Inevitable ambiguity in perturbation around flat space-time

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Perturbation of general-relativistic predictions around flat geometry, in general, introduces inevitable ambiguity. The ambiguity reflects the geometrical nature of general relativity and is never a difficulty of it. We explain it by taking a concrete example of the radar-echo experiment.

I. INTRODUCTION

There are famous experiments to check the effects of (classical) general relativity: (1) the deflection of light passing near the Sun, (2) the perihelion shift of Mercury's orbit, (3) the time delay of radar echoes between a planet (Mercury, Venus, or Mars) and Earth, and (4) the gravitational redshift of the electromagnetic-wave frequency. [Rigorously the last one (4) is not a test of general relativity but that of Einstein's equivalence principle, see, for example, Ref. 1.] We obtain some values (experimental data) of angles, time, or frequency for each experiment. The theoretical predictions for those values are often, except for (4), given in perturbative forms around flat space-time. For the predictions of (4), an exact formula is known.[2]

A concrete outcome of the present investigation is that the perturbative formula of (3) inevitably has ambiguity; it has no definite meaning. The reason is as follows. In the perturbation procedure it is necessary to compare points in two different geometries: i.e., curved space-time and flat space-time. The comparison, however, is nonsense geometrically; in other words, it has no intrinsic meaning. Therefore, we are forced to introduce, by hand, an identification rule in order to do the comparison. As a result, perturbative formulas, in general, inevitably contain ambiguity originating from the freedom of the identification rule.

There are special cases where perturbation has no ambiguity. Those examples are tests (1) and (2). In those cases observed quantities are defined in the asymptotic region where space-time is flat. Because of this special condition we can expand the exact expression with respect to a parameter (M: solar mass) of geometry without any comparison of curved and flat space-times in a nonasymptotic region.

In order to demonstrate the "inevitable ambiguity," we will closely examine the perturbative form of the radar-echoes delay time. Here we must point out that so far there are, at least, three (post-Newtonian) formulas for the time delay of radar echoes, even in the same coordinate system. That is, using the same standard coordinate system, there exist those formulas given by (1) Shapiro,[3] Ross and Schiff,[4] Dyson[3] (Misner, Thorne, and Wheeler,[2] Will[1]), (2) Weinberg,[2] Wald,[2], and (3) Logunov and Loskutov,[5] where the authors in the parentheses in (1) utilize essentially the same calculational method as the other authors in (1) except for making use of the isotropic coordinate system. Obviously the discrepancy stems from calculational or interpretative mistakes. Especially we should note that they use *plane trigonometry* (including the Pythagorean theorem) without sufficient consideration in deriving the formulas. Our secondary purpose is to give a systematic derivation of the perturbative formula, when an identification rule is given, without using plane trigonometry and to examine the validity of these three formulas. We will see, as a final result, (1) is right while the others are wrong. Furthermore the post-post-Newtonian order will be treated.

We will take the following three coordinate systems in the concrete calculations: (i) the "standard" coordinate system (t,r_S,θ,ϕ), (ii) the "de Donder" one (t,r_D,θ,ϕ), and (iii) the "isotropic" one (t,r_I,θ,ϕ). These three coordinate systems are defined in Schwarzschild geometry as

$$
\begin{aligned}
ds^2 &= -\left[1-\frac{2M}{r_S}\right]dt^2+\left[1-\frac{2M}{r_S}\right]^{-1}dr_s^2+r_S^2(d\theta^2+\sin^2\theta\,d\phi^2)\\
&= -\frac{r_D-M}{r_D+M}dt^2+\frac{r_D+M}{r_D-M}dr_D^2+(r_D+M)^2(d\theta^2+\sin^2\theta\,d\phi^2)\\
&= -\frac{\left[1-\dfrac{M}{2r_I}\right]^2}{\left[1+\dfrac{M}{2r_I}\right]^2}dt^2+\left[1+\frac{M}{2r_I}\right]^4(dr_I^2+r_1^2d\theta^2+r_I^2\sin^2\theta\,d\phi^2)\ ,
\end{aligned}
\tag{1.1}
$$

where M is the mass of a spherical body, and both the Newtonian gravitational constant and the velocity of light are put to unity, $G=c=1$. Coordinates t, θ, and ϕ are common in these three coordinate systems. As is easily confirmed, coordinates r_S, r_D, and r_I are related to each other by simple relations:

$$r_S=r_D+M=r_I+M+\frac{M^2}{4r_I} . \tag{1.2}$$

Note that the relations (1.2) are one-to-one outside the black hole ($r_S>2M, r_D>M, r_I>M/2$). We restrict consideration to the space-time region $r_S>2M$.

Here we make two points clear.

A. Criterion of nonambiguousness

We take the standpoint that nonambiguous quantities must satisfy the following two conditions.

Condition 1: There must not be geometrically nonsense procedures in their definitions.

Before stating condition 2, we note the above three coordinates (1.1) satisfy the following properties: (a) In the transformation equations among those coordinates, the spatial part (in the above case r, θ, and ϕ) does not contain the time coordinate; (b) in the asymptotic region (in the above case, the large-r region), they approach the flat form with the correct Newtonian approximation.

Condition 2: Nonambiguous quantities must be invariant (or covariant) under the transformations among those coordinates characterized by (a) and (b).

In the present paper, emphasis is placed on condition 1. (For a thorough investigation of condition 2, see the original version of this paper, Ref. 6.)

B. Good parameters and variables to specify the configuration of a gravitational system

We have the freedom to choose parameters and variables to specify the configuration of a gravitational system. It is, however, technically important to choose the best ones in order to clearly discuss the present problem.

1. *M: solar mass*

S, D, and I coordinates have a common asymptotic region for $r_i\to\infty$. On the condition that all asymptotic types of behavior must be the correct Newtonian form

$$g_{00}\sim -1+\frac{2M}{r}, \quad r\to\infty , \tag{1.3}$$

it is guaranteed that we may choose a common parameter M in the expression of the line element written by S, D, and I coordinates (1.1). The parameter M will play the role of the expansion parameter for all tests (1)–(3).

2. *b: relativistic energy; h: angular momentum*

A gravitational system in the Schwarzschild geometry has two constants of motion. We choose those constants as

$$b=b_S=b_D=b_I, \quad h=h_S=h_D=h_I , \tag{1.4}$$

where

$$b_S\equiv\left[1-\frac{2M}{r_S}\right]\frac{dt}{d\lambda}, \quad h_S\equiv r_S^2\sin^2\theta\frac{d\phi}{d\lambda} ,$$

$$b_D\equiv\frac{r_D-M}{r_D+M}\frac{dt}{d\lambda}, \quad h_D\equiv(r_D+M)^2\sin^2\theta\frac{d\phi}{d\lambda} , \tag{1.5}$$

$$b_I\equiv\left[\frac{1-\dfrac{M}{2r_I}}{1+\dfrac{M}{2r_I}}\right]^2\frac{dt}{d\lambda}, \quad h_I\equiv\left[1+\frac{M}{2r_I}\right]^4 r_I^2\sin^2\theta\frac{d\phi}{d\lambda} .$$

λ is an affine parameter describing the path $x^\mu=x^\mu(\lambda)$. This choice is guaranteed either by the requirement of coincidence of constants in the asymptotic region or by the fact that this choice is independent of transformations among S, D, and I coordinates.

Although we will not describe it explicitly, nonambiguousness (check of condition 2) of the tests (1) and (2) can be easily proved by using the result of Sec. II and the above choice of parameters. (See Ref. 6.)

3. *ψ: "angle" between Earth and a planet*

In test (3) we must use position variables of Earth and a planet. In addition to two position variables (r_A, r_B; A=Earth, B=planet), we will choose the "angle" ψ between the two objects. The "angle" ψ is introduced as a variable independent of (M, r_A, r_B) and common to S, D, and I coordinates. The use of ψ and its characteristic separation (ψ_{Ai}, ψ_{Bi}; $i=S,D,I$) will become a key tool of Sec. III.

After reviewing briefly the tests (1) and (2), for completeness, in Sec. II, we examine in detail the perturbative formula of test (3) in Sec. III. Remarks and conclusions are given in Sec. IV. Detailed explanations and calculations are relegated to Appendixes A–D. Appendix A explains general features of the orbit of light in the Schwarzschild geometry. Appendixes B, C, and D are detailed calculations used in the light deflection angle (Sec. II A), the perihelion shift angle (Sec. II B), and the time delay of radar echoes (Sec. III), respectively.

II. TWO NONAMBIGUOUS EXAMPLES

A. The deflection of light

When light grazes the Sun (mass M), it deflects as shown in Fig. 1. Using the geodesic equation $\delta\int ds=0$ and the null condition $ds^2=0$, we obtain the exact expression for the deflection angle[7] δ in the standard coordinate system as follows:

$$\pi+\delta=2\int_0^{v_0}\frac{dv}{[F(v)]^{1/2}} , \tag{2.1}$$

$$F(v)\equiv -v^2+1+2mv^3 , \tag{2.2}$$

where v_0 is a positive zero point of F:

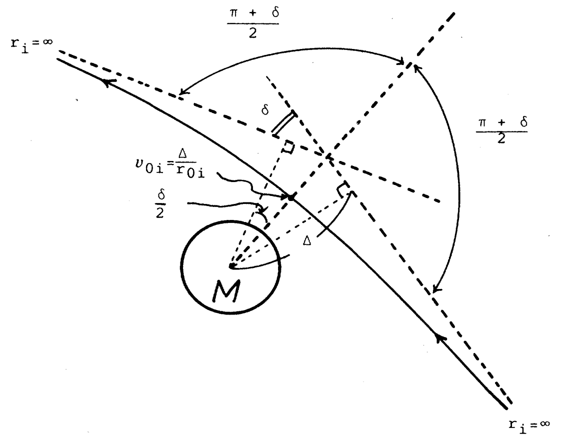

FIG. 1. The deflection of light by the Sun (mass M).

$$F(v_0)=0 \ . \tag{2.3}$$

(For derivation, see Appendix B.) The following notation is used for simplicity:

$$v\equiv\frac{\Delta}{r}, \quad m\equiv\frac{M}{\Delta}, \quad \Delta\equiv\frac{h}{b} \ . \tag{2.4}$$

Here b and h are constants of motion and have physical meaning of the relativistic energy and the angular momentum, respectively. Thus the ratio $\Delta=h/b$ has physical meaning of the impact parameter.[8] In general the functions $F(v)$ can have many zero points, but it is known that the positive solution v_0 is unique unless light is trapped by the black hole.[9] Needless to say, this solution v_0 corresponds to the point of nearest approach of light to the Sun.

Perturbing with respect to m, we can easily obtain

$$\delta=4m+\frac{15\pi}{4}m^2+O(m^3) \tag{2.5}$$

in the post-post-Newtonian approximation[10,11] (see Appendix B). The first-order term $4m$ amounts to 1.75 arcsec at the solar limb, on the other hand, the second-order term $(15\pi/4)m^2$ reaches 1.09×10^{-5} arcsec, whose effect is over 3 orders of magnitude below the sensitivity of the most accurate deflection experiment to date. But it is possible to check this order in the near future.[10,12] (Perhaps, this is the most promising experiment checking the post-post-Newtonian effect.)

B. The perihelion shift of mercury

The orbit of Mercury deviates from an ellipse in general relativity (Fig. 2). Using the geodesic equation, we obtain the exact expression for the angle[7] δ of perihelion shift in the standard coordinate system as follows:

$$2\pi+\delta=2\int_{v_1}^{v_2}\frac{dv}{[G(v)]^{1/2}} \ , \tag{2.6}$$

$$G(v)\equiv-v^2+2v+2\epsilon+2m^2v^3 \ , \tag{2.7}$$

where v_1 and v_2 are positive zero points of G,

$$G(v_1)=G(v_2)=0, \quad v_2>v_1 \ . \tag{2.8}$$

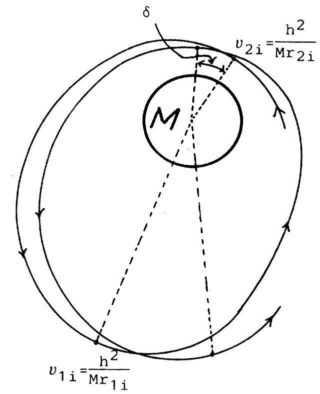

FIG. 2. The perihelion shift of Mercury.

(For derivation, see Appendix C.) The following notation is used for simplicity:

$$v\equiv\frac{h^2}{Mr}, \quad m\equiv\frac{M}{h}, \quad \epsilon\equiv\frac{b^2-1}{2m^2} \ . \tag{2.9}$$

Here b and h are relativistic energy and the angular momentum (per unit proper mass of Mercury), respectively.[13]

In general the functions $G(v)$ can have many zero points, but it is easily known that there are always two positive solutions corresponding to v_1 and v_2, as far as the orbit of "Mercury" is restricted in a finite region outside the black hole.[14] All the other solutions are negative or do not correspond to a finite orbit outside the black hole. Thus the solutions v_1 and v_2 are uniquely determined, which, of course, correspond to the aphelion and the perihelion, respectively.

Perturbation with respect to m (with ϵ fixed[15]) gives

$$\delta=6\pi m^2+\frac{15\pi}{2}m^4(7+2\epsilon)+O(m^6) \tag{2.10}$$

in the post-post-Newtonian approximation[11] (see Appendix C). The first-order term $6\pi m^2$ amounts to 43.03 arcsec per century for Mercury and 4.2 deg per year for the binary pulsar PSR 1913+16. On the other hand, the second-order term $(15\pi/2)m^4(7+2\epsilon)$ amounts to $\sim10^{-6}$ arcsec per century for Mercury and $\sim10^{-2}$ arcsec per year for the binary pulsar PSR 1913+16. The post-post-Newtonian effect is too small to be detected in the near future.[12]

III. THE TIME DELAY OF RADAR ECHOES

A. Exact expression

A radar signal, sent across the solar system past the Sun to a planet (or satellite) and returned to Earth suffers an "additional non-Newtonian delay" in its round-trip travel time [Fig. 3(a)]. In this section we consider the three-coordinate systems defined by (1.1). Putting aside the motions of Earth (A) and the planet (B) (Ref. 16), we obtain the exact expression for the round-trip travel time T_i expressed in the i coordinate as[17]

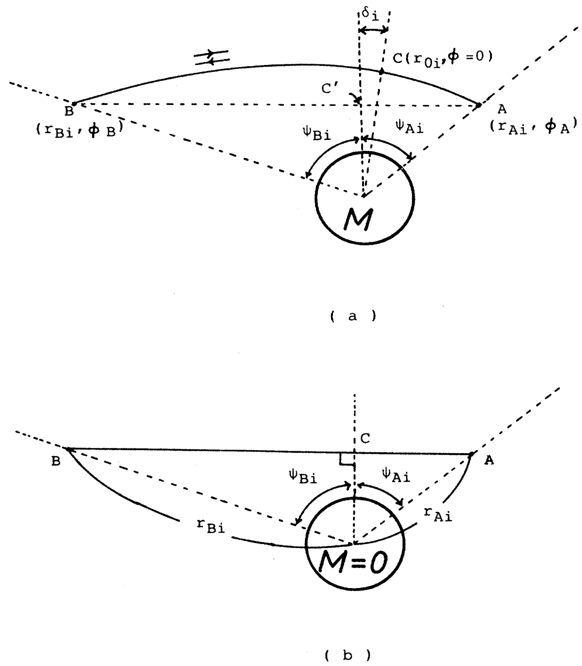

FIG. 3. The orbit of radar signals emitted from Earth (A), reflected by the planet (B) and back to Earth (A) again. C represents a point of nearest approach to the Sun. (a) The real situation with $M\neq 0$. (b) The ideal situation with $M=0$.

$$T_i=2\int_{r_{0i}}^{r_{Ai}}\frac{dr_i}{[H_i(r_i)]^{1/2}}+2\int_{r_{0i}}^{r_{Bi}}\frac{dr_i}{[H_i(r_i)]^{1/2}}\ ,\qquad i=S,D,I \tag{3.1}$$

(for derivation, see Appendix D), where $H_i(r_i)$ is defined by

$$\begin{aligned}
H_S(r_S)&\equiv\left[1-\frac{2M}{r_S}\right]^2\left[1-\frac{\Delta^2}{r_S^2}\left[1-\frac{2M}{r_S}\right]\right]\ ,\\
H_D(r_D)&\equiv\frac{(r_D-M)^2}{(r_D+M)^2}\left[1-\Delta^2\frac{r_D-M}{(r_D+M)^3}\right]\ ,\\
H_I(r_I)&\equiv\frac{\left[1-\dfrac{M}{2r_I}\right]^2}{\left[1+\dfrac{M}{2r_I}\right]^6}\left[1-\frac{\Delta^2}{r_I^2}\frac{\left[1-\dfrac{M}{2r_I}\right]^2}{\left[1+\dfrac{M}{2r_I}\right]^6}\right]\ .
\end{aligned} \tag{3.2}$$

r_{Ai} and r_{Bi} are the coordinate values of Earth and the planet in the i-coordinate system. r_{0i} is a positive zero point of H_i:

$$H_i(r_{0i})=0\ . \tag{3.3}$$

[For simplicity we consider only the orbits of type (III) in the classification of Appendix A (Ref. 18). In this case a unique r_{0i} exists for each orbit. It corresponds to the "perihelion" or "aphelion." See Appendix A for further discussions on the orbits of light.] r_{0i} is related with Δ, defined by (2.4), as follows:

$$\Delta^2=\frac{r_{0S}^2}{1-\dfrac{2M}{r_{0S}}}=\frac{(r_{0D}+M)^3}{r_{0D}-M}=r_{0I}^2\frac{\left[1+\dfrac{M}{2r_{0I}}\right]^6}{\left[1-\dfrac{M}{2r_{0I}}\right]^2}\ . \tag{3.4}$$

(Note that Δ has no dependence on the coordinate choice.) Because r_{0i} is always outside the black hole, $\Delta^2>0$ is assured in (3.4), which is consistent with the definition (2.4).

We can check the coordinate independence of the total time $T_S=T_D=T_I$ using Eqs. (3.1)–(3.4). (There is sometimes a confusion in the interpretation of the prediction of the radar-echo delay. A cause of the confusion stems from the misidentification of r_{Ai} (r_{Bi}) with the *distance*[19] between the Sun and Earth (planet). With this identification, $T_S=T_D=T_I$ do not hold. Because r_{Ai} (r_{Bi}), in fact, is not the distance but the *name* of the place of Earth (planet), the above way of thinking is incorrect from the outset. A point named r in the standard radial coordinate has names $r-M$ and $\frac{1}{2}\{r-M+[r(r-M)]^{1/2}\}$ in the de Donder and isotropic radial coordinates, respectively.)

B. The inevitable ambiguity in the comparison of curved and flat space-times

We cannot obtain a perturbative formula for (3.1) in the same way as given in Sec. II. First we see the difference in the technical aspect. In Eq. (2.1) or (2.6), all the end points of integrals are given by a trivial constant (0) and the zeros of $F(v)$ [Eq. (2.2)] or $G(v)$ [Eq. (2.7)] which can be determined by the constants of motion (m and ϵ). Therefore, the integrals (2.1) and (2.6) can be perturbatively expanded with respect to M (or m) without any comparison of curved and flat space-times, whereas, in the case (3.1), the end points r_{Ai} and r_{Bi} are not such constants. They are names of positions in curved space-time. Then we are forced to compare points in curved space-time and points in flat space-time in order to obtain a perturbative formula.

The situation can be explained in a more general way as follows. In the previous cases, i.e., the deflection of light and the perihelion shift of Mercury, the physical quantities are angles defined in the asymptotic region. (The angle of the perihelion shift can be regarded as the angle between the two geodesics which pass through the periherions and the center of the Sun; thus we can say it is defined in the asymptotic region.) Since the asymptotic region is flat, the comparison of the general-relativistic predictions with the Newtonian ones has a definite meaning. This is why we do not encounter any difficulty in the previous examples. In the general situation, however, such a comparison has *no* definite meaning, because we cannot compare quantities defined in different geometries. The radar-echo travel time, which is not what is defined in the asymptotic region, provides an example of that situation.

If we want to compare the two different geometries (in order to "clarify" the difference between two theories based on different geometries, for example), we must specify, *by hand*, how to identify points on these two geometries.[20] Such an "identification rule," which is obviously not unique, has no intrinsic meaning from the general-relativistic viewpoint. The resultant perturbative formulas, based on two different identification rules, do *not* coincide with each other, reflecting the degrees of freedom of the identification rule; this is what we call the "inevitable ambiguity." In the next subsection, we will verify this fact in a concrete manner.

C. The time-delay formula in the post-post-Newtonian approximation

Taking into account the above points, let us obtain the time-delay formula explicitly.

First we must specify an identification rule between the Schwarzschild and the Minkowski geometries. As for coordinates except for the radial one, we can naturally identify the coordinate variables in (1.1) with the corresponding flat polar ones because of the spherical symmetry of the system. As for the radial coordinate, there is no such natural rule except for the asymptotic region. We should specify the rule in such a way that the radial coordinate of the Schwarzschild geometry reduces to the corresponding flat one at the asymptotic region as well as $M=0$. For example, the rule (i) $r_S=r$ is an admissible one, where r is the flat polar radial distance. (This rule implies that the point on the Minkowski geometry, apart from the origin by the distance r, is identified with the point on the Schwarzschild geometry, named r in the standard coordinate system.) As another example, there are (ii) $r_S=r+M$ (i.e., $r_D=r$) or (iii) $r_S=r+M+M^2/4r$ (i.e., $r_I=r$). Though the available rules are not restricted to the above three examples, we concentrate only on them in the following discussion.

For the time being, let us adopt $r_S=r$ for the identification rule. [We, however, continue to use the suffix S (D,I) without omission in order to remember the adopted identification rule.] In order to make perturbation, we must clarify what are the independent quantities. Given the solar mass M, the radial coordinates r_{AS}, r_{BS}, and the "angle" ψ between Earth and the planet, then the orbit of light is determined[21] [Fig. 3(a)]. Because of the identification rule, r_S is identified with r, which is *independent* of M. The "angle" ψ is also independent of M (because of the trivial identification rule). Thus the above four quantities (M,r_{AS},r_{BS},ψ) are independent under the adopted rule $r_S=r$. (Notice that if we adopt another rule, e.g., $r_D=r$, then r_S depends on M, and r_D becomes independent of M [see (1.2)].)

First we consider the case $M=0$ [Fig. 3(b)]. Since the space is Euclidean in this case, we can use the plane trigonometry. Thus the angles ψ_{AS} and ψ_{BS}, defined as in Fig. 3(b) (with $i=S$), satisfy

$$\psi=\psi_{AS}+\psi_{BS},\quad r_{AS}\cos\psi_{AS}=r_{BS}\cos\psi_{BS}\ . \tag{3.5}$$

Solving these equations, we get

$$\begin{aligned}\psi_{AS}&=\arctan\left[\frac{r_{AS}-r_{BS}\cos\psi}{r_{BS}\sin\psi}\right],\\ \psi_{BS}&=\arctan\left[\frac{r_{BS}-r_{AS}\cos\psi}{r_{AS}\sin\psi}\right].\end{aligned} \tag{3.6}$$

When $M\neq 0$, the above method loses its validity because r_S , in general, does not represent distance and furthermore the plane trigonometry does not hold. But, under the identification rule $r_S=r$, we can naturally *define* a separation of ψ into ψ_{AS} and ψ_{BS} by Eqs. (3.6) in the $M\neq 0$ case also, because this separation does not depend on M; viz., ψ_{AS} and ψ_{BS} are independent of M. This separation of angle ψ defines a geodesic $\overline{MC'}$ [see Fig. 3(a)]. Obviously the geodesic $\overline{MC'}$, in general, does not coincide with another geodesic $\overline{MC}$ in the $M\neq 0$ case. The angle between these geodesics is denoted by δ_S in the following. Needless to say, $\delta_S=0$ holds in the $M=0$ case [see (3.12)].

Next, it is important to notice that both r_{0S} and Δ are *dependent* on M (even if r_{AS} and (r_{BS} are assumed to be independent of M) (Ref. 22). How they depend on M are determined from the fact that the points (r_{AS},ϕ_A) and (r_{BS},ϕ_B) must lie on the orbit of light[21] (see Appendix D):

$$\begin{aligned}v_S&=\cos\phi+m(1+\sin^2\phi)\\&\quad-\frac{m^2}{8}(3\sin\phi\sin2\phi-30\phi\sin\phi-20\cos\phi)+O(m^3)\ ,\end{aligned} \tag{3.7}$$

where $\phi=0$ corresponds to the point of nearest approach to the Sun and the notation of (2.4) is employed. (The orbit of light is restricted in the $\theta=\pi/2$ plane without loss of generality.) Finally we obtain

$$\begin{aligned}\Delta&=r_{AS}\cos\phi_A+M\frac{1+\sin^2\phi_A}{\cos\phi_A}-\frac{M^2}{r_{AS}}\left[\frac{(1+\sin^2\phi_A)^2}{\cos^3\phi_A}+\frac{3\sin\phi_A\sin2\phi_A-30\phi_A\sin\phi_A-20\cos\phi_A}{8\cos^2\phi_A}\right]+O(M^3)\\&=(A\to B\text{ in the above formula})\ .\end{aligned} \tag{3.8}$$

M dependence of r_{0S} is determined from (3.4) and (3.8). The last equality of (3.8) gives a relation between ϕ_A and ϕ_B.

Here we are ready to calculate the post-post-Newtonian formula. First, expanding (3.1) (with $i=S$) with respect to M suppressing the M dependence of Δ, we obtain (see Appendix D)

$$\begin{aligned}T&=2(r_{AS}^2-\Delta^2)^{1/2}+2M\left[\frac{r_{AS}}{(r_{AS}^2-\Delta^2)^{1/2}}+2\ln\frac{r_{AS}+(r_{AS}^2-\Delta^2)^{1/2}}{\Delta}\right]\\&\quad+M^2\left[\frac{7r_{AS}^2-8\Delta^2}{(r_{AS}^2-\Delta^2)^{3/2}}+\frac{15}{2\Delta}\left[\pi-2\,\mathrm{arccsc}\frac{r_{AS}}{\Delta}\right]\right]+(A\leftrightarrow B)+O(M^3)\ .\end{aligned} \tag{3.9}$$

Next, we insert the expression (3.8) into Δ of (3.9) and obtain

$$T=2r_{AS}\sin\phi_A+2M\left[-\sin\phi_A+\ln\frac{1+\sin\phi_A}{1-\sin\phi_A}\right]+\frac{M^2}{r_{AS}}\left[-\frac{1}{2}\left[\frac{8}{\cos^2\phi_A}+1\right]\sin\phi_A+\frac{15\phi_A}{2\cos\phi_A}\right]+(A\leftrightarrow B)+O(M^3)\ . \tag{3.10}$$

Finally we may write the relations

$$\phi_A=\psi_{AS}-\delta_S,\quad \phi_B=\psi_{BS}+\delta_S\ , \tag{3.11}$$

which clarify the M dependence of ϕ_A and ϕ_B, and are useful for subtracting the round-trip travel time in the absence of the Sun [see Fig. 3(a)]. The relation (3.8) between ϕ_A and ϕ_B gives δ_S as

$$\delta_S=M\delta_{1S}+O(M^2),\quad \delta_{1S}=\frac{1+\sin^2\psi_{BS}}{\cos\psi_{BS}(r_{AS}\sin\psi_{AS}+r_{BS}\sin\psi_{BS})}-(A\leftrightarrow B)\ . \tag{3.12}$$

Because of the second relation in (3.5), the $O(\delta_S)$ term cancels out within the "Newtonian term" of (3.10). Thus $O(M^2)$ terms in (3.12) are irrelevant for getting the post-post-Newtonian formula. As a final result we obtain

$$\begin{aligned}T=&2r_{AS}\sin\psi_{AS}+2M\left[-\sin\psi_{AS}+\ln\frac{1+\sin\psi_{AS}}{1-\sin\psi_{AS}}\right]\\&+M^2\left[\frac{1}{r_{AS}\sin\psi_{AS}+r_{BS}\sin\psi_{BS}}\left[\frac{2}{\cos\psi_{AS}}-\cos\psi_{AS}\right]\left[\frac{2}{\cos\psi_{AS}}-\cos\psi_{AS}-\frac{2}{\cos\psi_{BS}}+\cos\psi_{BS}\right]\right.\\&\left.-\frac{\sin\psi_{AS}}{2r_{AS}}\left[\frac{8}{\cos^2\psi_{AS}}+1\right]+\frac{15\psi_{AS}}{2r_{AS}\cos\psi_{AS}}\right]+(A\leftrightarrow B)+O(M^3)\ .\end{aligned} \tag{3.13}$$

Remember that we have adopted the identification rule $r_S=r$. Of course, we could adopt another identification rule such as $r_D=r$ or $r_I=r$. Then we would obtain

$$\begin{aligned}T=&2r_{AD}\sin\psi_{AD}+2M\ln\frac{1+\sin\psi_{AD}}{1-\sin\psi_{AD}}\\&+M^2\left[\frac{4}{\cos\psi_{AD}(r_{AD}\sin\psi_{AD}+r_{BD}\sin\psi_{BD})}\left[\frac{1}{\cos\psi_{AD}}-\frac{1}{\cos\psi_{BD}}\right]-\frac{\sin\psi_{AD}}{2r_{AD}}\left[\frac{8}{\cos^2\psi_{AD}}+1\right]\right.\\&\left.+\frac{15\psi_{AD}}{2r_{AD}\cos\psi_{AD}}\right]+(A\leftrightarrow B)+O(M^3)\end{aligned} \tag{3.14}$$

or

$$\begin{aligned}T=2r_{AI}\sin\psi_{AI}+2M\ln\frac{1+\sin\psi_{AI}}{1-\sin\psi_{AI}}+M^2&\left[\frac{4}{\cos\psi_{AI}(r_{AI}\sin\psi_{AI}+r_{BI}\sin\psi_{BI})}\left[\frac{1}{\cos\psi_{AI}}-\frac{1}{\cos\psi_{BI}}\right]-\frac{4\sin\psi_{AI}}{r_{AI}\cos^2\psi_{AI}}\right.\\&\left.+\frac{15\psi_{AI}}{2r_{AI}\cos\psi_{AI}}\right]+(A\leftrightarrow B)+O(M^3)\ ,\end{aligned} \tag{3.15}$$

respectively. [In the past literature, coordinates (x_{Ai},x_{Bi}) defined by $x_{Ai}\equiv r_{Ai}\sin\psi_{Ai}$, $x_{Bi}\equiv r_{Bi}\sin\psi_{Bi}$ are often used, instead of (ψ_{Ai},ψ_{Bi}).]

The M^0 term in (3.13)–(3.15) represents the round-trip travel time "in the absence of the Sun." The M^1 term in (3.13), which is the post-Newtonian time delay, agrees with the result given by Shapiro,[3] Ross and Schiff,[4] and Dyson.[3] The post-Newtonian term in [(3.14) and (3.15)] agrees with the result given by Ross and Schiff,[4] Misner, Thorne, and Wheeler,[2] and Will.[1] (In the post-Newtonian approximation, the isotropic formula has the same form as the de Donder one.) Though our result finally coincides with the ones obtained by those above, we should note that they do not correctly appreciate the implicit assumption of the identification rule in their calculation. Furthermore, their calculation appears disputable.[23] The result given by Weinberg[2] and Wald[2,22] and Logunov and Loskutov[5,24] is incomplete in the separation of the total time into the Newtonian and post-Newtonian parts, and does not agree with our result. The M^2 term in (3.13)–(3.15) represents the post-post-Newtonian time delay. [In Ref. 10, Epstein and Shapiro state that they obtained a post-post-Newtonian formula (unfortunately unpublished). A post-post-Newtonian formula is obtained also in Ref. 11, but the separation into the Newtonian, post-Newtonian, post-post-Newtonian, . . .

terms is incomplete.] The decomposition (3.13)–(3.15) does not coincide with each other.

It is useful to estimate the order of each term. If we take Mars as a planet (B), under a suitable identification rule, the Newtonian term amounts to ~ 2500 s, the post-Newtonian term ~ 250 μs, and the post-post-Newtonian term some tens of picoseconds, at a superior conjunction. Because the inevitable ambiguity amounts to some tens of *microseconds*, the above estimation must be used carefully.[25] As is well known, the post-Newtonian effect is confirmed by the experiment using the Viking spacecraft with high accuracy.[26,27] Though the post-post-Newtonian effect is very small, this level of accuracy also is not always impossible from the technical viewpoint. Unfortunately, such an experiment is impossible in the near future, mainly from economical reasons. For further detail about experiments, see Ref. 12.

Finally note that the post-post-Newtonian term in these formulas is divergent in the limit $\psi \to \pi$ (i.e., the superior conjunction), in which limit $\psi_{Ai} \to \pi/2$ and $\psi_{Bi} \to \pi/2$ as is known from (3.6) and the similar expressions for $i = D, I$. Thus this term becomes the same order of magnitude with the post-Newtonian term if

$$|\pi - \psi| \lesssim \left[\frac{M}{r_{Ai}} \right]^{1/2} \tag{3.16}$$

is satisfied. [Fortunately, for the (real) Sun such an extreme condition is not realized because of its large radius.] In this case the perturbation expansion, with respect to M, of T breaks down. In order to deal with such an extreme case, we must rely on another suitable approximation rather than the perturbation with respect to M.

IV. REMARKS AND CONCLUSIONS

We have some remarks about the results of this paper. (1) The different results of (3.13)–(3.15) should not be regarded as coordinate dependence or ambiguity of the prediction of general relativity. It is merely due to the freedom of identification rules in perturbation around flat space-time. The inevitable ambiguity reflects merely the geometrical nature of general relativity. It is never a difficulty of the theory. In this point, we do not agree with the statement of Ref. 5. (2) As well as the identification rule, such a notion as the "additional non-Newtonian delay" has no intrinsic meaning. Note that the quantity obtained by the radar-echo experiment is not the delay time but the total trip time, which is uniquely predicted. Quantities obtained by experiments can be completely explained by the correct theory (general relativity) only and require no comparison with another incorrect theory (Newton theory).

We have pointed out inevitable ambiguity in perturbation around flat space-time. Although the problem appears in a familiar gravitational system such as the radar-echo test, it has been overlooked so far. The ambiguity comes from the freedom of the identification rules in the comparison between curved and flat space-times. To show this fact explicitly, we have concretely calculated the radar-echo delay-time formula. Some technical and interpretative mistakes of past works about the formula have been pointed out. The formula has been systematically obtained up to the post-post-Newtonian order.[28]

Note added. After completion of the present work, we have become aware that Zel'dovich and Grishchuk[29] discuss, following the standpoint of Ross and Schiff,[4] a similar problem and criticize Logunov and Loskutov.[5] In addition, a counterargument to Zel'dovich and Grishchuk is presented by Logunov[30] himself recently. A thought experiment proposed by Logunov,[30] which "demonstrates" the arbitrariness in the predictions of general relativity, merely demonstrates the inevitable ambiguity of "the non-Newtonian delay" discussed in the present paper (see Sec. III B), which is by no means a difficulty of general relativity. A critique for the present view[6] has also appeared recently.[31] We consider the critique to be unreasonable.

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APPENDIX A: A BRIEF NOTE ON THE ORBIT OF LIGHT

In the Schwarzschild geometry the orbits of light can be classified into three types: (I) $m > 1/3\sqrt{3}$, (II) $m = 1/3\sqrt{3}$, and (III) $m < 1/3\sqrt{3}$ cases, where m is defined by (2.4). The orbits of type (I) connect points at infinity with the "center" of the black hole. These orbits have no turning point. The orbit of type (II) represents a circle with the "radius" $r_S = 3M$ [$r_D = 2M$, $r_I = (1+\sqrt{3}/2)M$]. Because this orbit is unstable, it can easily change into the orbits of types (I) or (III) by small perturbations. The orbits of type (III) are composed of two cases: the orbits restricted outside (case α) or inside (case β) the "critical sphere," $r_S = 3M$ [$r_D = 2M$, $r_I = (1+\sqrt{3}/2)M$]. These orbits have only one turning point, corresponding to the "perihelion" (case α) or "aphelion" (case β). Ends of the orbits are points at infinity (case α) or "center" of the black hole (case β), respectively. It should be noted that the light, which cuts across the critical sphere from outside into inside, never escapes from that sphere.

When two points A, B are given around a black hole (with mass M), is the orbit of light passing through them uniquely determined? The answer is *no.* There are infinitely many orbits, in general. Examples of them are drawn in Figs. 4(a)–4(d). Perhaps no further explanation is necessary. The orbits are classified by an integer n, "winding number." The point of nearest approach to the black hole is denoted by C, whose radial i coordinate is r_{0i}. In general, the larger $|n|$, the smaller $r_{0S} - 3M$ [$r_{0D} - 2M$, $r_{0I} - (1+\sqrt{3}/2)M$]. As far as the (real) Sun is concerned, only the orbit of $n=0$ is realistic because the points C of the orbits of $n\neq 0$ are inside the Sun. The orbits of $n\neq 0$, however, must be considered in the general situation. It is important to note that the post-post-Newtonian orbit of light (3.7) is an approximation of the orbit of $n=0$. Therefore, Δ (or r_{0i}) is uniquely determined in (3.8). Because the orbit of $n\neq 0$ is intrinsically nonperturbative, the exact solution (or a suitably approximated solution) is necessarily used in order to get r_{0i} concerning them. It is very interesting that the orbits of $n\neq 0$ exist even if A and B are the same point. [In Fig. 4 we depict only the cases in which both the points A and B are outside the critical sphere (and the point C lies in between A and B). Because the other cases are easily discussed in the same manner, we omit them.]

See Sec. 25.6 of the text by Misner, Thorne, and Wheeler,[2] for further detail.

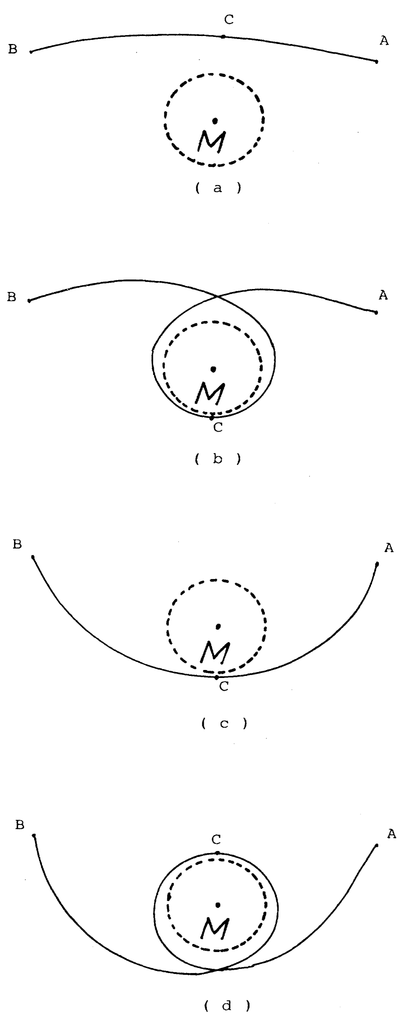

FIG. 4. Examples of possible orbits of light around a black hole (mass M): The orbits are classified by an integer n, "winding number." C represents a point of nearest approach to the black hole. The critical sphere is depicted by a dashed line. (a) $n=0$. (b) $n=1$. (c) $n=-1$. (d) $n=-2$.

APPENDIX B: CALCULATIONS OF THE LIGHT DEFLECTION ANGLE

In this appendix we give detailed derivation of Eqs. (2.1) and (2.5), which appear in the text (Sec. II A) without derivation.

1. Derivation of (2.1)

The motion of light in the gravitational field of the Sun (mass M) is determined by the action

$$S=\int \frac{ds}{d\lambda}d\lambda\ ,$$
$$\left(\frac{ds}{d\lambda}\right)^2=-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]\ , \tag{B1}$$

where the variable λ (affine parameter) parametrizes the light path $x^\mu=x^\mu(\lambda)$ and we have taken the standard coordinate system (t,r_S,θ,ϕ), defined in (1.1). (In this appendix we omit the suffix S for the sake of simplicity.) Variations of (B1) with respect to t, θ, and ϕ, respectively, give the (geodesic) equations

$$\frac{d}{d\lambda}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}\right]=0\ ,$$
$$r^2\left(\frac{d\phi}{d\lambda}\right)^2\sin\theta\cos\theta-\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right)=0\ , \tag{B2a}$$
$$\frac{d}{d\lambda}\left(r^2\sin^2\theta\frac{d\phi}{d\lambda}\right)=0\ .$$

Instead of the equation derived from the variation with respect to r, we may take the null condition

$$\left(\frac{ds}{d\lambda}\right)^2=-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]=0\ . \tag{B2b}$$

Because of the spherical symmetry of the metric, light moves on a plane, which is assured by the second equation of (B2a). We may take it to be the $\theta=\pi/2$ plane. Then, the second equation of (B2a) is trivially satisfied, and the first and the third ones are easily integrated out to be

$$\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}=b\ \ (\text{const}),\quad r^2\frac{d\phi}{d\lambda}=h\ \ (\text{const})\ , \tag{B3}$$

where the integration constants b and h have the physical meaning of the relativistic energy and the angular momentum, respectively. Using Eq. (B3) and the relation, derived from (B3),

$$\frac{dr}{d\lambda}=\frac{dr}{d\phi}\frac{d\phi}{d\lambda}=\frac{h}{r^2}\frac{dr}{d\phi}\ , \tag{B4}$$

we can obtain the equation of the orbit of light $r=r(\phi)$ from (B2b) as

$$-\left[1-\frac{2M}{r}\right]^{-1}b^2+\left[1-\frac{2M}{r}\right]^{-1}\frac{h^2}{r^4}\left[\frac{dr}{d\phi}\right]^2+r^{-2}h^2=0\ . \tag{B5}$$

In terms of the variables (v,m,Δ), defined in (2.4), (B5) can be rewritten in the form

$$\frac{d\phi}{dv}=\pm\frac{1}{F(v)^{1/2}},\quad F(v)=-v^2+1+2mv^3\ . \tag{B6}$$

From the definition of the deflection angle δ (see Fig. 1), its (exact) expression is obviously given by (2.1).

2. Derivation of (2.5)

We perturbatively evaluate the right-hand side of Eq. (2.1) (with the standard coordinate system), i.e.,

$$I(m)\equiv 2\int_0^{v_0}\frac{dv}{F(v)^{1/2}},\quad F(v)=-v^2+1+2mv^3\ , \tag{B7}$$

with respect to m $(0<m\ll 1)$. If we denote three roots of the cubic equation $F(v)=0$ as v_0, $-v_1$, and $\alpha/2m$, $F(v)$ can be written as

$$F(v)=(v_0-v)(v+v_1)(\alpha-2mv)\ . \tag{B8a}$$

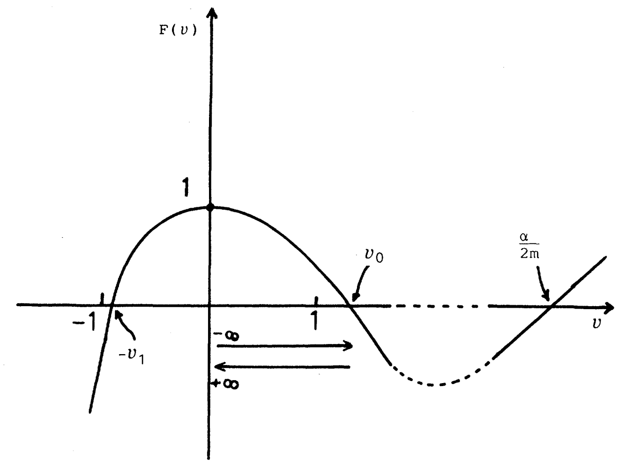

FIG. 5. Graph of the function $F(v)$, which appears in calculations of the light deflection angle [see (B8a)].

The three roots v_0, v_1, and α are related through the relation

$$\alpha=1+2m(v_1-v_0)=\frac{2mv_0v_1}{v_0-v_1}=\frac{1}{v_0v_1}\ . \tag{B8b}$$

The graph of $F(v)$, in the present case $(0<m\ll 1)$, is drawn in Fig. 5. Inserting (B8a) into (B7), and expanding with respect to m, we obtain

$$\begin{aligned}I(m)&=\frac{2}{\alpha^{1/2}}\int_0^{v_0}\frac{1}{[(v_0-v)(v+v_1)]^{1/2}}dv+\frac{2m}{\alpha^{3/2}}\int_0^{v_0}\frac{v}{[(v_0-v)(v+v_1)]^{1/2}}dv+\frac{3m^2}{\alpha^{5/2}}\int_0^{v_0}\frac{v^2}{[(v_0-v)(v+v_1)]^{1/2}}dv+O(m^3)\\&\equiv\frac{2}{\alpha^{1/2}}I_0(m)+\frac{2m}{\alpha^{3/2}}I_1(m)+\frac{2m^2}{\alpha^{5/2}}I_2(m)+O(m^3)\ .\end{aligned} \tag{B9}$$

The three integrals contained in (B9) are easily integrated out to be

$$\begin{aligned}I_0(m)&\equiv\int_0^{v_0}\frac{1}{[(v_0-v)(v+v_1)]^{1/2}}dv=2\arcsin\left[\frac{v_0}{v_1+v_0}\right]^{1/2},\\I_1(m)&\equiv\int_0^{v_0}\frac{v}{[(v_0-v)(v+v_1)]^{1/2}}dv=(v_0v_1)^{1/2}+(v_0-v_1)\arcsin\left[\frac{v_0}{v_1+v_0}\right]^{1/2},\\I_2(m)&\equiv\int_0^{v_0}\frac{v^2}{[(v_0-v)(v+v_1)]^{1/2}}dv\\&=-\left[\frac{v_1+v_0}{2}\right]^2\left[\frac{\pi}{4}-\tfrac{1}{4}\sin\left[2\arcsin\frac{v_1-v_0}{v_1+v_0}\right]-\tfrac{1}{2}\arcsin\frac{v_1-v_0}{v_1+v_0}\right]\\&\quad+(v_0-v_1)(v_0v_1)^{1/2}+(v_0^2+v_1^2)\arcsin\left[\frac{v_0}{v_1+v_0}\right]^{1/2}.\end{aligned} \tag{B10}$$

On the other hand, the two roots v_0 and $-v_1$ of $F(v)=0$ can be iteratively solved as

$$\begin{aligned}v_0&=1+m+\tfrac{5}{2}m^2+O(m^3)\ ,\\-v_1&=-1+m-\tfrac{5}{2}m^2+O(m^3)\ .\end{aligned} \tag{B11a}$$

Making use of the relation (B8b) with (B11a), α is evaluated to be

$$\alpha=1-4m^2+O(m^4)\ . \tag{B11b}$$

Therefore, (B10) with (B11a) and (B11b) results in

$$I_0(m)=\frac{\pi}{2}+m+O(m^3),\quad I_1(m)=1+\frac{\pi}{2}m+O(m^2)\ ,\qquad \text{(B12)}$$
$$I_2(m)=\frac{\pi}{4}+O(m)\ .$$

Inserting (B11b) and (B12) into (B9), we finally obtain

$$I(m)=\pi+4m+\frac{15\pi}{4}m^2+O(m^3)\ . \qquad \text{(B13)}$$

Therefore, the deflection angle δ [$\equiv I(m)-\pi$] is given by (2.5).

APPENDIX C: CALCULATIONS OF THE PERIHELION SHIFT ANGLE

In this appendix we give detailed derivation of Eqs. (2.6) and (2.10), which appear in the text (Sec. II B) without derivation.

1. Derivation of (2.6)

The orbit of "Mercury" in the gravitational field of the Sun is determined by the geodesic equation derived from (B1). (As in Appendix B, we take the standard coordinate system and omit the suffix S.) The different point, from the case of light (Appendix B), is that we cannot use the null condition (B2b) but can choose, as the affine parameter λ, the proper time $\lambda=\tau(=-s)$. Therefore, the equations of motion, in the "Mercury" case, are given by (B2a) and

$$-1=\frac{ds^2}{d\lambda^2}=-\left[1-\frac{2M}{r}\right]\left[\frac{dt}{d\lambda}\right]^2+\left[1-\frac{2M}{r}\right]^{-1}\left[\frac{dr}{d\lambda}\right]^2+r^2\left[\left[\frac{d\theta}{d\lambda}\right]^2+\sin^2\theta\left[\frac{d\phi}{d\lambda}\right]^2\right]\ . \qquad \text{(C1)}$$

As in Appendix B, we may choose $\theta=\pi/2$, and Eqs. (B2a) are solved as (B3). Inserting (B3) and (B4) into (C1), we obtain the equation of the orbit of "Mercury":

$$1=\frac{b^2}{1-\frac{2M}{r}}-\frac{1}{1-\frac{2M}{r}}\frac{h^2}{r^4}\left[\frac{dr}{d\phi}\right]^2-\frac{h^2}{r^2}\ . \qquad \text{(C2)}$$

In terms of the new variables (v,m,ϵ) defined in (2.9), Eq. (C2) is rewritten in the form

$$\frac{d\phi}{dv}=\pm\frac{1}{G(v)^{1/2}},\quad G(v)=-v^2+2v+2\epsilon+2m^2v^3\ . \qquad \text{(C3)}$$

By the definition of the angle δ of the perihelion shift (see Fig. 2), we easily obtain its exact expression (2.6) (for the case of the standard coordinate system).

2. Derivation of (2.10)

We evaluate, perturbatively with respect to m ($\ll 1$), the right-hand side of Eq. (2.6), i.e.,

$$J(m)\equiv 2\int_{v_1}^{v_2}\frac{dv}{G(v)^{1/2}}\ ,\qquad G(v)=-v^2+2v+2\epsilon+2m^2v^3\ , \qquad \text{(C4)}$$

where v_1 and v_2 are two roots of $G(v)=0$ ($v_2>v_1$). If we write another root as $\alpha/2m^2$, $G(v)$ can be written in the form

$$G(v)=(v-v_1)(v_2-v)(\alpha-2m^2v)\ . \qquad \text{(C5)}$$

v_1, v_2, and α are related by

$$\alpha=1-2m^2(v_1+v_2)=\frac{2(1-m^2v_1v_2)}{v_1+v_2}=-\frac{2\epsilon}{v_1v_2}\ . \qquad \text{(C6)}$$

The graph of $G(v)$ is depicted in Fig. 6. In order for the "Mercury" to be bounded around the Sun, ϵ must be restricted within the region

$$-\tfrac{1}{2}<\epsilon<0\ . \qquad \text{(C7)}$$

Inserting (C5) into (C4), and expanding with respect to m, we obtain

$$J(m)=\frac{2}{\alpha^{1/2}}\int_{v_1}^{v_2}\frac{1}{[(v-v_1)(v_2-v)]^{1/2}}\times\left[1+\frac{m^2}{\alpha}v+\frac{3}{2}\frac{m^4}{\alpha^2}v^2\right]dv+O(m^6)\ . \qquad \text{(C8)}$$

In order to evaluate the above expression, we use the integral formulas

$$\int_{v_1}^{v_2}\frac{dv}{[(v-v_1)(v_2-v)]^{1/2}}=\pi\ ,$$
$$\int_{v_1}^{v_2}\frac{v\,dv}{[(v-v_1)(v_2-v)]^{1/2}}=\frac{\pi}{2}(v_1+v_2)\ , \qquad \text{(C9)}$$
$$\int_{v_1}^{v_2}\frac{v^2dv}{[(v-v_1)(v_2-v)]^{1/2}}=\frac{\pi}{4}[\tfrac{3}{2}(v_1^2+v_2^2)+v_1v_2]\ .$$

Then, (C8) becomes

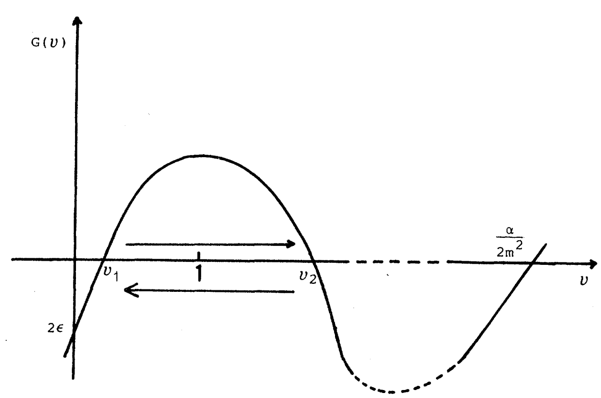

FIG. 6. Graph of the function $G(v)$, which appears in calculations of the perihelion shift angle [see (C5)].

$$J(m)=\frac{2\pi}{\alpha^{1/2}}+\frac{\pi}{\alpha^{3/2}}m^2(v_1+v_2)+\frac{3\pi}{4}\frac{m^4}{\alpha^{5/2}}[\tfrac{3}{2}(v_1^2+v_2^2)+v_1v_2]+O(m^6)\ . \qquad \text{(C10)}$$

On the other hand, by solving the equation $G(v)=0$, we obtain the iterative solutions for the two roots, v_1 and v_2, as

$$v_1=1-(1+2\epsilon)^{1/2}-m^2\frac{[1-(1+2\epsilon)^{1/2}]^3}{(1+2\epsilon)^{1/2}}+O(m^4)\ ,$$
$$v_2=1+(1+2\epsilon)^{1/2}+m^2\frac{[1+(1+2\epsilon)^{1/2}]^3}{(1+2\epsilon)^{1/2}}+O(m^4)\ . \qquad \text{(C11a)}$$

From the relation (C6) with (C11a) we obtain

$$\alpha=1-4m^2-8m^4(2+\epsilon)+O(m^6)\ . \qquad \text{(C11b)}$$

Finally, inserting (C11a) and (C11b) into (C10) we get

$$J(m)=\pi[2+6m^2+\tfrac{15}{2}m^4(7+2\epsilon)]+O(m^6)\ , \qquad \text{(C12)}$$

which leads to the result (2.10), with $2\pi+\delta=J(m)$.

APPENDIX D: CALCULATIONS OF TIME DELAY OF RADAR ECHOES

In this appendix we give detailed derivation of Eqs. (3.1), (3.7), and (3.9), which appear in the text (Sec. III) without derivation.

1. Derivation of (3.1)

The equations of motion of light are already given in Appendix B, viz., (B2a) and (B2b), which are solved as (B3) with $\theta=\pi/2$. (As in Appendixes B and C, we take the standard coordinate system and omit the suffix S.) Because the physical quantity which we want to measure is the time interval, we use the following relation instead of (B4):

$$\frac{dr}{d\lambda}=\frac{dr}{dt}\frac{dt}{d\lambda}=\frac{b}{1-\dfrac{2M}{r}}\frac{dr}{dt}\ , \qquad \text{(D1)}$$

where (B3) has been used. Using Eqs. (B3) and (D1), Eq. (B2b) (with $\theta=\pi/2$) becomes

$$\frac{dt}{dr}=\pm\frac{1}{H(r)^{1/2}}\ ,$$
$$H(r)=\left[1-\frac{2M}{r}\right]^2\left[1-\frac{\Delta^2}{r^2}\left[1-\frac{2M}{r}\right]\right]\ , \qquad \text{(D2)}$$

where we have put $\Delta=h/b$ as in (2.4). Then the definition of the round-trip travel time [see Fig. 3(a)] gives the expression (3.1) with $i=S$. The similar method can be applied to derive the $i=D$ or $i=I$ formulas.

2. Derivation of (3.7)

We have only to solve the equation of the orbit of light (B6), i.e.,

$$\left[\frac{dv}{d\phi}\right]^2-F(v)=\left[\frac{dv}{d\phi}\right]^2+v^2-1-2mv^3=0\ , \qquad \text{(D3a)}$$

which is already derived in Appendix B. As the initial condition we may choose

$$\left.\frac{dv}{d\phi}\right|_{\phi=0}=0\ , \qquad \text{(D3b)}$$

which specifies the origin of the angle ϕ.

We seek a solution of (D3a) in the form

$$v(\phi)\equiv v_0(\phi)+mv_1(\phi)+m^2v_2(\phi)+O(m^3)\ . \qquad \text{(D4)}$$

Inserting (D4) into (D3a), differential equations for the functions $v_0(\phi)$, $v_1(\phi)$, and $v_2(\phi)$ are obtained as

$$\left[\frac{dv_0}{d\phi}\right]^2+v_0^2-1=0\ , \qquad \text{(D5a)}$$

$$\frac{dv_0}{d\phi}\frac{dv_1}{d\phi}+v_0v_1=v_0^3\ , \qquad \text{(D5b)}$$

$$\frac{dv_0}{d\phi}\frac{dv_2}{d\phi}+v_0v_2=3v_0^2v_1-\frac{1}{2}\left[v_1^2+\left[\frac{dv_1}{d\phi}\right]^2\right]\ . \qquad \text{(D5c)}$$

These equations can be solved successively:

$$\begin{aligned} v_0(\phi)&=\sin(\phi+\alpha)\ ,\\ v_1(\phi)&=1+\cos^2(\phi+\alpha)+\beta\cos(\phi+\alpha)\ ,\\ v_2(\phi)&=-\tfrac{1}{2}(\beta^2-5)\sin(\phi+\alpha)-\beta\sin[2(\phi+\alpha)]\\ &\quad-\tfrac{3}{8}\cos(\phi+\alpha)\sin[2(\phi+\alpha)]\\ &\quad-\tfrac{15}{4}(\phi+\alpha)\cos(\phi+\alpha)+\gamma\cos(\phi+\alpha)\ , \end{aligned} \qquad \text{(D6)}$$

where α, β, and γ are arbitrary integral constants. They are determined by the initial condition (D3b):

$$\alpha=\frac{\pi}{2},\quad \beta=0,\quad \gamma=\frac{15\pi}{8}\ . \qquad \text{(D7)}$$

Putting back (D7) into (D6), we obtain

$$\begin{aligned} &v_0(\phi)=\cos\phi,\quad v_1(\phi)=1+\sin^2\phi\ ,\\ &v_2(\phi)=\tfrac{15}{4}\phi\sin\phi+\tfrac{5}{2}\cos\phi-\tfrac{3}{8}\sin\phi\sin(2\phi)\ . \end{aligned} \qquad \text{(D8)}$$

Equation (D4) with (D8) is nothing but (3.7).

3. Derivation of (3.9)

We evaluate the expression

$$T=2\int_{r_0}^{r_A}\frac{dr}{H(r)^{1/2}}+(A\leftrightarrow B)\ ,$$
$$H(r)=\left[1-\frac{2M}{r}\right]^2\left[1-\frac{r_0^2}{r^2}\frac{1-\dfrac{2M}{r}}{1-\dfrac{2M}{r_0}}\right]\ , \qquad \text{(D9)}$$

which is obtained from (3.1) and (3.2) (with the standard coordinate system), where use has been made of the relation (3.4): i.e.,

$$\Delta^2=\frac{r_0^2}{1-\frac{2M}{r_0}} . \tag{D10}$$

Although r_0 can be expressed, by solving (D10), in terms of M and Δ, we postpone explicitly writing it out until Eq. (D14).

We consider the case in which light moves only outside the critical sphere ($r_A, r_B \geq r \geq r_0 > 3M$; see Appendix A). Expanding the integrand of (D9) with respect to M, we obtain

$$\frac{T}{2}=\int_{r_0}^{r_A}\frac{1}{[1-(r_0/r)^2]^{1/2}}\left[1+M\left[\frac{2}{r}+\frac{r_0}{(r+r_0)r}\right]+M^2\left[\frac{4}{r^2}+\frac{2r_0}{(r+r_0)r^2}+\frac{2}{(r+r_0)r}+\frac{3}{2}\frac{r_0^2}{(r+r_0)^2r^2}\right]\right]dr+(A\leftrightarrow B)+O(M^3) . \tag{D11}$$

In order to evaluate (D11) explicitly, we use the integral formulas

$$\begin{aligned}
&\int_{r_0}^{r_A}\frac{dr}{[1-(r_0/r)^2]^{1/2}}=(r_A^2-r_0^2)^{1/2}, \quad \int_{r_0}^{r_A}\frac{dr}{(r^2-r_0^2)^{1/2}}=\ln\left[\frac{r_A+(r_A^2-r_0^2)^{1/2}}{r_0}\right],\\
&\int_{r_0}^{r_A}\frac{r-r_0}{(r^2-r_0^2)^{3/2}}dr=\frac{1}{r_0}\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}, \quad \int_{r_0}^{r_A}\frac{dr}{(r^2-r_0^2)^{1/2}r}=\frac{1}{r_0}\left[\frac{\pi}{2}-\operatorname{arccsc}\frac{r_A}{r_0}\right],\\
&\int_{r_0}^{r_A}\frac{dr}{(r+r_0)^{3/2}(r-r_0)^{1/2}r}=\frac{1}{r_0^2}\left[\frac{\pi}{2}-\operatorname{arccsc}\frac{r_A}{r_0}-\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}\right],\\
&\int_{r_0}^{r_A}\frac{dr}{(r+r_0)^{3/2}(r-r_0)^{1/2}}=\frac{1}{r_0}\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2},\\
&\int_{r_0}^{r_A}\frac{dr}{(r+r_0)^{5/2}(r-r_0)^{1/2}r}=\frac{1}{r_0^3}\left[\frac{\pi}{2}-\operatorname{arccsc}\frac{r_A}{r_0}-\frac{4}{3}\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}-\frac{1}{3}\frac{r_0}{r_A+r_0}\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}\right],
\end{aligned} \tag{D12}$$

where $0\leq \operatorname{arccsc}(r_A/r_B)\leq \pi/2$. Using these formulas, (D11) is completely integrated out to be

$$\frac{T}{2}=(r_A^2-r_0^2)^{1/2}+M\left[2\ln\left[\frac{r_A+(r_A^2-r_0^2)^{1/2}}{r_0}\right]+\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}\right]+\frac{M^2}{r_0}\left[\frac{15}{4}\pi-\frac{15}{2}\operatorname{arccsc}\frac{r_A}{r_0}-2\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}-\frac{1}{2}\frac{r_0}{r_A+r_0}\left[\frac{r_A-r_0}{r_A+r_0}\right]^{1/2}\right]+(A\leftrightarrow B)+O(M^3) . \tag{D13}$$

Now remember r_0 depends on Δ and M through (D10). Solving (D10) with respect to Δ and M, we obtain

$$r_0=\Delta-M-\frac{3}{2\Delta}M^2+O(M^3) . \tag{D14}$$

Substituting this expression into (D13), and expanding with respect to M once more, we finally reach the result

$$\frac{T}{2}=(r_A^2-\Delta^2)^{1/2}+M\left[\frac{r_A}{(r_A^2-\Delta^2)^{1/2}}+2\ln\frac{r_A+(r_A^2-\Delta^2)^{1/2}}{\Delta}\right]+M^2\left[\frac{1}{(r_A^2-\Delta^2)^{3/2}}\left(\frac{7}{2}r_A^2-4\Delta^2\right)+\frac{1}{\Delta}\left[\frac{15}{4}\pi-\frac{15}{2}\operatorname{arccsc}\frac{r_A}{\Delta}\right]\right]+(A\leftrightarrow B)+O(M^3) , \tag{D15}$$

which is nothing but (3.9).

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[7]In general (viz., if the gravitational source is a black hole), δ can take any positive value (cf. Appendix A).

[8]Exactly speaking, we cannot compare the orbit in the presence of the gravitational field with that without it directly because the gravitational field implies distortion of space-time. In this respect the situation is very different from scattering of particles in flat space-time by a nongravitational field. Thus the meaning of the impact parameter is well defined only in the asymptotic region where space-time is flat and we can compare the orbits. Remember that Fig. 1 is inaccurate in this point (cf. Sec. III B).

[9]A necessary (but not sufficient) condition for light not to be trapped by the black hole is $m < 1/3\sqrt{3}$. (See also Appendix A.)

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[13]Because the proper energy (per unit proper mass) is 1 in our unit, $E := b - 1$ can be interpreted as the Newtonian energy. When the relativistic effect is small, $E \ll 1$, the approximation $E \simeq (b^2 - 1)/2$ is very good. Note that ϵ in (2.9) is essentially E in this approximation.

[14]A necessary (but not sufficient) condition for the finite orbit to exist is $m \leq 1/2\sqrt{3}$.

[15]Note that there are two kinds of parameters m and ϵ, which, intuitively speaking, parametrize general-relativistic effects of the Sun's gravity and special-relativistic ones of Mercury's motion, respectively. For our cases, perturbation with respect to ϵ is not suitable because $\epsilon \sim 1$ corresponds to the Newtonian approximation when the orbit of "Mercury" is bounded. On the other hand, m is very small in the usual situation (for example, $m \sim 10^{-4}$ for Mercury).

[16]These motions are by no means negligible, but can be taken into account in a straightforward way in reducing the observational data. [In addition to them, there are some important factors, such as the effect of solar corona, etc., which have to be considered in the practical experiment (Refs. 26 and 27).]

[17]The *coordinate time* expression is used for simplicity, though the *proper time* one (on Earth, for example) must be used for physical interpretation of "time." (The coordinate time is not the "time" but the "name.")

[18]Furthermore, we restrict ourselves to the case that the point C, corresponding to r_{0i}, lies in between Earth (A) and the planet (B) on the orbit of light, such as Fig. 3(a) or 4. When this condition is not satisfied, a slight modification is necessary for expression (3.1). [Note that the above condition is not satisfied by the orbit of $n=0$ (see Appendix A) with $r_A \neq r_B$ when the arrangement of Earth and the planet is near the inferior conjunction.]

[19]The true (i.e., proper) distance, which is meaningful only when a reference frame is specified, is defined by the integration of $dl \equiv (\sum_{m,n=1}^{3} \gamma_{mn} dx^m dx^n)^{1/2}$, where γ_{mn} is the metric of a three-dimensional space (reference frame) which is determined by the metric $g_{\mu\nu}$ of four-dimensional space-time (see Landau-Lifshitz or Møller of Ref. 2). As far as the coordinate systems belonging to the *same* reference frame are concerned, dl is independent of the coordinate choice. [A proof is presented in Sec. 9.16 (or Appendix 4 in the first edition) of Møller's textbook (Ref. 2).] The physically meaningful distance is defined only outside the black hole. Usually it is said that the Sun would become a black hole if it were crushed into 3 km, which is the "Schwarzschild radius" of the Sun. Thereby it is very important to realize that this value 3 km means *not* the real distance but merely the name in the standard coordinate system. (If we adopt the de Donder coordinate system or the isotropic one, the "Schwarzschild radius" of the Sun becomes $\frac{3}{2}$ or $\frac{3}{4}$ km, respectively.)

[20]In order to avoid a misunderstanding, we make a comment. For definiteness, consider the case of the radar-echo prediction: The total trip time between two points A and B on the flat geometry *can* be numerically compared with the one between two points A' and B' on the curved geometry, of course. But we can never take $A = A'$, $B = B'$ because of the difference of geometry. On the other hand, when we want to derive "the non-Newtonian delay," we must *assume* $A = A'$, $B = B'$ somehow. The way of that assumption is nothing but the "identification rule" discussed in the text.

[21]The orbit of light is not unique in general. Only the simplest orbit, corresponding to Eq. (3.7), is depicted in Fig. 3(a). See Appendix A in this respect.

[22]The statement given by Wald (Ref. 2), that either r_{0i} or Δ can be regarded as independent of M, is *incorrect.* Because of this fact, the separation into Newtonian and post-Newtonian parts in incomplete in his result. [He incorrectly considers that $(r_{AS}^2 - r_{0S}^2)^{1/2} + (r_{BS}^2 - r_{0S}^2)^{1/2}$ represents the Newtonian time delay. The same mistake is shared by Weinberg (Ref. 2). The method of Logunov and Loskutov (Ref. 5) is an improved version. See also Ref. 24.]

[23]Many authors make use of the *plane trigonometry,* which is not correct in curved space-time, to derive the post-Newtonian time delay. Since the deviation from the plane tri-

gonometry amounts to $O(M)$, such a method seems nonsense at first. Their explanation (not the calculation) contains *double mistakes* and the result luckily becomes correct: First they say that a straight line can be used as an approximate orbit because of Fermat's principle. Next they use plane trigonometry. Careful examination, however, shows that their calculation is based not on the straight orbit but on the curved one defined so that the plane trigonometric formulas hold formally. (A real straight line, if any, is never accommodated in the four-dimensional curved space-time in general.) Owing to Fermat's principle, such a (slight) modification does not influence the round-trip travel time in the post-Newtonian approximation. Their method cannot be applied in order to derive the post-post-Newtonian formula.

[24]Logunov and Loskutov do not calculate the full post-Newtonian formula. They directly give a post-Newtonian formula based on the approximation $r_{AS}, r_{BS} \ll r_{0S}$. But when estimating $(r_{AS}^2 - r_{0S}^2)^{1/2}$ and $(r_{BS}^2 - r_{0S}^2)^{1/2}$, they implicitly assume that δ_S, in (3.12), is zero (this means $r_{AS} = r_{BS}$), and use plane trigonometry. Fortunately, their result is very similar to the correct one with $r_{AS}, r_{BS} \ll r_{0S}$.

[25]In Ref. 26, two kinds of anomalous time delay are reported: (1) The measured time delay is about 5 μs systematically larger than the theoretical prediction over the one-month period centered at the superior conjunction; (2) the measured data, even after adding 5 μs constant, differs from the theoretical one (systematically) near the superior conjunction (5 days) about 5–15 μs. The present authors suggested, in the original manuscript of the present paper, the possibility that the anomalous time delay of (1) may stem from the unsuitable treatment of the inevitable ambiguity. This possibility, however, has been denied. In Ref. 27, which is the complete version of Ref. 26, the anomalous time delay of (1) *vanishes.* (Unfortunately, this important fact is not explicitly stated in Ref. 27.) As for the anomalous time delay of (2), no definitive explanation has been obtained.

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[28]Because of the rapid progress of measurement techniques, the days when the post-post-Newtonian calculations become urgent are not so far in the future. See Ref. 12 for the possible experiments in the near future.

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