial}{\partial\epsilon}=\int_0^1 d\tau\frac{\partial e(\tau)}{\partial\epsilon}\frac{\delta}{\delta e(\tau)} . \tag{4.26}$$

So, in the e-coordinate basis,

$$K_{\rm gf}(\tau)=L_{\xi}e \ . \tag{4.27}$$

To proceed further, we must make a choice for ξ, specifying its dependence on e. One such choice of dependence has already been discussed, i.e., the one obtained in Sec. II B on the basis of considerations of global stability, Eq. (2.16). In what follows, we will be working near the "gauge slice," i.e., at $\chi(\tau)\approx 1$. In this regime, using Eqs. (2.17, (2.18), and (2.21), Eq. (2.16) reduces to

$$K_{\rm gf}(\tau)=2G(\lambda)\lambda^2\frac{\partial^2\tilde{\chi}(\tau)}{\partial\tau^2} \ . \tag{4.28}$$

We can also arrive at a choice for $K_{\rm gf}$ from heuristic considerations of tensor "index balancing." If we want a world-line vector related to $e_a{}^{\alpha}$, we could of course use the inverse einbein $e^a{}_{\alpha}(\tau)$. This would lead to a vanishing component (4.27) of the gauge-fixing force in the e direction as well, since $e_a{}^{\alpha}(\tau)e_{\beta}{}^{b}(\tau)=e(\tau)(e(\tau))^{-1}=1$. If we allow ourselves a single world-line derivative operator, we can obtain

$$\xi^a(\tau)=F(\lambda)\delta^{\alpha\beta}e^a{}_{\alpha}e^b{}_{\beta}e^c{}_{\gamma}\nabla_b e_c{}^{\gamma} \ , \tag{4.29}$$

or, dropping superfluous indices,

$$\xi(\tau)=F(\lambda)e^{-3}(\tau)\nabla e(\tau) \ , \tag{4.30}$$

where $\nabla=\nabla_a$ is the world-line covariant derivative operator, and $F(\lambda)$ is any function of λ. [Note that the introduction of $\delta^{\alpha\beta}$ in (4.29) selects a preferred world-line frame. One might contract the indices differently in (4.29), or start from the covariant derivative of the inverse einbein, but the final form (4.30) will be the same.] Using (3.20), (4.27), and (4.30),

$$\begin{aligned}K_{\rm gf}(\tau)&=F(\lambda)\frac{\partial}{\partial\tau}(e^{-2}(\tau)\nabla e)\\&=\frac{F(\lambda)}{\lambda}\frac{\partial}{\partial\tau}(\chi^{-2}\nabla\chi) \ .\end{aligned} \tag{4.31}$$

In the regime $\chi(\tau)\approx 1$, $K_{\rm gf}$ in (4.31) corresponds to $K_{\rm gf}$ in (4.28) with $F(\lambda)=2G(\lambda)\lambda^3$. [Of course, it is the form of $K_{\rm gf}$ in Eq. (2.16) which we know to have the global stability properties; we do not know whether this holds for the form (4.31).]

$K_{\rm gf}$ is a contravariant vector in superspace, so to directly transform the components (4.31) in the e-coordinate basis into the components in the $\{\lambda,\chi^m\}$ coordinate basis, we would need the "partial derivatives" $\delta\lambda/\delta e(\tau)$. It is simpler to first multiply the contravariant components by the metric (3.29a)–(3.29c) to obtain the covariant e-basis components, transform the covariant components from the e basis to the $\{\lambda,\chi^{Am}\}$ basis, then obtain the contravariant components in that basis using the inverse-metric components (4.9a)–(4.9c). The result is

$$K_{\rm gf}{}^{\lambda}=0 \ , \tag{4.32}$$

$$K_{\rm gf}{}^{Am}=\sum_{B,n}\phi^{Am,Bn}\frac{F(\lambda)}{\lambda^2}\int_0^1 d\tau\, h_{Bn}\chi^{-1}\frac{\partial}{\partial\tau}(\chi^{-2}\nabla\chi) \ . \tag{4.33}$$

As discussed in Sec. III B, the coordinates χ_n^* are diffeomorphism invariant; so

$$K_{\rm gf}{}^{\mu Am}=0 \ . \tag{4.34}$$

D. The Fokker-Planck equation

Using (2.4)–(2.6), (4.9), (4.13), (4.24), and (4.32)–(4.34), we find the time-independent ($\partial P/\partial t=0$) Fokker-Planck equation in the $\{\lambda,\chi^{Am},x^{\mu Am}\}$ basis to be

$$\begin{aligned}0=&\frac{\partial}{\partial\lambda}\left[|\lambda|^{-1}|\phi|\left[-|\lambda|\frac{\partial}{\partial\lambda}+\sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2m^2}{\lambda}(x^{\mu Am})^2-\frac{\lambda M^2}{2}\right]P\right]\\&+\sum_{\mu=1}^{D}\sum_{\substack{A,m\\B,n}}\frac{\partial}{\partial x^{Am}}\left[|\lambda|^{-1}|\phi|^{1/2}\phi^{Am,Bn}\left[-|\lambda|^{-1}\frac{\partial}{\partial\chi^{Bn}}+\alpha\frac{F(\lambda)}{\lambda^2}\int_0^1 d\tau\, h_{Bn}\chi^{-1}\frac{\partial}{\partial\tau}(\chi^{-2}\nabla\chi)\right]P\right]\\&+\sum_{A,m}\frac{\partial}{\partial x^{Am}}\left[|\lambda|^{-1}|\phi|^{1/2}\left[-|\lambda|^{-1}\frac{\partial}{\partial x^{\mu Am}}+\frac{4\pi^2m^2}{\lambda^2}x^{\mu Am}\right]P\right] \ ,\end{aligned} \tag{4.35}$$

where

$$\sum_{A,m}f_{A,m}\equiv\sum_{m=0}^{\infty}f_{C,m}+\sum_{m=1}^{\infty}f_{S,m} \ . \tag{4.36}$$

Following Ref. 3 we will solve this in the limit of large gauge-fixing force; that is, in the limit $\alpha\to\infty$.

Write P as

$$\begin{aligned}P\underset{\alpha\to\infty}{\sim}&N\exp[-\alpha\Gamma_0(\lambda,\chi,x)-\Gamma_1(\lambda,\chi,x)\\&+O(\alpha^{-1})] \ .\end{aligned} \tag{4.37}$$

N is a normalization constant independent of λ,χ, and x, but in general dependent on α.

As noted in Sec. III B, the configuration space consists of two regions characterized by the sign of λ. An important question is whether the diffusion starting from either $\lambda>0$ or $\lambda<0$ ever reaches $\lambda=0$. The limit point $\lambda=0$ is strongly repulsive from both regions, as one sees from Eq. (4.24a) for $K_{\rm cl}{}^{\lambda}$, the classical force in the λ direction. Its behavior for small λ is

$$K_{\rm cl}^{\lambda}\sim 1/\lambda$$

which suggests that $\lambda=0$ is never reached. If this is in fact the case, the Fokker-Planck equation is reducible and describes two independent stochastic processes, one in each fundamental region. (A simple example of a reducible stochastic process is a random walk on the integers, where each step is restricted to ± 2. In this case the even and odd integers form two disjoint regions.) In what follows, we will find a normalized solution in the fundamental region $\lambda>0$ and we will see that it vanishes as λ approaches zero; which indicates that no transitions occur between the two fundamental regions, at least in the $\alpha\to\infty$ limit.

Note also that the integral curves of $K_{\rm gf}$ do not cross from one fundamental region to another [Fig. 1(a)]. [Indeed, if $e(\tau)$ evolves through fictitious time t only under the influence of $K_{\rm gf}$—i.e., $K_{\rm cl}$ and the noise force η are set to zero, as in (2.20)—then λ does not change with t. To see this, integrate both sides of (2.20) over τ, use the definition of λ and the fact that the right-hand side of (2.20) is a total derivative.] This is to be contrasted with the case of Yang-Mills theory in Landau-Lorentz gauge $\partial_\mu A^\mu=0$. There the embedding in superspace of the submanifold $\partial_\mu A^\mu=0$ [analogous to the submanifold $e(\tau)=\lambda$ for the Polyakov point particle] is such that integral curves of continuous gauge transformations do connect one Gribov region to another [Fig. 1(b)].

At any rate, because of the singularities at $\lambda=0$, the FP equation (4.35) is really two equations, one for $\lambda>0$, the other for $\lambda<0$, with separate solutions for each of these two sectors.

Using (4.37) in (4.35), we find in either sector, to leading order in α,

$$0=P\alpha^2\left\{|\lambda|\left[\frac{\partial\Gamma_0}{\partial\lambda}\right]^2+\sum_{A,m}\left[\sum_{B,n}\phi^{Am,Bn}\left[\frac{\partial\Gamma_0}{\partial\chi^{Am}}\right]\left[\frac{\partial\Gamma_0}{\partial\chi^{Bn}}\right]+\frac{\partial\Gamma_0}{\partial\chi^{Am}}K_{\rm gf}{}^{Am}\right]+\sum_{\mu=1}^{D}\sum_{A,m}\left[\left[\frac{\partial\Gamma_0}{\partial x^{\mu Am}}\right]^2\right]\right\}. \tag{4.38}$$

So, either $P=0$, or, in the notation of (4.7),

$$\sum_{i,j}\frac{\partial\Gamma_0}{\partial Z^i}G^{ij}\frac{\partial\Gamma_0}{\partial Z^i}+\sum_i\frac{\partial\Gamma_0}{\partial Z^i}K^i_{\rm gf}=0\ . \tag{4.39}$$

Since G is positive definite [see (3.29a)–(3.29c)] we can conclude that

$$K^i_{\rm gf}=0\Longrightarrow\frac{\partial\Gamma_0}{\partial Z^i}=0\ . \tag{4.40}$$

At the same time, from (4.4a) and (4.32)–(4.34) we see that

$$\chi^{Am}=0\Longrightarrow K_{\rm gf}{}^i=0\ . \tag{4.41}$$

The above two relations tell us that the Taylor expansion of Γ_0 in χ^{Am} about $\chi^{Am}=0$ has the form

$$\Gamma_0(\lambda,\chi,x)=\gamma_0+\tfrac{1}{2}\sum_{\substack{A,m\\B,n}}\chi^{Am}M_{Am,Bn}(\lambda,x)\chi^{Bn}+O((\chi^{Am})^3)\ . \tag{4.42}$$

γ_0 is independent of λ, χ, and x except in that, as mentioned above, it may have different values in the two sectors $\lambda<0$ and $\lambda>0$. Using (4.42) and working to lowest order in χ^{Am}, (4.38) becomes

$$0=\bar{\chi}^TMG^{-1}M\chi+\chi^Tm\mathbf{K}_{\rm gf}\ . \tag{4.43}$$

Here we have suppressed indices (Am) in favor of matrix notation: e.g., $\mathbf{K}_{\rm gf}\equiv K_{\rm gf}{}^{Am}$. The terms discarded in going from (4.38) to (4.42) are all $O((\chi^{Am})^3)$; note that, from (4.42), $\partial\Gamma_0/\partial\lambda$ and $\partial\Gamma_0/\partial x^{\mu Am}$ are $O(\chi^2)$, and, from (4.33), that $K_{\rm gf}{}^{Am}$ is linear in χ [the Christoffel symbol in ∇ is proportional to $(\partial/\partial\tau)\chi(\tau)=(\partial/\partial\tau)(\sum_{A,\mu\neq 0}h^{Am}\chi_{Am})$]. Write $\mathbf{K}_{\rm gf}$ as

$$\mathbf{K}_{\rm gf}=N\cdot\chi+O(\chi^2)\ , \tag{4.44}$$

where

$$N_{Am\,Bn}=-\frac{F(\lambda)}{\lambda^2}4\pi^2m^2\delta^A_B\delta^m_n\ . \tag{4.45}$$

We will from this point on drop the $O(\chi^2)$ parts of $K_{\rm gf}{}^{\rm Am}$; the justification for this is one of self-consistency, in that our ultimate solution will have support only at $\chi^{Am}=0$. Then (4.43) becomes

$$0=\chi^T(MG^{-1}M+MN)\chi\ . \tag{4.46}$$

This holds for arbitrary χ, so the symmetric part of the quantity in parentheses in (4.46) vanishes:

$$0=2MG^{-1}M+MN+N^TM\ . \tag{4.47}$$

Following Ref. 3 we find

$$M_{Am,Bn}=\frac{F(\lambda)}{|\lambda|}4\pi^2m^2\delta_{AB}\delta_{mn}\ . \tag{4.48}$$

Using (4.37), (4.42), and (4.48), we find

$$P\underset{\alpha\to\infty}{\sim}N\exp\left[-\alpha\gamma_0-\frac{\alpha}{2}\chi^TM\chi-\Gamma_1(\lambda,\chi,x)+O(\chi^3)+O(\alpha^{-1})\right]\ . \tag{4.49}$$

Until this point, the function $F(\lambda)$ has been arbitrary. We now impose the following condition:

$$\mathrm{sgn}[F(\lambda)]=\mathrm{sgn}(\lambda)\ . \tag{4.50}$$

This choice clearly *breaks* modular invariance, and it is this choice, which we are perfectly free to make in the context of stochastic gauge fixing—the gauge-fixing force is of necessity a gauge-dependent quantity—which, as we can see immediately, brings about a restriction of the λ integration in (4.14) to the modular region $\lambda>0$. For, if $F(\lambda)$ satisfies (4.50) then $-\chi^TM\chi>0$ when λ is negative, and P is unnormalizable; unless, that is, the factor $Ne^{-\alpha\gamma_0}$ (which, as we recall, can change value at $\lambda=0$), *vanishes identically* for $\lambda<0$. [Further discussion of the nature of the allowed choices for $F(\lambda)$ follows the solution for P, Eq. (4.66).]

Since N and γ_0 are otherwise arbitrary, we can thus write

$$P \underset{\alpha\to\infty}{\sim} \theta(\lambda)\tilde{N}\exp\left[-\frac{\alpha}{2}\chi^T M\chi - \Gamma_1(\lambda,\chi,x) + O(\alpha^{-1})\right] . \tag{4.51}$$

Clearly, the $O(\alpha)$ term in the exponent of (4.51) will, as α grows larger and larger, give a δ-function concentration in the χ^{Am} direction; so we have dropped the term "$+O(\chi^3)$" in going from (4.49) to (4.51), and, indeed, the use of a Taylor expansion in χ about $\chi=0$ has been *a posteriori* justified. Recalling that[12]

$$\lim_{a\to\infty}\left[\frac{a}{\pi}\right]^{1/2}\exp(-ax^2) = \delta(x) , \tag{4.52}$$

let us define the quantity $Q(\lambda,\chi,x)$ by the relation

$$\exp[-\Gamma_1(\lambda,\chi,x)] = \left[\prod_{A=C,S}\prod_{m=1}^{\infty}\frac{F(\lambda)}{|\lambda|}2\pi m^2\right]^{1/2} Q(\lambda,\chi,x) . \tag{4.53}$$

From (4.52), we see that P will approach a well-defined distribution in χ as $\alpha\to\infty$ provided $\tilde{N}$ is of the form

$$\tilde{N}(\alpha) = \tilde{\tilde{N}}\prod_{A=C,S}\prod_{m=1}^{\infty}\alpha^{1/2} = \tilde{\tilde{N}}\alpha^{-1/2} , \tag{4.54}$$

where $\tilde{\tilde{N}}$ is independent of α, and where we have used (4.12). Using (4.53) and (4.54) in (4.51),

$$P \underset{\alpha\to\infty}{\sim} \theta(\lambda)\tilde{\tilde{N}}\Delta(\chi,\alpha)Q(\lambda,\chi,x) , \tag{4.55}$$

where

$$\Delta(\chi,\alpha) \equiv \alpha^{-1/2}\left[\prod_{m=1}^{\infty}\frac{F(\lambda)}{|\lambda|}2\pi m^2\right]\exp\left[-\frac{\alpha}{2}\chi^T M\chi\right] , \tag{4.56}$$

so

$$\lim_{\alpha\to\infty}\Delta(\chi,\alpha) = \prod_{A=C,S}\prod_{m=1}^{\infty}\delta(\chi^{Am}) \equiv \delta(\chi) , \tag{4.57}$$

$$\begin{aligned}\lim_{\alpha\to\infty} P &= \theta(\lambda)\tilde{\tilde{N}}\delta(\chi)Q(\lambda,\chi,x)\\ &= \theta(\lambda)\tilde{\tilde{N}}\delta(\chi)Q(\lambda,0,x) .\end{aligned} \tag{4.58}$$

We now take the form (4.55) as an ansatz for P, and insert that into the exact FP equation (4.35). In those terms not involving explicit factors of α or derivatives with respect to χ we can immediately take the $\alpha\to\infty$ limit; using (4.57), (4.35) becomes, for $\lambda>0$ and $\alpha\to\infty$,

$$\begin{aligned}0 = {}&\delta(\chi)\frac{\partial}{\partial\lambda}\left[|\lambda|^{-1}|\phi|^{1/2}\left[-|\lambda|\frac{\partial}{\partial\lambda} + \sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2 m^2}{\lambda}(x^{\mu Am})^2 - \frac{\lambda M^2}{2}\right]Q(\lambda,0,x)\right]\\ &+ \sum_{\mu=1}^{D}\sum_{\substack{A,m\\B,n}}\frac{\partial}{\partial\chi^{\mu Am}}\left[|\lambda|^{-1}|\phi|^{1/2}\phi^{Am,Bn}\left[-|\lambda|^{-1}\frac{\partial}{\partial\chi^{Am}} + \alpha\frac{F(\lambda)}{\lambda^2}\int_0^1 d\tau\, h_{Am}\chi^{-1}\frac{\partial}{\partial\tau}(\chi^{-2}\nabla\chi)\right]\Delta(\chi,\alpha)Q(\lambda,\chi,x)\right]\\ &+\delta(\chi)\sum_{A,m}\frac{\partial}{\partial\chi^{Am}}\left[|\lambda|^{-1}|\phi|^{1/2}\left[-|\lambda|^{-1}\frac{\partial}{\partial x^{\mu Am}} - \frac{4\pi^2 m^2}{\lambda^2}x^{\mu Am}\right]Q(\lambda,0,x)\right] .\end{aligned} \tag{4.59}$$

We verify that the term in (4.59) involving the product $\alpha\Delta(\chi,\alpha)$ approaches a well-defined distribution as $\alpha\to\infty$, and vanishes at $\chi^{Am}\to\pm\infty$ (it clearly does for finite α). This term is a sum of terms containing factors of

$$\alpha(\chi^{Am})^N\Delta(\chi,\alpha) = \alpha(\chi^{Am})^N\prod_{B=C,S}\prod_{n=1}^{\infty}\left[\left[\alpha\frac{F(\lambda)}{|\lambda|}2\pi m^2\right]^{1/2}\exp\left[-\alpha\frac{f(\lambda)}{|\lambda|}2\pi^2 m^2(\chi^{Am})^2\right]\right] , \tag{4.60}$$

where $N=1,2,\ldots$. We have

$$\lim_{\alpha\to\infty}\alpha(\chi^{Am})^N\Delta(\chi,\alpha) = \prod_{(B,n)\neq(A,m)}\delta(\chi^{Bn})\lim_{\alpha\to\infty}\alpha^{3/2}(\chi^{Am})^N\left[\frac{F(\lambda)}{|\lambda|}2\pi m^2\right]^{1/2}\exp\left[-\alpha\frac{F(\lambda)}{|\lambda|}2\pi^2 m^2(\chi^{Am})^2\right] \tag{4.61}$$

but

$$\lim_{a\to\infty}a^{3/2}xe^{-ax^2} = -\Gamma(\tfrac{3}{2})\delta'(x),\quad \lim_{a\to\infty}a^{3/2}x^2e^{-ax^2} = \Gamma(\tfrac{3}{2})\delta(x),\quad \lim_{a\to\infty}a^{3/2}x^n e^{-ax^2} = 0,\quad n=3,4,\ldots .$$

For example, for $N=2$, (4.61) is equal to

$$-\prod_{(B,n)\neq(A,m)}\delta(\chi^{Bn})\frac{\Gamma(\frac{3}{2})}{\pi^{1/2}}\left[\frac{F(\lambda)}{|\lambda|}2\pi^2 m^2\right]^{-1}\delta'(\chi^{Am}) . \tag{4.62}$$

Thus, integrating (4.59) over all χ^{Am} eliminates the $\partial/\partial x^{Am}$ term, and gives

$$0=\frac{\partial}{\partial\lambda}\left[|\lambda|^{-1}\left[-|\lambda|\frac{\partial}{\partial\lambda}+\sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2m^2}{\lambda}(x^{\mu Am})^2-\frac{\lambda M^2}{2}\right]Q(\lambda,0,x)\right]$$
$$+\sum_{\mu=1}^{D}\sum_{A,m}\frac{\partial}{\partial x^{\mu Am}}\left[|\lambda|^{-1}\left[-|\lambda|^{-1}\frac{\partial}{\partial x^{\mu am}}-\frac{4\pi^2m^2}{\lambda^2}x^{\mu Am}\right]Q(\lambda,0,x)\right]. \tag{4.63}$$

Hence, a solution to (4.63) is ($\lambda>0$)

$$Q=\exp[-S_{\rm cl}(\lambda,x)]=\exp\left[-\sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2m^2}{\lambda}(x^{\mu Am})^2-\frac{\lambda}{2}M^2\right]. \tag{4.64}$$

The complete solution for P is

$$P=\tilde{\tilde{N}}\theta(\lambda)\delta(\chi)\exp[-S_{\rm cl}(\lambda,x)] \tag{4.65}$$

and the expression for the path integral is

$$\int De\,Dx\,P=\tilde{\tilde{N}}\int_{-\infty}^{+\infty}d\lambda|\lambda|^{-1}\prod_{\mu=1}^{D}\int_{-\infty}^{+\infty}dx^{\mu 0}\prod_{A=C,S}\prod_{m=1}^{\infty}\int_{-\infty}^{+\infty}d\chi^{Am}\prod_{\mu=1}^{D}dx^{\mu Am}\theta(\lambda)\delta(\chi)\exp[S_{\rm cl}(\lambda,x)]$$
$$=\tilde{\tilde{N}}\Omega_D\int_0^{+\infty}d\lambda\,\lambda^{-1}\sum_{A=C,S}\prod_{m=1}^{\infty}\prod_{\mu=1}^{D}\int_{-\infty}^{+\infty}dx^{\mu Am}\exp\left[-\frac{\lambda}{2}m^2-\sum_{A=C,S}\sum_{m=1}^{\infty}\frac{2\pi^2M^2}{\lambda}(x^{\mu Am})^2\right]$$
$$=\tilde{\tilde{N}}'\Omega_D\int_0^{+\infty}d\lambda\,\lambda^{-1}\exp\left[-\frac{\lambda M^2}{2}\right]\left[\prod_{m=1}^{\infty}\lambda^{1/2}M^{-1}\right]^{2D}=\tilde{\tilde{N}}''\Omega_D\int_0^{+\infty}d\lambda\,\lambda^{-1}\exp\left[-\frac{\lambda M^2}{2}\right]\lambda^{-D/2}$$

or

$$\int De\,Dx\,P=\tilde{\tilde{N}}''\Omega_D\int_0^{+\infty}d\lambda\,\lambda^{-1-D/2}\exp\left[-\frac{\lambda m}{2^2}\right], \tag{4.66}$$

where Ω_D = spacetime volume. Equation (4.66) is the correct answer for the one-loop amplitude. (See, e.g., Ref. 11 and references therein.)

We can see now that only the sign of $F(\lambda)$ has any effect on the solution $P(\lambda,\chi,x)$ in the $\alpha\to\infty$ limit. We reiterate in more general terms the argument presented after Eq. (4.50). If $F(\lambda)$ is negative *anywhere* within a region $\lambda>0$ or $\lambda<0$, then $P(\lambda,\chi,x)$ is an exponentially rising function of χ at that value of λ and hence is unnormalizable unless the overall coefficient $\tilde{N}=Ne^{-\alpha\gamma 0}$ is zero. (It is true that $\tilde{N}$ will vanish as $\alpha\to\infty$ like $\alpha^{-1/2}$—see Eq. (4.54)—but this is irrelevant, since the divergence of the χ integral of $\exp[-(\alpha/2)\chi^TM\chi]$, for $F(\lambda)<0$, occurs for all finite α and has nothing to do with the $\alpha\to\infty$ limit.) But $Ne^{-\alpha\gamma 0}$ can only change its value at $\lambda=0$; this follows from the requirement that P satisfy the FP equation. So, if $F(\lambda)<0$ *anywhere* in a region, P must vanish *everywhere* in that region. Thus, any functional form whatever for $F(\lambda)$ will lead to one of four possible types of solutions.

(1) $F(\lambda)$ is everywhere positive; $P(\lambda,\chi,x)$ has support for $\lambda>0$ and $\lambda<0$ (the solution depends only on $|\lambda|$). $F(\lambda)$'s in this class thus *respect* modular invariance (in the $\alpha\to\infty$ limit) and, therefore, fail to fix the gauge with respect to modular invariance. (An analogous situation would be to attempt to do gauge fixing in a Yang-Mills theory by adding to the action a "gauge-fixing term" which was gauge invariant.) To avoid overcounting in this case, it would be necessary to resort to the same sort of fix which is employed in the usual (nonstochastic) Polyakov quantization, i.e., restricting by hand the integration over the Teichmüller parameter λ to a single modular region, $\lambda>0$ or $\lambda<0$.

(2) $F(\lambda)$ is positive for $\lambda>0$ and negative for at least one value of $\lambda<0$; $P(\lambda,\chi,x)$ has support for $\lambda>0$ only. $F(\lambda)$'s in this class thus *break* modular invariance and *do* fix the gauge completely.

(3) $F(\lambda)$ is positive for $\lambda<0$ and negative for at least one value of $\lambda>0$; $P(\lambda,\chi,x)$ has support for $\lambda<0$ only. Trivially equivalent to case (2) above.

(4) $F(\lambda)$ is negative for at least one value of $\lambda>0$, and for at least one value of $\lambda<0$; $P(\lambda,\chi,x)$ vanishes identically.

Case (2), of course, is the one we have considered in detail in this section, but the changes in the analysis for cases (1), (3), and (4) are trivial. One might also ask: what if $F(\lambda)$ is identically zero? That is, we might perform stochastic quantization *without* in any way fixing the gauge. Quantization without the need for gauge fixing was one of the original motivations behind the stochastic approach,[13] and has been applied to some aspects of string quantization,[1] but does not seem to be of any help regarding the problem of modular overcounting.

V. THE PROPAGATOR

In this section we will present the calculation of P for the case of a world line with open (arbitrary) end points; that is, the calculation of the propagator for a relativistic free particle. Most of the qualitative features that arose in the loop calculation of the previous section still remain in the present situation. For this reason, we will here mainly point out the particular aspects in which the propagator calculation differs.

The first observation to make is that the positions of the particle at the end points, x_i^μ and x_f^μ, will *not* be regarded as coordinates of the space through which the probability diffuses during fictitious-time evolution, but rather as fixed external parameters. Hence it is convenient to split $x^\mu(\tau)$ into

$$x^\mu(\tau)=\bar{x}^\mu(\tau)+\tilde{x}^\mu(\tau) , \tag{5.1}$$

where

$$\frac{d^2\bar{x}^\mu(\tau)}{d\tau^2}=0, \quad \bar{x}^\mu(0)=x_i^\mu, \quad \bar{x}^\mu(1)=x_f^\mu , \tag{5.2a}$$

which implies

$$\bar{x}^\mu(\tau)=x^\mu(0)+[x^\mu(1)-x^\mu(0)]\tau \tag{5.2b}$$

and

$$\tilde{x}^\mu(\tau)=\sum_{m=1}^{\infty}\tilde{x}^{m}\tilde{h}_{Sm}(\tau) , \tag{5.3}$$

where

$$\tilde{h}_{Sm}(\tau)=\sqrt{2}\sin(\pi m\tau) . \tag{5.4}$$

For $\chi(\tau)$ we have, in this case [making use of (4.2)],

$$\chi(\tau)=1+\sum_{m=1}^{\infty}\chi^m\tilde{h}_{Cm}(\tau) , \tag{5.5}$$

where

$$\tilde{h}_{Cm}=\sqrt{2}\cos(\pi m\tau) . \tag{5.6}$$

So, the coordinates which determine a holonomic basis for tensors in superspace are now

$$\{Z^i\}\equiv\{\lambda,\chi^m,\tilde{x}^{\mu m};\ m=1,2,\ldots\} . \tag{5.7}$$

We obtain for the nonzero components of the metric tensor in these coordinates

$$G_{\lambda\lambda}=|\lambda|^{-1} , \tag{5.8a}$$

$$G_{m,n}=|\lambda|\tilde{\phi}_{mn} , \tag{5.8b}$$

$$G_{\mu n,\nu m}=|\lambda|\delta_{\mu\nu}\delta_{mn} , \tag{5.8c}$$

where

$$\tilde{\phi}_{mn}\equiv\int_0^1 d\tau\frac{\tilde{h}_{Cm}\tilde{h}_{Cn}(\tau)}{|\chi(\tau)|} . \tag{5.8d}$$

For its inverse,

$$G^{\lambda\lambda}=|\lambda| , \tag{5.9a}$$

$$G^{m,n}=|\lambda|^{-1}\tilde{\phi}^{mn} , \tag{5.9b}$$

$$G^{\mu m,\nu n}=|\lambda|^{-1}\delta_{\mu\nu}\delta_{mn} , \tag{5.9c}$$

where $\tilde{\phi}^{mn}$ is the inverse of $\tilde{\phi}_{mn}$. The determinant of the metric is then

$$\det G=G_{\lambda\lambda}\prod_{m=1}^{\infty}\left[G_{m,m}\prod_{\mu=1}^{D}G_{\mu m,\mu m}\right]=|\lambda|^{-(3+D)/2}\tilde{\phi} . \tag{5.10a}$$

where

$$\tilde{\phi}\equiv\det\tilde{\phi}_{mn} . \tag{5.10b}$$

Thus, the measure now becomes

$$\int\prod_i D\phi^i\sqrt{\det G}$$
$$\to\int_{-\infty}^{+\infty}d\lambda|\lambda|^{-(3+D)/4}$$
$$\times\prod_{m=1}^{\infty}\left[\int_{-\infty}^{+\infty}d\chi^m\tilde{\phi}(\chi)\prod_{\mu=1}^{D}d\tilde{x}^{\mu m}\right] . \tag{5.11}$$

Next, we will compute the classical drift force, for which we need to compute the volume of the group of gauge transformations. As discussed in Sec. III C, world-line vector fields that generate diffeomorphisms have to satisfy the boundary conditions

$$\xi^a(0)=\xi^a(1)=0$$

and, hence, they can be expanded in sine series:

$$\xi(\tau)=\sum_{m=1}^{\infty}\xi^{Sm}\tilde{h}_{Sm} . \tag{5.12}$$

Following the reasoning of Sec. IV B, and noticing that now there is *no* zero mode in (5.12), we obtain the expression for $\det H$:

$$\det H=\prod_{n=1}^{\infty}|\lambda|^3\int_0^1 d\tau(h_{Sm})^2\chi^3$$
$$=|\lambda|^{-3/2}\prod_{m=1}^{\infty}\int_0^1 d\tau(h_{Sm})^2\chi^3$$
$$\equiv|\lambda|^{-3/2}\Omega[\chi] . \tag{5.13}$$

So, $\det H$ depends on λ and χ:

$$\det H=\det H(\lambda,\chi) . \tag{5.14}$$

We can thus write V_g as

$$V_g=\int\prod_{m=1}^{\infty}d\xi^m\sqrt{\det H(\lambda,\chi)} .$$

So (see Sec. II A),

$$\tilde{V}_g=|\lambda|^{-3/4} . \tag{5.15}$$

The classical action in this case is given by

$$S_{\rm cl}=\mathrm{sgn}(\lambda)\bar{S}_{\rm cl}+\frac{\pi^2}{2|\lambda|}\sum_{m=1}^{\infty}m^2\tilde{x}^{\mu m}\tilde{x}^{\mu m} , \tag{5.16}$$

where

$$\mathrm{sgn}(\lambda)=\begin{cases}1, & \lambda>0 ,\\ -1, & \lambda<0 ,\end{cases}$$

and

$$\bar{S}_{\rm cl}=\frac{1}{2\lambda}(x_f^\mu-x_i^\mu)(x_f^\mu-x_i^\mu)+\frac{\lambda M^2}{2} . \tag{5.17}$$

Using (2.10), (2.11), (5.9a)–(5.9c), (5.16), and

$$\frac{\partial}{\partial\lambda}\ln\tilde{V}_g=\frac{1}{\tilde{V}_g}\frac{\partial\tilde{V}_g}{\partial\lambda}=\frac{-3}{4}|\lambda|^{-1}\mathrm{sgn}(\lambda) , \tag{5.18}$$

we obtain

$$K_{\rm cl}^{\lambda}=-2|\lambda|\delta(\lambda)\bar{S}_{\rm cl}+\frac{(x_f-x_i)^2}{2\lambda}-\frac{\lambda M^2}{2}+\frac{\pi^2}{2\lambda}\sum_{m=1}^{\infty}m^2\tilde{x}^{\mu m}\tilde{x}^{\mu m}+\tfrac{3}{4}\mathrm{sgn}(\lambda) , \tag{5.19a}$$

$$K_{\rm cl}^{m}=0 , \tag{5.19b}$$

$$K_{\rm cl}^{\mu m}=-\frac{\pi^2 m^2}{\lambda^2}\tilde{x}^{\mu m} . \tag{5.19c}$$

For the components gauge-fixing force we follow the reasoning of sec. IV C and obtain

$$K_{\rm gf}{}^{\lambda}=0 , \tag{5.20a}$$

$$K_{\rm gf}{}^{m}=\sum_n\tilde{\phi}^{mn}\frac{F(\lambda)}{\lambda^2}\int_0^1 d\tau\,\tilde{h}_{Cn}\chi^{-1}\frac{\partial}{\partial\tau}(\chi^{-2}\nabla\chi) , \tag{5.20b}$$

$$K_{\rm gf}{}^{\mu m}=0 . \tag{5.20c}$$

Hence, the Fokker-Planck equation for this case is

$$\begin{aligned}0=&\frac{\partial}{\partial\lambda}\left[|\lambda|^{-(3+D)/4}|\tilde{\phi}|^{1/2}\left[-|\lambda|\frac{\partial}{\partial\lambda}+\sum_{m=1}^{\infty}\frac{\pi^2m^2}{2\lambda}\tilde{x}^{\mu m}\tilde{x}^{\mu m}-2|\lambda|\delta(\lambda)\bar{S}_{\rm cl}+\frac{(x_f-x_i)^2}{2\lambda}-\frac{\lambda M^2}{2}+\tfrac{3}{4}\mathrm{sgn}(\lambda)\right]P\right]\\&+\sum_{m,n=1}^{\infty}\frac{\partial}{\partial\chi^m}\left[|\lambda|^{-(3+D)/4}|\tilde{\phi}|^{1/2}\tilde{\phi}^{mn}\left[-|\lambda|^{-1}\frac{\partial}{\partial\chi^n}+\frac{\alpha F(\lambda)}{\lambda^2}\int_0^1 d\tau\,\tilde{h}_{Cn}\chi^{-1}\frac{\partial}{\partial\tau}(\chi^2\nabla\chi)\right]P\right]\\&+\sum_{\mu=1}^{D}\sum_{m=1}^{\infty}\frac{\partial}{\partial\tilde{x}^{\mu m}}\left[|\lambda|^{-(3+D)/4}|\tilde{\phi}|^{1/2}\left[-|\lambda|^{-1}\frac{\partial}{\partial\tilde{x}^{\mu m}}-\frac{\pi^2m^2}{\lambda^2}\tilde{x}^{\mu m}\right]P\right] .\end{aligned} \tag{5.21}$$

Reasoning as in Sec. IV D, we obtain the following solution for P in the $\alpha\to\infty$ limit:

$$P_{\alpha\to\infty}\sim N\exp\left[-\alpha\gamma_0-\frac{\alpha}{2}\chi^T\cdot M\cdot\chi-\Gamma_1(\lambda,\chi,x)+O(\chi^3)+O(\alpha^{-1})\right] , \tag{5.22}$$

where

$$M_{m,n}=\frac{F(\lambda)}{|\lambda|}\pi^2m^2\delta_{mn} . \tag{5.23}$$

So, if we impose the same condition on $F(\lambda)$ as in the preceding section, Eq. (4.49), we obtain again a solution of the form (4.51), which *explicitly* avoids overcounting:

$$P_{\alpha\to\infty}\sim\theta(\lambda)\tilde{N}\exp\left[-\frac{\alpha}{2}\chi^T\cdot M\cdot\chi-\Gamma_1(\lambda,\chi,x)+O(\alpha^{-1})\right] . \tag{5.24}$$

The analysis that led from Eq. (4.51) to Eq. (4.63) still applies, mutatis mutandis, once we make the relevant definitions in this case:

$$e^{-\Gamma_1(\lambda,\chi,x)}=\left[\prod_{m=1}^{\infty}\left[\frac{F(\lambda)}{2|\lambda|}\pi m^2\right]^{1/2}\right]Q(\lambda,\chi,x) , \tag{5.25}$$

$$\tilde{N}(\alpha)=\tilde{\tilde{N}}\prod_{m=1}^{\infty}\alpha^{1/2}=\tilde{\tilde{N}}\alpha^{-1/4} , \tag{5.26}$$

$$P_{\alpha\to\infty}\sim\theta(\lambda)\tilde{\tilde{N}}\Delta(\chi,\alpha)Q(\lambda,\chi,x) . \tag{5.27}$$

$\tilde{\tilde{N}}$ is an α- independent normalization constant:

$$\Delta(\chi,\alpha)\equiv\prod_{m=1}^{\infty}\left[\frac{\alpha F(\lambda)\pi m^2}{2|\lambda|}\right]^{1/2}\exp\left[\frac{-\alpha}{2}\chi^TM\chi\right] , \tag{5.28a}$$

$$\lim_{\alpha\to\infty}\Delta(\chi,\alpha)=\prod_{m=1}^{\infty}\delta(\chi^m)\equiv\delta(\chi) . \tag{5.28b}$$

We then have that $Q(\lambda,0,x)$ satisfies (for $\lambda>0$)

$$0=\frac{\partial}{\partial\lambda}\left[\lambda^{-(3+D)/4}\left[-\lambda\frac{\partial}{\partial\lambda}+\sum_{m=1}^{\infty}\frac{\pi^2m^2}{2\lambda}\tilde{x}^{\mu m}\tilde{x}^{\mu m}+\frac{(x_f-x_i)^2}{2}-\frac{\lambda M^2}{2}+\frac{3}{4}\right]Q(\lambda,0,x)\right]$$
$$+\sum_{\mu=1}^{D}\sum_{m=1}^{\infty}\frac{\partial}{\partial\tilde{x}^{\mu m}}\left[\lambda^{-(3+D)/4}\left[-\lambda^{-1}\frac{\partial}{\partial\tilde{x}^{\mu m}}-\frac{\pi^2m^2}{\lambda^2}\tilde{x}^{\mu m}\right]Q(\lambda,0,x)\right]. \tag{5.29}$$

Hence, a solution is

$$Q(\lambda,0,x)=\lambda^{3/4}\exp[-S_{\rm cl}(\lambda,\tilde{x})]=\lambda^{3/4}\exp\left[-\frac{l}{2\lambda}(x_f^{\mu}-x_i)^2-\frac{\lambda M^2}{2}-\frac{\pi^2}{2\lambda}\sum_{m=1}^{\infty}m^2\tilde{x}^{\mu m}\tilde{x}^{\mu m}\right]. \tag{5.30}$$

The complete solution in the $\alpha\to\infty$ limit is then

$$P=\tilde{\tilde{N}}\theta(\lambda)\delta(\chi)\lambda^{3/4}\exp\left[-\frac{1}{2\lambda}(x_f-x_i)^2-\frac{\lambda M^2}{2}-\frac{\pi^2}{2\lambda}\sum_{m=1}^{\infty}m^2\tilde{x}^{\mu m}\tilde{x}^{\mu m}\right]. \tag{5.31}$$

Finally, the path integral is given by

$$\int De\,Dx\sqrt{\det G}\;P=\tilde{\tilde{N}}\int_{-\infty}^{+\infty}d\lambda|\lambda|^{-(3+D)/4}\prod_{m=1}^{\infty}\int d\chi^m\prod_{\mu=1}^{D}d\tilde{x}^{\mu m}\theta(\lambda)\delta(\chi)\lambda^{3/4}\exp[-S_{\rm cl}(\lambda,\tilde{x})]$$
$$=\tilde{\tilde{N}}\int_0^{+\infty}d\lambda\,\lambda^{-D/4}\exp\left[-\frac{1}{2\lambda}(x_f-x_i)^2-\frac{\lambda M^2}{2}\right]\prod_{m=1}^{\infty}\prod_{\mu=1}^{D}\int_{-\infty}^{+\infty}d\tilde{x}^{\mu m}\exp\left[-\frac{\pi^2}{2\lambda}\sum_{m=1}^{\infty}m^2\tilde{x}^{\mu m}\tilde{x}^{\mu m}\right]$$
$$=\tilde{\tilde{N}}'\int_0^{+\infty}d\lambda\,\lambda^{-D/4}\exp\left[-\frac{1}{2\lambda}(x_f-x_i)^2-\frac{\lambda M^2}{2}\right]\left[\prod_{m=1}^{\infty}(\lambda^{1/2}m^{-1})\right]^D$$
$$=\tilde{\tilde{N}}''\int_0^{+\infty}d\lambda\,\lambda^{-D/4}\exp\left[-\frac{1}{2\lambda}(x_f-x_i)^2-\frac{\lambda M^2}{2}\right]\lambda^{-D/4}$$

or

$$\int De\,Dx\sqrt{G}\;P$$
$$=\tilde{\tilde{N}}''\int_0^{+\infty}d\lambda\,\lambda^{-D/2}\exp\left[-\frac{1}{2\lambda}(x_f-x_i)^2-\frac{\lambda M^2}{2}\right] \tag{5.32}$$

which is, if course, the correct answer. (See, e.g., Ref. 11 and references therein.)

VI. DISCUSSION

In this paper we have dealt with the Polyakov particle in order to illustrate in the simplest possible case how stochastic quantization may be used in a theory which is diffeomorphism invariant (see also Ref. 14). For a physically less trivial system, the success of the method depends upon finding a Liapunov function so that the gauge-fixing force from which it is derived will have desirable global restoring properties, as discussed in Sec. II B. We shall display a simple Liapunov function for the bosonic string in first quantization.

If we wish to follow the usual approach, and fix only the invariance with respect to coordinate transformations, but leave the Weyl rescaling invariance unfixed (see, for example, Green, Schwarz, and Witten,[7] Sec. 3.1.1—the two ghost fields c^+ and c^+ correspond to the two reparametrizations), then we should choose a Liapunov function which is invariant under Weyl rescaling. A simple choice is

$$I[g]=\int d^2\sigma\sqrt{g}\,(\delta_{\alpha\beta}g^{\alpha\beta}),$$

where $\delta_{\alpha\beta}$ could be replaced by a more general background metric $g_{B\alpha\beta}$ if desired. Note that this expression is positive, and therefore bounded from below, because the metric tensor is a positive matrix. A gauge-fixing force derived from this Liapunov function will concentrate the probability distribution around the coordinate choice which minimizes this function. An infinitesimal coordinate transformation $\delta\sigma^\alpha=\xi^\alpha$ induces the change in metric tensor

$$\delta g^{\alpha\beta}=\nabla^\alpha\xi^\beta+\nabla^\beta\xi^\alpha .$$

The functional $I[g]$ is stationary under this variation if the metric tensor g satisfies the coordinate condition

$$\nabla^\alpha(\delta_{\alpha\beta}-\tfrac{1}{2}\delta_{\gamma\epsilon}g^{\beta\epsilon}g_{\alpha\beta})=0 .$$

Positivity of the second variation of I with respect to changes in g induced by coordinate transformations provides an additional condition which can serve to fix the modular region.

The goal, of course, is not simply to again compute the infinite-α limit and recover the string's modular regions. The method of stochastic quantization is valid for all values of α; nonsingular values of α will presumably yield amplitudes explicitly dependent on the gauge degrees of freedom χ. A second-quantized field theory based on such a first quantization would likely retain the freedom from modular overcounting of the first-quantized template,[16] most likely by means of a weighted average over several modular regions, rather than the complete restriction to a single region as in the $\alpha \rightarrow \infty$ case or the usual *ad hoc* procedure. Work on the first-quantized and second-quantized string theories is currently in progress.

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[16]We remind the reader that, as pointed out in the Introduction, the possibility of a connection between, on the one hand, the ability of stochastic quantization to prevent modular overcounting, in a systematic manner, in the first-quantized point-particle theory and, on the other hand, the absence of modular overcounting in a covariant second-quantized string theory based upon a stochastic first quantization is, at present, purely conjectural (though highly plausible). Covariant second-quantized string theories have been proposed in Ref. 15 which do not involve stochastic quantization, and which do yield the correct modular regions for tree-level processes, but lack of modular overcounting at the loop level has yet to be demonstrated for these theories.