# Is minisuperspace quantization valid?: Taub in mixmaster

Karel V. Kuchař
*Physics Department, University of Utah, 201 James Fletcher Building, Salt Lake City, Utah 84112*

Michael P. Ryan, Jr.
*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, México 04510, Distrito Federal, Mexico*

(Received 10 August 1989)

The paper addresses quantitatively the question of the validity of physical predictions based on minisuperspace quantization of Einstein's theory of gravitation. It studies a homogeneous, anisotropic cosmological model of higher symmetry (the Taub model) embedded in one of lesser symmetry (the mixmaster model). The comparison of the physical behavior of these two models is based on the construction of a non-negative probability density and the associated conserved inner product which allow a consistent probabilistic interpretation of the state function of the Universe in the interesting regime of deep channel penetration. It is shown that the respective behavior is widely different. A program is set for investigating a hierarchy of models with higher symmetry embedded in models of lesser symmetry to spell out the criteria under which minisuperspace quantum results can be expected to make meaningful predictions about full quantum gravity.

## I. INTRODUCTION

### A. Minisuperspaces—a minihistory

The first attempt at minisuperspace quantization seems to be due to DeWitt,[1] although the name "minisuperspace" was coined by Misner only five years later.[2] At that time Wheeler had introduced the idea of superspace,[3] the space of all three-geometries as the arena in which geometrodynamics develops, a particular four-geometry being represented by a trajectory in this space. Misner had just finished applying the Hamiltonian formulation of gravity, developed in the late 1950s and early 1960s, to cosmological models with an eye toward quantization of these cosmologies as model theories of general relativity, and he invented "minisuperspace" and "minisuperspace quantization" or "quantum cosmology" to describe the evolution of cosmological spacetimes as trajectories in the finite-dimensional sector of superspace related to the finite number of parameters that describe $t=$const slices of the models, and the quantum version of such models, respectively.

Cosmological minisuperspaces and their quantum versions were extensively studied in the early 1970s (Ref. 4), but interest in them waned after about 1975 and little new work was done until Hawking revived the field in the 1980s (Ref. 5), emphasizing path-integral approaches. This started a lively resurgence of interest in minisuperspace quantization in this decade.

### B. The physical meaning of quantum minisuperspaces

Perhaps one of the greatest difficulties with quantum cosmology has always been the seductive character of its results. It is obvious that taking the metric of a cosmological model which is truncated by an enormous degree of imposed symmetry and simply plugging it into a quantization procedure *cannot* give an answer that is in any way a quantum gravity solution. What is being done in quantum cosmology is to assume that one can essentially represent a metric as a series expansion in space-dependent modes, where the cosmological model is the homogeneous mode, and that in some sense one can ignore the dependence of the wave function of the Universe on all inhomogeneous modes. This artificial "freezing" (i.e., setting equal to zero) of the modes before quantization is an obvious violation of the uncertainty principle and cannot lead to an exact solution. While this basic idea is so well understood that it should not be necessary to state it, and it has been mentioned by almost every author who has written on quantum cosmology, the results of applying this untenable quantization procedure have always seemed to predict such reasonable and internally consistent behavior of the Universe that it has been difficult to believe that they have no physical content.

One way to reconcile the physically unsound quantization procedure with the plausible results is to conjecture that somehow quantum cosmologies (and other minisuperspace quantization problems) are *approximations* to some true quantum gravity solution that would represent the closest one could come to a cosmological model in quantum geometrodynamics. Unfortunately, unless one can completely quantize gravity in order to find the necessary solutions, it is impossible to check this conjecture directly. The best one can do is to investigate a hierarchy of (classically and quantum mechanically) exactly soluble models, each of which has as a special case another exactly soluble model of higher symmetry. We plan a series of papers in which this set of models is used as a laboratory for testing the possibility that quantum solutions of the highest-symmetry models are approximations to solutions in the models with less symmetry.

Since the soluble models are all minisuperspace models, we will introduce the term "microsuperspace" for the higher-symmetry models contained in them.

In this paper we will consider the simplest possible combination of minisuperspace and microsuperspace, a homogeneous anisotropic cosmological model embedded in another homogeneous model with a lower degree of symmetry. The Taub model[6] is a special case of the diagonal Bianchi type-IX cosmological model (the "mixmaster" universe[7]). In certain regions of the minisuperspace it is possible to find an exact solution both to the quantum Taub model and to the full diagonal type-IX model, and these solutions can be compared. The details of the models and their quantization will be discussed below.

### C. Minisuperspaces as approximate solutions

The major difficulty in considering quantum minisuperspaces as approximations to true quantum gravity solutions is to define the sense in which the word "approximate" is to be taken. This led us to consider the problem of justifying minisuperspace quantization.[8] In ordinary quantum mechanics there are a number of well-known approximation schemes, such as perturbation theory, WKB methods, or the adiabatic and sudden approximations, that yield solutions that are in some sense close to exact solutions in the full theory. If quantum minisuperspaces are to be quantum approximations, the sense of the approximation must be different from the ones listed above. If one thinks of superspace as the arena in which a solution of the Einstein equations is a trajectory, then we must think of minisuperspace as a lower-dimensional sheet of superspace, and a minisuperspace solution as a trajectory confined to this sheet. One possible definition of a quantum minisuperspace solution as an "approximation" would be that a properly constructed "wave packet" centered around a minisuperspace three-geometry would tend to follow closely and without much spreading into the surrounding minisuperspace the "shadow" packet that defines the minisuperspace solution. One could also ask whether such a wave packet could keep pace with the minisuperspace shadow, at least for a certain "time," so that the minisuperspace sector could predict the true behavior in some "useful" interval. Here the problem would reduce to determining what length of time (and for what definition of "time") would be necessary to be useful. An example might be a length of time from a cosmological singularity to some finite time when quantum gravity was no longer important. In terms of cosmic time $t$ this would be a short interval indeed, but in terms of a logarithmic time $t'$ (i.e., if the singularity were at cosmic time $t=0$, and if one chose $t'=\ln t$) the amount of $t'$ time between the singularity and any finite $t$ would be infinite.

A closer look at this formulation reveals that the condition that the wave packet not spread too much outside of the minisuperspace sheet is not really essential.[3] From the minisuperspace wave function one can predict only the behavior of a limited number of dynamical variables: namely, those which are constructed entirely from the minisuperspace coordinates and momenta. When the measurement of all other variables is ignored, the superspace wave function can be reduced to a density operator which, at any instant, can be decomposed into an incoherent combination of projectors (pure minisuperspace states). The first question one must ask is whether the transitions among the terms of such a decomposition are negligible, at least for some period of time. If the answer is no, minisuperspace dynamics cannot be consistently formulated at all. If the answer is yes, each term in the decomposition evolves according to an autonomous "Schrödinger equation" which does not care about the presence or absence of the remaining terms. In this case it is meaningful to ask how much the structure of this Schrödinger equation (the metric of its kinetic term and its potential) differs from that of the minisuperspace Schrödinger equation. If the corrections are small, the minisuperspace solution yields approximately the same statistics of the minisuperspace variables as the wave function of the full theory.

Note that the answer to both questions (about the transition coefficients and about the structure of the Schrödinger equation) depends on the superspace wave function which is the starting point of the decomposition. It turns out, however, that for the validity of the minisuperspace prediction it is not vital that the function not spread outside the minisuperspace sheet, but rather that the gradient of the potentials in the regions to which it spreads do not appreciably influence its propagation along this sheet.

We have analyzed the conceptual status of this approach on simple examples taken from ordinary quantum mechanics and field theory and showed under what conditions such "minisuperspace models" can be regarded as approximations to the full quantum theory.[8] For such examples, described by a time-independent Schrödinger equation, the minisuperspace approximation turns out to be akin to the Born-Oppenheimer approximation. In canonical gravity, however, the Wheeler-DeWitt equation is intrinsically time dependent and it resembles a Klein-Gordon equation. The conditions under which the density operator is approximately evolved by the minisuperspace Wheeler-DeWitt equation governing its projector term remains to be worked out.

Another approach to the question of what is meant by a minisuperspace approximation would start by constructing a "coordinate system" in superspace, that is, expanding all three-geometries (actually all three-metrics, but with due attention to coordinate invariance) as a series in some complete set of tensor eigenstates where the minisuperspace sector is some limited set of these states. If we make an artificial restriction to finite three-spaces (a sort of "box normalization") we can achieve a countable infinity of coordinate "axes" that can be used to describe any three-metric. Of course, any of the previous definitions of approximation can be reduced to statements couched in the language of such a coordinate system (the minisuperspaces studied in this paper will be an example), but there are some definitions of approximation, especially one due to Misner,[9] that are most naturally stated when referred to the mode system. Misner conjectured that near a cosmological singularity the "energy" of the gravitational system tends to flow into the homogeneous mode, with essentially all of it in this mode very near the singularity. The major stumbling block in this definition is that it requires a careful definition of the concept of "energy" that one is using.

The last meaning of the word "approximation" we will consider is perhaps not exactly an "approximation" at all. It is that the minisuperspace serves as a sort of "square well" potential, a simple concept that is often used in ordinary quantum mechanics as a catchall potential that is supposed to give one an overall *qualitative* picture of the important parts of the behavior of some physical system without reproducing the *exact* behavior of the system. The conjecture here about minisuperspaces is that they would correspond to something like a one-dimensional slice of a three-dimensional harmonic oscillator in ordinary quantum mechanics. If we were to write the potential of such a harmonic oscillator as $V(x,y,z)=\omega_1^2x^2+\omega_2^2y^2+\omega_3^2z^2$, a "minisuperspace" would be the sector $y=z=0$ with $V(x)=\omega_1^2x^2$. The details of the behavior of the one-dimensional harmonic oscillator would be different from that of the complete system because of the different frequencies in the different directions, but the qualitative behavior would be the same. There could be a number of three-dimensional potentials with the same minisuperspace $V(x)=\omega_1^2x^2$, such as

$$V(x,y,z)=\begin{cases}\omega_1^2x^2, & -L<y<+L,\ -L<z<+L,\\ \infty & \text{otherwise,}\end{cases} \tag{1.1}$$

for example, that would give the same qualitative behavior, while there would be potentials [such as $V(x,y,z)=\omega_1^2x^2-\omega_2^2y^2-\omega_3^2z^2$] that would give the same minisuperspace behavior, but whose full behavior would not even have a qualitative relationship with that of the minisuperspace. The final question here would be whether general-relativistic minisuperspaces give such a qualitative picture of the complete system, and, if not, under what circumstances one can trust the minisuperspace predictions.

In this paper we will study a particular system (the diagonal Bianchi type-IX cosmology) and try to answer some of the questions posed above in the limited context of a soluble problem.

### D. The mixmaster and Taub universes—homogeneous models within homogeneous models

One of the simplest minisuperspaces that has a soluble microsuperspace and allows a nontrivial choice of internal time is the mixmaster universe. In a parametrization due to Misner[7] one can write the metric of this model as

$$ds^2=-N^2dt^2+e^{-2\tau(t)}e^{2\beta(t)}_{ij}\omega^i\omega^j , \tag{1.2}$$

where $\tau(t)$ is a scalar, $\beta_{ij}(t)$ is a diagonal $3\times3$ traceless matrix, and the $\omega^i$ are the invariant one-forms of SO(3):

$$\omega^1=-\sin\psi\, d\theta+\sin\theta\cos\psi\, d\phi , \tag{1.3a}$$

$$\omega^2=\cos\psi\, d\theta+\sin\theta\sin\psi\, d\phi , \tag{1.3b}$$

$$\omega^3=\cos\theta\, d\phi+d\psi . \tag{1.3c}$$

The $\beta$ matrix can be parametrized as

$$\beta=\mathrm{diag}[x+\sqrt{3}y, x-\sqrt{3}y, -2x] . \tag{1.4}$$

The full minisuperspace is given by the space of all $x^A=(\tau,x,y)$. In fact, this is a slightly "unfolded" version of minisuperspace since there are additional symmetries of this metric (generated by discrete basis rotations), but this parametrization has proved more useful for quantization. The classical vacuum Einstein equations allow a consistent set of solutions for (1.2). A subset of these metrics which also allow consistent vacuum solutions is the family of Taub models which correspond to $y=0$, giving a microsuperspace parametrized by $x^A=(\tau,x)$.

We want to compare the quantum behavior of the Taub model with that of the full mixmaster model, where we will define "quantum behavior" as the evolution of solutions of the Wheeler-DeWitt equations for the respective models. There are a number of ways of arriving at the Wheeler-DeWitt equation for these metrics, which vary from path-integral approaches to canonical methods. We will use the conformal super-Hamiltonian approach of Misner.[2] The action for general relativity has the form

$$S=\frac{1}{16\pi}\int(\pi^{ij}\dot g_{ij}-NH_\perp-N^iH_i)dt\wedge\omega^1\wedge\omega^2\wedge\omega^3 . \tag{1.5}$$

For Bianchi type-IX models the $H_i$ are identically zero, so we will drop them from the action and assume (as is obvious from our form of the metric) that $N^i=0$. If we insert $g_{ij}$ from (1.2) and (1.4) into (1.5) and collect the linear combinations of $\pi^{ij}$ that multiply $\dot\tau,\dot x,\dot y$ to define $p_\tau,p_x,p_y$ [after integrating over the space variables using $\int\omega^1\wedge\omega^2\wedge\omega^3=(4\pi)^2$], $S$ reduces to

$$S=\int(p_x\dot x+p_y\dot y+p_\tau\dot\tau-NH_\perp)dt , \tag{1.6}$$

where

$$H_\perp=(6\pi)^{1/2}e^{3\tau}[-p_\tau^2+p_x^2+p_y^2+e^{-4\tau}W(x,y)] , \tag{1.7}$$

with

$$W(x,y)=12\pi^2\{e^{-8x}-4e^{-2x}\cosh(2\sqrt{3}y)+2e^{4x}[\cosh(4\sqrt{3}y)-1]\} . \tag{1.8}$$

The Misner[2] "supertime" gauge consists of scaling $H_\perp$ to the super-Hamiltonian $H$, where

$$H=-p_\tau^2+p_x^2+p_y^2+e^{-4\tau}W(x,y) . \tag{1.9}$$

As we will mention in more detail below, the Wheeler-DeWitt equation is the result of turning $\tau,x,y,p_\tau,p_x,p_y$, into operators satisfying the commutation relations $[\hat\tau,\hat p_\tau]=i$, $[\hat x,\hat p_x]=i$, $[\hat y,\hat p_y]=i$ and imposing the super-Hamiltonian constraint $H=0$ as a restriction $\hat H\Psi=0$ on the state function of the Universe $\Psi(\tau,x,y)$. Note that the equation $\hat H\Psi=0$ implies a certain factor ordering of the "traditional" Wheeler-DeWitt equation based on the operator version of $H_\perp$, that is, the equation $\hat H_\perp\Psi=0$ contains a factor of $e^{3\tau}$, and it is not clear how much of this factor should be folded into $-\hat p_\tau^2$. In principle one could write the first term in $\hat H_\perp$ as $-e^{(3-B)\tau}\hat p_\tau e^{B\tau}\hat p_\tau$, where $B$ is some constant, to give a class of possible equations. This is essentially the approach of Hawking,[5] where the constant $B$ is linearly related to the constant $p$ which he defines. In our case we will adopt the Misner[2] standpoint that the conformal invariance of the super-Hamiltonian under changes of $N$ implies that we must work with conformally related equivalence classes of superspace metrics rather than with rigid choices of the metric, and that the form of $H$ given in (1.9) must be representative of its class, and is as good as any other choice in both the classical and quantum regimes.

While we will not make much use of it, we will mention, in relation to the discussion of the meaning of approximate solution, the Arnowitt-Deser-Misner (ADM) Hamiltonian for this system. The ADM Hamiltonian is minus the conjugate of any variable that is chosen to mark an internal time for the system. For the time choices that we will consider, $\tau$ and $v=\tau+x$, the ADM Hamiltonian will be $-p_\tau$ and $(p_x+p_\tau)/2$, respectively.

### E. The scheme of the paper

In the body of the paper we will first display an exact classical solution to our problem in order to establish a framework for the quantum solutions. We will then derive the Wheeler-DeWitt equation corresponding to the super-Hamiltonian $H$. While giving the general solution to this equation is quite difficult, we can find an exact solution corresponding to large $x$ and small $y$ that will be sufficient for our purposes. This solution can easily be reduced to one for $y=0$ which corresponds to the Taub model. Before these solutions can be compared it is necessary to define a consistent, conserved, non-negative probability density that will give us the possibility of comparing physically reasonable predictions in the two cases. In Sec. III we will define such a probability density. In Sec. IV we will construct superspace wave packets, one in the microsuperspace (Taub) sector and another peaked around this sector, and then compare the evolution of these two packets.

In the final section we will discuss ways in which the microsuperspace packet approximates the behavior of the minisuperspace packet, and show that for this one case none of the definitions of approximation (with the possible exception of the minisuperspace packet staying near the track of the microsuperspace packet for a length of time) is valid. The lesson is that a minisuperspace example will not always reflect the behavior of a system in superspace. We need a criterion to tell us when minisuperspace predictions are useful. We will not attempt to provide such a criterion here; there exists a wide range of minisuperspaces that must be studied before we have enough information to be able to discern a pattern in their behavior. We must first cover a sufficiently comprehensive sample of microsuperspaces embedded in minisuperspaces to allow us to construct tests that can tell whether or not a particular minisuperspace gives reasonable physical predictions. The present paper shows that microsuperspace predictions are unreliable when the isotropy class is widened. An entirely different, much more difficult, but also much more interesting situation arises when homogeneous cosmological models of a given isotropy class are embedded in inhomogeneous models of the same isotropy class. This is the logical next step in our program.

## II. EQUATIONS OF MOTION OF THE MODEL AND THEIR CLASSICAL AND QUANTUM SOLUTIONS

We start from the super-Hamiltonian for diagonal Bianchi type-IX universes given by Eq. (1.9). The Taub model corresponds to $y=0$, $p_y=0$ which leads to the reduced potential

$$W(x)=12\pi^2(e^{-8x}-4e^{-2x}) . \tag{2.1}$$

As we have said, the region of the potential that we are interested in for our microsuperspace-minisuperspace comparison will be the "channel" at very large $x$ (at large $\tau$), and we will assume (and later show) that we can restrict our interest to small $y$. In this case the full potential (1.8) reduces to

$$e^{-4\tau}W(x,y)\simeq(24\pi)^2e^{-4\tau}e^{4x}y^2\equiv m^2(\tau,x,y) , \tag{2.2}$$

for the mixmaster model and it vanishes for the Taub model:

$$e^{-4\tau}W(x)\simeq 0\equiv m^2(\tau,x) . \tag{2.3}$$

In our approximation, the potential term is non-negative and thus resembles a (position-dependent) mass term in the Klein-Gordon super-Hamiltonian. To stress this resemblance we have called the potential term $m^2(\tau,x,y)$. The non-negative nature of the potential term is vital for the probabilistic interpretation of the wave function which we adopt in Sec. III.

In the microsuperspace (Taub) sector the Einstein action becomes

$$I=\int(p_x dx+p_\tau d\tau),\quad p_\tau^2=p_x^2 . \tag{2.4}$$

The classical equations of motion are

$$p_\tau=\pm p_x=\text{const},\quad \frac{dx}{d\tau}=\pm 1 . \tag{2.5}$$

If we choose the positive sign the universe point moves toward positive $x$ with velocity $(dx/d\tau)$ equal to one. It has long been known[10] that Eqs. (2.5) correspond to those of a photon moving in one dimension. Thus in $x\tau$ space (see Fig. 1) the trajectory of the universe point lies along the "light cone" given by the solid lines at 45° to the axes in the figure.

Classically there is no loss of generality in going to superspace "light cone" or "double null" coordinates. Note that the "light cone" is defined by the DeWitt supermetric[1] which has the signature $(-+)$ for the $\tau,x$ coordinates. If we define the variables $u$ and $v$ by $u=\tau-x$, $v=\tau+x$ and their conjugate momenta by $p_u=\frac{1}{2}(p_\tau-p_x)$, $p_v=\frac{1}{2}(p_\tau+p_x)$, the action reduces to

$$I=\int(p_u du+p_v dv) , \tag{2.6}$$

where the Hamiltonian constraint is $-4p_up_v=0$. The classical solution we are interested in is the one where $p_u=0$ and $u=u_0$. Classically there is nothing remark-

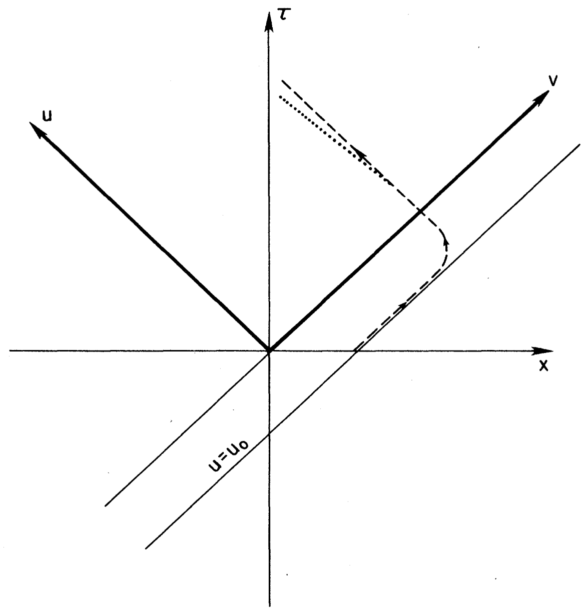


FIG. 1. The $x\tau$ (and $uv$) plane. The darker lines represent the "light cone." The solid line at $u=u_0$ ($u_0$ large and negative) represents a classical microsuperspace (Taub) trajectory, while the dashed line is an approximation to the true classical minisuperspace trajectory. The dotted line represents a better approximation for large positive $u$.

able about either set of coordinates, $x\tau$ or $uv$; the solution is exactly the same, the straight line in Fig. 1.

For the more general case of the mixmaster model (2.2) the new coordinates allow one to find an analytic solution. The small-$y$, large-$x$ case gives us

$$I=\int (p_x dx+p_y dy+p_\tau d\tau) , \tag{2.7}$$

where

$$p_\tau^2=p_x^2+p_y^2+(24\pi)^2 e^{-4(\tau-x)}y^2 . \tag{2.8}$$

The transformation from $\tau,x,y$ to $v,u,y$ gives

$$I=\int (p_u du+p_y dy+p_v dv) \tag{2.9}$$

and the Hamiltonian constraint yields

$$p_v=\frac{1}{4p_u}[p_y^2+(24\pi)^2 e^{-4u}y^2] . \tag{2.10}$$

If we calculate $dp_v/dv$ using the variational equations of motion for $u,y,p_u,p_y$, we find that it is zero. One can now use this fact and Hamilton's equations for $dy/dv$ and $dp_y/dv$ coupled with the fact that $du/dv=p_v/p_u$ to write

$$\frac{d^2y}{du^2}=-\frac{1}{p_v^2}(12\pi)^2 e^{-4u}y . \tag{2.11}$$

This is readily soluble, giving Chitre's[11] solution for $y$,

$$y=Z_0\left[\frac{6\pi}{p_v}e^{-2u}\right] , \tag{2.12}$$

where $Z_0$ is any zero-order Bessel function.

What seems not to have been noticed is that this form for $y$ allows one to integrate the equation for $u(v)$ directly (at least in implicit form), since

$$\frac{dv}{du}=\frac{1}{4p_v^2}[p_y^2+(24\pi)^2 e^{-4u}y^2] . \tag{2.13}$$

Substituting $p_y=2p_v(dy/du)$ from the variational equations along with the expression for $du/dv$, and the explicit expression for $y$, one finds

$$\frac{dv}{du}=\left[\frac{12\pi}{p_v^2}\right]e^{-4u}\left[Z_1^2\left[\frac{6\pi}{p_v}e^{-2u}\right]+Z_0^2\left[\frac{6\pi}{p_v}e^{-2u}\right]\right] . \tag{2.14}$$

The integrals necessary to solve this equation by quadratures are tabulated and

$$v=c_0-\left[\frac{6\pi}{p_v}\right]^2 e^{-4u}\left[2Z_1^2\left[\frac{6\pi}{p_v}e^{-2u}\right]+Z_0^2\left[\frac{6\pi}{p_v}e^{-2u}\right]-Z_0\left[\frac{6\pi}{p_v}e^{-2u}\right]Z_2\left[\frac{6\pi}{p_v}e^{-2u}\right]\right] , \tag{2.15}$$

where $c_0$ is a constant and $Z_m$ an $m$th-order Bessel function. Note that the above analysis seems somewhat clumsy, since one could take $u$ as time instead of $v$ and Eqs. (2.14) and (2.15) would follow directly. Unfortunately, this time choice does not allow the microsuperspace example as a special case, a problem that will be more vexing when we consider the quantum solutions.

The behavior of the universe point as given by (2.15) is best seen by considering the asymptotic form of the Bessel functions. From Fig. 1 note that if the universe point enters the channel at large $x$ and with $\tau$ not too large, $u$ is large and negative, so we are interested in the large argument asymptotic form for $Z_m$. If we take $Z_m=\tilde{A}(\cos\rho J_m+\sin\rho N_m)$, we find that the equation for $y$ reduces to

$$y=\frac{\tilde{A}\sqrt{p_v}}{\sqrt{3\pi}}e^{u}\cos\left[\frac{6\pi}{p_v}e^{-2u}-\frac{\pi}{4}-\rho\right] , \tag{2.16}$$

while the solution for $v(u)$ becomes

$$v=c_0-\frac{24\tilde{A}^2}{p_v}e^{-2u} , \tag{2.17}$$

and if $u=u_0$ at $v=v_0$ then

$$v=v_0+\frac{24\tilde{A}^2}{p_v}(e^{-2u_0}-e^{-2u}) . \tag{2.18}$$

This approximate solution follows the dashed line in Fig.

1. Notice that even this asymptotic solution has the behavior predicted by Misner[7] in that it represents the universe point moving into the channel (staying close to the $u=u_0$ line) for some time, then stopping (at the point where $dx/d\tau$ blows up; where $dv/du=1$), then moving out of the channel again (along a $v=$const line).

As the universe point moves out of the channel $u$ becomes large and positive, and the Bessel functions give

$$y=-\frac{4\tilde{A}}{\pi}u\sin\rho+\frac{2\tilde{A}}{\pi}\ln\left[\frac{6\pi}{p_v}\right]\sin\rho \tag{2.19}$$

and

$$v=v_0+\frac{24}{p_v}\tilde{A}^2e^{-2u_0}-\frac{16\tilde{A}^2}{\pi^2}u\sin^2\rho\ . \tag{2.20}$$

For small $\rho$ this represents a motion of the universe point following the dotted line in Fig. 1, that is, moving at a very small angle to the $v=$const line while $y$ grows linearly in $u$. Here we can see that our assumption of small $y$ is justified. For small enough $\tilde{A}$ and large negative $u$, $y$ stays small and the universe point never leaves the region where (2.2) is valid until long after it has exited from the channel.

From the behavior of the Bessel functions one can see that these asymptotic solutions give the essential features of the motion of the universe point. The details of the motion will imply at most small oscillations around the dashed and dotted lines in Fig. 1.

The $uv$ coordinates allow us to find an exact solution to the quantum problem. By turning the position variables into multiplication operators $\hat{x}$, $\hat{y}$, and $\hat{\tau}$ and the conjugate momenta into differentiation operators, $\hat{p}_x=-i\partial_x$, $\hat{p}_y=-i\partial_y$, and $\hat{p}_\tau=-i\partial_\tau$, the super-Hamiltonian can be interpreted as the operator

$$\hat{H}=-\hat{p}_\tau^{\,2}+\hat{p}_x^2+\hat{p}_y^{\,2}+m^2(\hat{\tau},\hat{x},\hat{y})\ , \tag{2.21}$$

and the constraint $H=0$ imposed as a restriction $\hat{H}\Psi=0$ on the wave function of the Universe. This procedure yields the Wheeler-DeWitt equation

$$-\hat{H}\Psi=-\frac{\partial^2\Psi}{\partial\tau^2}+\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-m^2(\tau,x,y)\Psi=0\ . \tag{2.22}$$

As in the classical case we are interested in solutions for large $x$ and $\tau$ near $y=0$. Equation (2.22) becomes

$$-\frac{\partial^2\Psi}{\partial\tau^2}+\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-(24\pi)^2e^{-4(\tau-x)}y^2\Psi=0\ . \tag{2.23}$$

The microsuperspace sector occurs when one puts $p_y=y=0$ before quantization and the Wheeler-DeWitt equation simplifies to

$$-\frac{\partial^2\Psi}{\partial\tau^2}+\frac{\partial^2\Psi}{\partial x^2}=0\ . \tag{2.24}$$

At this point we will go to the double-null coordinates $uv$ of the classical solution. One can either make the transformation $u=\tau-x$, $v=\tau+x$ as a simple change of variables in the Wheeler-DeWitt equation, or quantize directly the super-Hamiltonian constraint written in the new coordinates $u,v,y$. Either procedure leads to the *same* Wheeler-DeWitt equation, Eq. (2.23) assuming the form

$$-4\frac{\partial^2\Psi}{\partial u\,\partial v}=-\frac{\partial^2\Psi}{\partial y^2}+(24\pi)^2e^{-4u}y^2\Psi \tag{2.25}$$

and Eq. (2.24) the form

$$-4\frac{\partial^2\Psi}{\partial u\,\partial v}=0\ , \tag{2.26}$$

as one would expect for "double null" coordinates in the minisuperspace (and microsuperspace) metrics.

Equation (2.26) has the obvious general solution

$$\Psi=F(u)+G(v)\ , \tag{2.27}$$

where $F$ and $G$ are arbitrary functions of $u$ and $v$, respectively. While Eq. (2.25) is not easy to solve, a solution can be found by means of the method of Salusti and Zirilli[12] that was used by Berger[13] to solve the equations of the Gowdy model.[14]

To begin to solve (2.25) notice that the super-Hamiltonian is independent of $v$ so we may define an "energy operator" $\hat{E}=i\partial_v$. Because $\hat{E}$ commutes with $\hat{H}$ we can find solutions of the Wheeler-DeWitt equation (2.25) which are eigenstates of $\hat{E}$:

$$\Psi_E=\phi_E(u,y)e^{-iEv}\ ; \tag{2.28}$$

and the $\phi_E$ obey the equation

$$4iE\frac{\partial\phi_E}{\partial u}=-\frac{\partial^2\phi_E}{\partial y^2}+(24\pi)^2e^{-4u}y^2\phi_E\ . \tag{2.29}$$

There is a family of solutions to this equation that can be constructed from a "base state" solution by the repeated application of a $u$-dependent raising operator (see Berger[13]). For our purposes we will need only the "base state," although we will make reference to the "higher" states in our discussion of the probability density for the Universe without displaying them explicitly.

The "base state" is found by taking the ansatz

$$\phi_E=e^{-C(u)}e^{-\lambda A(u)y^2}\ , \tag{2.30}$$

where $\lambda$ is a constant. Inserting this form into (2.29) one finds that only terms independent of $y$ and terms proportional to $y^2$ appear. Equating these separately we obtain (an overdot represents $d/du$)

$$-4iE\lambda\dot{A}=-4\lambda^2A^2+(24\pi)^2e^{-4u}\ , \tag{2.31a}$$

$$-iE\dot{C}=\tfrac{1}{2}\lambda A\ . \tag{2.31b}$$

Equation (2.31a) is a Riccati equation and the usual procedure of taking $A\equiv\dot{U}/U$ leads to an equation where all nonlinear terms can be eliminated by taking $\lambda=-iE$. The final equation for $U$ is

$$4E\ddot{U}+(24\pi)^2e^{-4u}U=0\ , \tag{2.32}$$

which has as its solutions $U=Z_0[(6\pi/E)e^{-2u}]$, $Z_0$ any zero-order Bessel function. We will show in Sec. IV that the solution that most closely represents the classical solution given above has

$$U = H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]. \tag{2.33}$$

The function $C$ can now be found by quadratures to be

$$C = \ln\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}. \tag{2.34}$$

The final form for $\phi_E$ is

$$\phi_E = \frac{1}{\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}}\exp(iE\dot{U}/Uy^2). \tag{2.35}$$

Wave packets can be built up out of these $\phi_E$ that have the form

$$\Psi(u,y,v) = \int \frac{e^{-iEv}f(E)}{\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}}\exp[-\omega^2(u)y^2], \tag{2.36}$$

where

$$-\omega^2(u) \equiv iE(\dot{U}/U)$$
$$= 12\pi i e^{-2u}H_1^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{-1},$$

where we have deliberately left the range of integration over $E$ unspecified since it is an important point that will be covered in the next section. In that section we will discuss the construction of a probability density for these models.

## III. PROBABILISTIC INTERPRETATION OF COSMOLOGICAL MODELS

It is clear that the wave function (2.27) of the microsuperspace model as $v \to \infty$ widely differs from the wave function (2.33), (2.35), and (2.36) of the minisuperspace model even if they develop from closely corresponding initial data. (We shall elaborate on this point in Sec. IV.) This indicates that the microsuperspace results do not approximate the minisuperspace quantum mechanics. To see how badly the physical predictions based on the microsuperspace calculation differ from the minisuperspace ones, we must relate the two wave functions to observable quantities.

There is no generally accepted interpretation of the wave functional of the three-geometry or of its minisuperspace truncation in terms of idealized measurements. The Wheeler-DeWitt equation resembles the Klein-Gordon equation with a dynamical (and indefinite) mass term. Whatever internal geometric variable is chosen to play the role of time, the modulus $|\Psi|^2$ of the wave function does not yield a conserved probability in the remaining variables. When one attempts to treat $|\Psi|^2$ as a probability density for the three-geometry to have a definite value at a given instant of an external time (analogous to the proper time of relativistic particle mechanics) one must violate the super-Hamiltonian constraint. We thus feel that those interpretations of the state function $\Psi$ which take $|\Psi|^2$ as the probability density[15] are conceptually untenable.

The Klein-Gordon systems always possess a conserved current. The time component of this current can unfortunately become negative and thus cannot serve as a probability density. The construction of a Hilbert space of states with a positive-definite inner product can be consistently carried out only for stationary Klein-Gordon systems with a positive mass term.[16] General relativity is not a stationary system in superspace[17] and the potential term in its super-Hamiltonian is not positive. It is thus not known how to turn the space of solutions of the Wheeler-DeWitt equation into a Hilbert space endowed with a positive-definite inner product. This led to suggestions that a "third quantization" of the metric field is needed before quantum gravity can be given a probabilistic interpretation.[18] Alternatively, proposals were made that quantum cosmology has a probabilistic interpretation only when the Universe is in a quasiclassical state.[19] Neither of these alternatives is suitable for the interpretation of the state functions of a second-quantized theory describing situations close to the cosmological singularity which are far from being quasiclassical. This is exactly the case of the solutions (2.27) and (2.33), (2.35), and (2.36) which we presented in the preceding section.

We shall base our interpretation of these solutions on the fact that the mixmaster universe which is running into one of the corners of the triangular potential well is passing through the regime in which the super-Hamiltonian *is* stationary and *has* a non-negative potential term. (This is true also for the Taub universe.) In such situations, there exists a one-system Hilbert space on which one can define appropriate position operators and construct a non-negative probability density for the system to be localized in a given configuration. This construction is a generalization of the procedures leading to the Newton-Wigner position operators and the corresponding probability density for a free relativistic particle.[20] The imposition of the Hilbert space structure on the space of solutions of the Klein-Gordon equation on stationary backgrounds was given by Magnon-Ashtekar,[21] Ashtekar and Magnon,[22] and Kay[23] (see also Kay and Wald[24]). Its connection with the Schrödinger evolution equation and the problem of time slicing is discussed in Kuchař.[16] The general construction of the position operators and probability densities for arbitrary stationary Klein-Gordon systems (with positive mass potential term) will be discussed elsewhere.[25] We shall limit our present discussion strictly to the kind of questions (and the type of systems) posed by our two models. For them, the (super)metric background is *flat*, the super-Hamiltonian is stationary with respect to a *null* Killing vector field, the explicit introduction of the position operators will be circumvented, and the positive probability density will be introduced only on a privileged foliation by null hypersurfaces.

To start our discussion, note that the super-Hamiltonian (1.9) has the form

$$H = -4p_u p_v + p_y^2 + m^2(u,y) \tag{3.1}$$

[for the Taub universe, $p_y=0$ and $m=m(u)$]. The mass term [cf. Eqs. (2.2) and (2.3)]

$$m^2=\begin{cases}(24\pi)^2 e^{-4u}y^2 & \text{for the mixmaster universe,}\\ 0 & \text{for the Taub universe,}\end{cases} \tag{3.2}$$

does not depend on the null coordinate $v$. In geometric language, the null Killing vector $\mathbf{v}=\partial_v$ of the supermetric leaves the mass term unchanged, $\partial_v m=0$. In canonical language, the "energy" variable

$$E\equiv -v^A p_A=-p_v \tag{3.3}$$

has vanishing Poisson bracket with the super-Hamiltonian $H$:

$$\{H,E\}=0\ . \tag{3.4}$$

We quantize the system by turning the coordinates $x^A=(u,v,y)$ into multiplication operators, the momenta $p_A=(p_u,p_v,p_y)$ into differentiation operators $\hat{p}_A=-i\partial_A$, and $H$ and $E$ into the operators

$$\hat{H}=4\partial_u\partial_v-\partial_y^2+m^2(u,y),\quad \hat{E}=i\partial_v\ . \tag{3.5}$$

The classical equation (3.4) passes thereby unchanged into quantum mechanics:

$$\frac{1}{i}[\hat{H},\hat{E}]=0\ . \tag{3.6}$$

The operators $\hat{H}$ and $\hat{E}$ thus commute and hence possess a common system of eigenfunctions. Let the space $\mathcal{F}^+$ be spanned by the eigenfunctions of $\hat{E}$ corresponding to positive eigenvalues $E$:

$$\hat{E}\Psi_E=E\Psi_E,\quad E>0\ . \tag{3.7}$$

Such eigenfunctions have the form (2.28):

$$\Psi_E=\phi_E(u,y)e^{-iEv}\ . \tag{3.8}$$

Those eigenfunctions from $\mathcal{F}^+$ which simultaneously satisfy the constraint

$$\hat{H}\Psi=0 \tag{3.9}$$

span the physical space $\mathcal{F}_0^+$. It is only this space which is going to be endowed with a Hilbert-space structure.

The constraint (3.9) leads to an eigenproblem for $\phi_E$,

$$\hat{p}_u\phi_E=-\frac{1}{4E}\widehat{h^2}\phi_E\ , \tag{3.10}$$

where $\widehat{h^2}$ is the operator

$$\widehat{h^2}=\hat{p}_y^{\,2}+m^2(u,y)=-\partial_y^2+m^2(u,y)\ . \tag{3.11}$$

To simplify the argument, assume that the mass term $m^2(u,y)$ by such that the eigenfunctions $\phi_E$ are integrable with respect to an auxiliary inner product

$$(\phi_1|\phi_2)\equiv\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dy\ \phi_1^*\phi_2 \tag{3.12}$$

and thus belong to a discrete spectrum $E_n$ of energies. [This assumption is not satisfied for our models, Eq. (3.2); when applying our results below, we must make the appropriate modifications to the continuous spectrum.] Note that the operator $\widehat{h^2}$ is positive definite and Hermitian with respect to the auxiliary inner product (3.12).

The energy levels are, of course, degenerate. We shall span the space of solutions to the energy level $E_n$ by a countable basis $\phi_{nd}$ of functions. (We shall discuss their normalization and orthogonality later.)

From two solutions $\Psi_1$ and $\Psi_2$ of the constraint equation (3.9) we can construct the current

$$J_{12}^A\equiv\frac{1}{2i}G^{AB}(\Psi_1^*\overleftrightarrow{\partial}_B\Psi_2)\ ; \tag{3.13}$$

here, $G^{AB}$ is the DeWitt supermetric identified from the kinetic term of the super-Hamiltonian. This current satisfies the continuity equation

$$J_{12,A}^A=0 \tag{3.14}$$

which implies that the integral

$$\langle\Psi_1|\Psi_2\rangle_\Sigma\equiv-\int_\Sigma d\Sigma_A J_{12}^A \tag{3.15}$$

has the same value on every (spacelike) hypersurface $\Sigma$. Here,

$$d\Sigma_A\equiv\epsilon_{AA_1A_2}d_{(1)}X^{A_1}d_{(2)}X^{A_2} \tag{3.16}$$

is the hypersurface area determined by the Levi-Civita pseudotensor $\epsilon_{A_1A_2A_3}$.

Using Eq. (3.15) as a starting point, one can introduce the probability density for the Universe to be localized here or there on an arbitrary fixed spacelike hypersurface $\Sigma$. The actual calculations are, however, much simpler when carried out along a one-parameter family of *null* hypersurfaces $v=$const. The null coordinate $v$ here plays the role of intrinsic time. Let us see what happens when we evaluate the integral (3.15) on a null hypersurface $v=$const.

We parametrize the hypersurface $v=$const by the coordinates $x^a=(u,y)$. The area element reduces in this parametrization to

$$d\Sigma_A=(0,-\tfrac{1}{2}du\ dy,0) \tag{3.17}$$

and hence

$$\langle\Psi_1|\Psi_2\rangle=-(\Psi_1|\hat{p}_u\Psi_2)=\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dy\ i\Psi_1^*\Psi_{2,u}\ . \tag{3.18}$$

We see that the integral (3.15) is the expectation value of $-\hat{p}_u$ with respect to the auxiliary inner product (3.12).

Let us check directly whether $\langle\Psi_1|\Psi_2\rangle_v$ is conserved in $v$ time. By using the continuity equation (3.14), we get

$$\frac{d}{dv}\langle\Psi_1|\Psi_2\rangle_v=-\tfrac{1}{2}\int_{-\infty}^{\infty}dy[J_{12}^u]_{u=-\infty}^{u=\infty}-\tfrac{1}{2}\int_{-\infty}^{\infty}du[J_{12}^y]_{y=-\infty}^{y=\infty}\ . \tag{3.19}$$

This expression vanishes only if the $u$ component $J_{12}^u$ of $J_{12}^A$ falls off sufficiently rapidly at $u\to\pm\infty$ and the $y$ component $J_{12}^y$ falls off sufficiently rapidly at $y\to\pm\infty$. This can be shown to be true for the solutions we will use.

The integral (3.15) or (3.18) is not positive on the space $\mathcal{F}_0$ of solutions to the Wheeler-DeWitt equation (3.9), but it becomes positive when restricted to the space of solutions $\mathcal{F}_0^+$ spanned by the positive-energy eigenfunctions (3.7). Let us first show that the energy eigenfunctions $\Psi_m$ and $\Psi_n$ belonging to different energy levels $E_m \neq E_n$ are mutually orthogonal under the inner product $\langle\ |\ \rangle$ and that the norm of any one of them is positive. Indeed, Eqs. (3.10) and (3.18) imply

$$\langle \Psi_m | \Psi_n \rangle = \frac{1}{4E_n} e^{i(E_m - E_n)v} (\phi_m | \hat{h}^2 \phi_n) . \tag{3.20}$$

Next, from the Hermiticity of $\hat{p}_u$ under the auxiliary inner product and from the eigenvalue equation (3.10) it follows that

$$\left[ \frac{1}{E_m} - \frac{1}{E_n} \right] (\phi_m | \hat{h}^2 \phi_n) = 0 . \tag{3.21}$$

For $E_m \neq E_n$ Eqs. (3.20) and (3.21) lead to the orthogonality condition

$$\langle \Psi_m | \Psi_n \rangle = 0 \quad \text{for } E_m \neq E_n . \tag{3.22}$$

On the other hand, for $E_m = E_n \equiv E$, Eq. (3.20) reads

$$\langle \Psi_E | \Psi_E \rangle = \frac{1}{4E} (\phi_E | \hat{h}^2 \phi_E) . \tag{3.23}$$

This expression is positive because $\hat{h}^2$ is a positive-definite operator under the auxiliary product $(\ |\ )$ and $E > 0$ on $\mathcal{F}_0^+$. This proves the desired properties of the energy eigenfunctions $\Psi_n$.

We can now choose the basis $\phi_{nd}$ in the space of eigenfunctions belonging to a fixed energy level $E_n$ such that it is orthonormal under the inner product (3.23) (this can be achieved by the standard orthonormalization of the original basis). We can then conclude that

$$\langle \Psi_{nd} | \Psi_{n'd'} \rangle = \delta_{dd'} \delta_{nn'} . \tag{3.24}$$

An arbitrary state function $\Psi \in \mathcal{F}_0^+$ from the physical space $\mathcal{F}_0^+$ can be decomposed in the energy basis $\Psi_{nd}$:

$$\Psi = \sum_{nd} c_{nd} \Psi_{nd} . \tag{3.25}$$

Because of Eq. (3.24), the $\langle\ |\ \rangle$ norm of $\Psi$ is

$$\langle \Psi | \Psi \rangle = \sum_{nd} |c_{nd}|^2 . \tag{3.26}$$

This expression is manifestly positive definite. The space $\mathcal{F}_0^+$ can thus be completed in the norm (3.26) to a Hilbert space with the positive-definite inner product (3.18).

The formalism we have developed enables us to introduce the probability density $\rho(u,y;v)$ for the Universe to be found in the cell $du\,dy$ about the point $x^a = (u,y)$ on the (null) hypersurface $v = \text{const}$. Let the state $\Psi(u,y;v)$ be normalized to one in the $\langle\ |\ \rangle$ inner product. By Eq. (3.18) this means that

$$\begin{aligned}\langle \Psi | \Psi \rangle &= -(\Psi | \hat{p}_u \Psi) \\ &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dy \tfrac{1}{2} i (\Psi^* \Psi_{,u} - \Psi^*_{,u} \Psi) = 1 .\end{aligned} \tag{3.27}$$

The integrand in Eq. (3.27) is real, but for a general $\Psi \in \mathcal{F}_0^+$ it may be positive in some regions and negative in others. For this reason it cannot serve as the probability density $\rho(u,y;v)$. However, the integral (3.27) can be rearranged so that its integrand becomes non-negative. For this purpose, decompose $\Psi$ into the energy eigenfunctions, Eq. (3.25). By the eigenproblem equation (3.10) $-\hat{p}_u$ is a positive-definite operator in the space $\mathcal{F}_0^+$ under the auxiliary inner product $(\ |\ )$. Define the square root $\hat{h}$ of the operator $\hat{h}^2$ by spectral analysis with respect to this auxiliary product and introduce $\sqrt{-\hat{p}_u}$ by its action on the basis functions $\Psi_{nd}$ (or $\phi_{nd}$):

$$\sqrt{-\hat{p}_u} \phi_{nd} \equiv \frac{1}{2\sqrt{E_n}} \hat{h} \phi_{nd} . \tag{3.28}$$

The normalization integral (3.27) can then be brought into the form

$$\langle \Psi | \Psi \rangle = (\bar{\Psi} | \bar{\Psi}) \quad \text{with } \bar{\Psi} = \sqrt{-\hat{p}_u} \Psi , \tag{3.29}$$

which yields the desired non-negative probability density

$$\begin{aligned}\rho(u,y;v) &\equiv |\bar{\Psi}(u,y;v)|^2 \\ &= \left| \sum_{nd} c_{nd} \frac{1}{2\sqrt{E_n}} \hat{h} \phi_{nd}(u,y) e^{-iE_n v} \right|^2 .\end{aligned} \tag{3.30}$$

It is this probability density and its evolution in $v$ time which enables us to compare the physical predictions based on the minisuperspace and microsuperspace quantum dynamics with each other.

We see that to extract physical predictions from the wave function of the Universe requires an indirect and rather involved procedure. The probability density (3.30) is a nonlocal functional of the state $\Psi(u,y;v)$. First, one must solve the eigenvalue problem for the $\hat{h}^2$ operator defined by Eq. (3.11) on the Hilbert space with the auxiliary inner product (3.12). Let $h_i^2$ be the eigenvalues of $\hat{h}^2$ and $\chi_{ik}(u,y)$ the corresponding eigenfunctions (the index $k$ takes care of the degeneracy of the $h_i^2$ levels). Second, the state function which one is interpreting must be decomposed into the energy basis, Eq. (3.25). Third, each of the energy eigenfunctions $\phi_{nd}(u,y)$ must in its turn be decomposed into the eigenfunctions $\chi_{ik}(u,y)$ of the $\hat{h}^2$ operator:

$$\phi_{nd}(u,y) = \sum_{ik} b_{ndik} \chi_{ik}(u,y) . \tag{3.31}$$

Only after all these steps are taken can one write down the explicit expression for the probability density:

$$\rho(u,y;v) = \left| \sum_{nd} \sum_{ik} c_{nd} b_{ndik} \frac{h_i}{2\sqrt{E_n}} \chi_{ik}(u,y) e^{-iE_n v} \right|^2 . \tag{3.32}$$

For a general state, the evaluation of the probability density (3.32) and the analysis of its evolution in $v$ may be quite difficult.

Fortunately, our task of interpreting the particular solutions (2.27), and (2.33), (2.35), and (2.36) of the Wheeler-DeWitt equation leads to considerable simplifications. First, these functions are the base states of the Berger ladder, and we thus do not need to superimpose the states with different degeneracy indices $d$. Second, their decomposition into the eigenstates $\chi_{ik}$ of $\hat{h}^2$

need, for the purposes of our discussion, only be carried out approximately. These circumstances make the interpretation of the quantum dynamics of our models based on the probability density (3.30) or (3.31) technically manageable.

Let us now apply the ideas developed above to both the general type-IX models and to the Taub model. For the Taub model the probability is extremely simple. We span the Hilbert space $\mathcal{F}_0^+$ on the positive-energy solutions corresponding to the Killing vector $\partial_v$ and consider a state in this space describing the Universe moving in the positive $x$ direction $[G(v)=0]$. Then

$$F(u)=\int_0^\infty f(k)e^{-iku}dk \ . \tag{3.33}$$

The probability density $\rho(u;v)$ for the Universe to be found at the time $v$ at the point $u$ is given by the expression (3.30), where $\rho(u,;v)=|\overline{\Psi}|^2$ is constructed from

$$\overline{\Psi}=\sqrt{-\hat{p}_u}\,\Psi(u)=\int_0^\infty \sqrt{k}\,f(k)e^{-iku}dk \ . \tag{3.34}$$

As an example one can take $f(k)=e^{-\delta k}e^{iku_0}$, which would give

$$\overline{\Psi}=\frac{\sqrt{\pi}}{2}\frac{1}{[\delta^2+(u-u_0)^2]^{3/4}} \times\exp\left[-\frac{3i}{2}\arctan\left[\frac{u-u_0}{\delta}\right]\right] . \tag{3.35}$$

Normalizing to $\int\overline{\Psi}^*\overline{\Psi}\,du=1$ gives

$$\rho(u;v)=\frac{\delta}{\sqrt{2}[\delta^2+(u-u_0)^2]^{3/2}} , \tag{3.36}$$

which not surprisingly represents a packet sharply peaked around $u=u_0$ that is independent of $v$. The packet moves into the channel to $x=\infty$ without spreading.

The general case is more difficult. As we mentioned in Sec. II we will be interested in one particular solution for $\phi_E$ given there. The family of solutions $\phi_{Ed}$ which span the Hilbert space $\mathcal{F}_0^+$ is made up of the "higher" states mentioned in Sec. II that could be constructed from $\phi_E=\phi_{E0}$ by means of $u$-dependent ladder operators. Here we apply the general analysis to $\phi_{E0}$ itself; i.e., we need not care about the degeneracy index $d$.

The $\hat{h}^2$ operator for our case is

$$\hat{h}^2=-\partial_y^2+(24\pi)^2e^{-4u}y^2 \ . \tag{3.37}$$

Since $u$ is a multiplication operator, its eigenstates are $\delta(u-u')$ and there is a family of eigenstates of $\hat{h}^2$, $\chi_{iu'}$, of the form

$$\chi_{iu'}=\delta(u-u')H_i(\sqrt{24\pi}e^{-u'}y)e^{-12\pi e^{-2u'}y^2} , \tag{3.38}$$

where the $H_i$ are Hermite polynomials. Because $\hat{h}^2\chi_{iu'}=(i+\frac{1}{2})24\pi e^{-2u'}\chi_{iu'}$,

$$\hat{h}\chi_{iu'}=(i+\tfrac{1}{2})^{1/2}\sqrt{24\pi}e^{-u'}\chi_{iu'} \ . \tag{3.39}$$

The continuous label $u'$ plays the role here of the degeneracy index $k$ of the general formalism. These eigenstates are complete, so $\phi_E$ can be expanded in them in the form

$$\phi_E(u,y)=\sum_i\int_{-\infty}^\infty du'[b_{Ei}(u')H_i(\sqrt{24\pi}e^{-u'}y)e^{-12\pi e^{-2u'}y^2}\delta(u-u')] \tag{3.40}$$

and

$$\hat{h}\phi_E=\sum_i\int_{-\infty}^\infty du'(i+\tfrac{1}{2})^{1/2}b_{Ei}(u')(\sqrt{24\pi}e^{-u'})H_i(\sqrt{24\pi}e^{-u'}y)e^{-12\pi e^{-2u'}y^2}\delta(u-u') \ . \tag{3.41}$$

The coefficients $b_{Ei}(u')$ can be found by multiplying $\phi_E$ by $\chi^*_{iu''}$ and integrating over $u$ and $y$. The necessary integrals are tabulated and one finds

$$b_{Ei}(u')=\frac{\sqrt{12\pi}e^{-u'}\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u'}\right]\right]^{-1/2}}{\sqrt{\omega^2(u')+12\pi e^{-2u'}}2^{2l}l!}\left[\frac{-\omega^2(u')}{\omega^2(u')+12\pi e^{-2u'}}\right]^l \tag{3.42}$$

for $i=2l$ and $b_{Ei}(u')=0$ for $i$ odd.

In principle the last thing we need is the normalization constant to produce the normalization of Eq. (3.23) and (3.24), modified to the continuous spectrum of $E$:

$$(\phi_E|\hat{h}^2\phi_{E'})=4E'\delta(E-E') \ . \tag{3.43}$$

For our purposes we will fold this normalization $N(E)$ into the $f(E)$ of Eq. (2.36) that is used to define $\Psi(u,y;v)$, so it is not necessary to go through the calculation needed to give it explicitly. Suffice it to say that if we write

$$\phi_E=\frac{N(E)}{\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}}\exp\left\{12\pi i e^{-2u}\left[H_1^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\Big/H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]y^2\right\} , \tag{3.44}$$

$(\phi_E|\hat{h}^2\phi_{E'})$ gives Eq. (3.43) if we take $N(E)=e^{-i\pi/4}2^{3/4}\pi$.

The probability density $\rho(u,y;v)$ that was defined previously is, for our case [using $\hat{h}\phi_E$ from Eq. (3.41)],

$$\rho=|\Psi|^2=\left|\int_0^{\infty}dE\frac{\tilde{f}(E)e^{-iEv}e^{-2u}\sqrt{2}(12\pi)}{\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}}\left[\sum_{l=0}^{\infty}\frac{(2l+\frac{1}{2})^{1/2}e^{12\pi e^{-2u}y^2}}{\sqrt{\omega^2(u)+12\pi e^{-2u}}2^{2l}l!}H_{2l}(\sqrt{24\pi}e^{-u}y)\left[\frac{-\omega^2(u)}{\omega^2(u)+12\pi e^{-2u}}\right]^l\right]\right|^2, \tag{3.45}$$

where $\tilde{f}(E)$ includes $N(E)/2\sqrt{E}$. In principle (3.45) gives the answer to the probability density for our case, and it could be evaluated by numerical methods for any $\tilde{f}(E)$, but an approximate analysis will give us more insight into the behavior of a wave packet built up around the microsuperspace solution. We will construct such a wave packet in the next section.

## IV. MINISUPERSPACE AND MICROSUPERSPACE BEHAVIOR

A typical microsuperspace wave packet was constructed in Eq. (3.35). As we mentioned in Sec. III this wave packet moves into the channel with no tendency to bounce, and eventually reaches $x=+\infty$ $(v=+\infty$ with the packet still centered around $u=u_0)$. The behavior of the minisuperspace wave packet is more complicated, and its fate is not obvious from the forms of $\phi_E$ and $\rho(u,y;v)$ we have given; we will need to make approximations to see what is happening. Obviously we must look at $\rho$ to get a true picture of the behavior of the probability of finding the Universe in any particular state.

We will begin to consider approximations to $\rho$ by studying the asymptotic forms of $\phi_E$. For small $E$ and large negative $u$ (the approximation most important for our purposes) we need the large-argument form of the Bessel functions that appear in these quantities, and $\phi_E$ takes the extremely simple form

$$\phi_E\approx N(E)(3\pi^2/E)^{1/4}e^{-u/2}\exp\left[\frac{3\pi i}{E}e^{-2u}\right]\times e^{-i\pi/8}e^{-12\pi e^{-2u}y^2}. \tag{4.1}$$

This approximate form of $\phi_E$ in (4.1) shows us why the "base state" solution is useful. It represents (for large negative $u$) a Gaussian strongly centered around $y=0$, an excellent candidate for a quantum model peaked around the microsuperspace sector.

While the form of $\phi_E$ is suggestive, we must see if the probability density for $\phi_E$ has the same behavior. In order to calculate $\rho$ from (3.45) we need the sum

$$S=\frac{e^{-12\pi e^{-2u}y^2}}{\sqrt{\omega^2(u)+12\pi e^{-2u}}}\times\sum_{l=0}^{\infty}\frac{(2l+\frac{1}{2})^{1/2}}{2^{2l}l!}H_{2l}(\sqrt{24\pi}e^{-u}y)\times\left[\frac{-\omega^2(u)}{\omega^2(u)+12\pi e^{-2u}}\right]^l. \tag{4.2}$$

For large negative $u$ we have $\omega\simeq 12\pi e^{-2u}$, which gives

$$S\approx\frac{e^{u}e^{-12\pi e^{-2u}y^2}}{\sqrt{24\pi}}\times\sum_{l=0}^{\infty}\frac{(2l+\frac{1}{2})^{1/2}H_{2l}(\sqrt{24\pi}e^{-u}y)}{2^{2l}l!}\left[\frac{-1}{2}\right]^l. \tag{4.3}$$

What we need to know is the behavior of $S$ as a function of $y$, since the rest of $\rho$ does not contain $y$. Also, the value of $S$ at $y=0$ will be important, since we expect the form of $\rho$ for the wave packet to be that sketched in Fig. 2, with a long "nose" extending into the channel and the cross section at $y=0$ governing the maximum penetration of probability into the channel.

For $y=0$ we find

$$S\approx\frac{e^u}{\sqrt{24\pi}}\sum_{l=0}^{\infty}\frac{(2l+\frac{1}{2})^{1/2}(2l)!}{2^{2l}(l!)^2}\left[\frac{1}{2}\right]^l. \tag{4.4}$$

The sum in (4.4) is a constant and it is only necessary to show that the series converges. In Appendix A we show that

$$\lim_{u\to-\infty}S\approx Qe^u, \tag{4.5}$$

where $Q$ is a small numerical constant introduced there. Thus for large negative $u$ and $\tilde{f}(E)$ of Eq. (3.12) with support only in the region of $E=0$ we obtain

$$\rho\approx\left|\int_0^{\infty}dE\frac{\sqrt{12\pi}Q\tilde{f}(E)e^{-iEv}e^{-u}}{\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}}\right|^2. \tag{4.6}$$

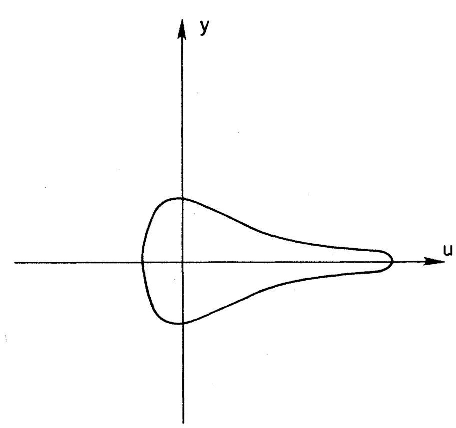


FIG. 2. The approximate form of the wave packet discussed in Sec. III represented by a contour of equal probability density in the $uy$ plane.

Here, of course, we must use the large-argument form of $H_0^{(2)}$ which yields

$$\rho \approx e^{-3u}\left|\int_0^\infty dE\frac{\tilde{f}(E)\pi^{1/2}3^{1/4}\sqrt{12\pi Q}}{E^{1/4}}e^{-iEv}\right.\left.\times\exp\left[\frac{3\pi i}{E}e^{-2u}\right]\right|^2$$
$$=e^{-3u}\left|\int_0^\infty dE\,F(E)e^{-iEv}\exp\left[\frac{3\pi i}{E}e^{-2u}\right]\right|^2 . \quad (4.7)$$

For $y\neq 0$ we must return to $S$. For very small $y$ the sum in $S$ should not differ too much from $S(y=0)$, that is, $S$ should be relatively flat near $y=0$. For very large $y$ we can take the maximum power of $y$ in $H_{2l}(\sqrt{24\pi}e^{-u}y)$ to represent $H_{2l}$, and we find

$$S\approx e^u e^{-12\pi e^{-2u}y^2}\sum_{l=0}^{\infty}\frac{(2l+\frac{1}{2})^{1/2}}{l!}\left[\frac{iE}{24\pi}y^2\right]^l . \quad (4.8)$$

If we could ignore the factor $(2l+\frac{1}{2})^{1/2}$, we would find $S$ to be $e^u\exp(-12\pi e^{-2u}y^2)\exp(iEy^2/24\pi)$. Since $(2l+\frac{1}{2})^{1/2}$ varies slowly compared to $l!$, we can expect only small corrections to $\exp(iEy^2/24\pi)$ to appear in $S$, and the dominant contribution to the envelope of the wave packet will be $\exp(-12\pi e^{-2u}y^2)$. As a function of $u$ and $y$ then $\rho$ should behave (up to small corrections) like

$$\rho\sim e^{-24\pi e^{-2u}y^2}e^{-3u}\left|\int_0^\infty dE\,F(E)e^{-iEv}\exp\left[\frac{3\pi i}{E}e^{-2u}\right]\right|^2 . \quad (4.9)$$

We will use specific functions $F(E)$ in (4.9) to construct approximate wave packets which for $y=0$ will be peaked around $u=u_0$ (where $u_0$, as before, is large and negative) at $v=v_0$ and fall off rapidly for all other values of $u$. If one considers probability densities which behave like (4.4), one can see that the quantity inside the absolute value signs will fall off rapidly as we move away from $u=u_0$, while the factor $\exp(-24\pi e^{-2u}y^2)$ will become narrower in $y$ as we go toward more negative $u$ and will expand as we go to less negative (and positive) values of $u$. From this we can see that a $\rho$=const contour at $v=v_0$ in the $uy$ plane will have the long narrow "nose" in the direction of large negative $u$ (large positive $x$) that is shown in Fig. 2 and will have a fatter tail as we go toward positive $u$ (negative $x$).

The major difficulty with the expression (4.9) is that it is only valid for large negative $u$. For positive $u$ we must return to the full expression for $\rho$. As before, we will consider $S$ for $y=0$ first. For large positive $u$ (large enough so that $e^{-2u}/E$ is small), $\omega^2(u)\to -2iE/[\ln(6\pi/E)-2u]$ and

$$S(y=0)\approx\sum_{l=0}^{\infty}\frac{(2l+\frac{1}{2})^{1/2}\sqrt{24\pi}(2l)!}{\omega(l!)^2 2^{2l}}\left[\frac{\omega^2}{\omega^2+12\pi e^{-2u}}\right]^l . \quad (4.10)$$

Since this series converges slowly, we must find a way to sum it. In Appendix A we show that one can approximate $S$ by

$$S(y=0)\approx L_0\omega e^{2u} , \quad (4.11)$$

where $L_0$ is a constant added there to absorb any defects of the approximation method. From the definition of $\rho$ in (3.12) and of $F(E)$ implicit in (4.7), we find that for large positive $u$

$$\rho\approx L_0^2\left|\int_0^\infty dE\frac{E^{1/4}F(E)\omega e^{-iEv}}{\left[H_0^{(2)}\left[\frac{6\pi}{E}e^{-2u}\right]\right]^{1/2}}\right|^2$$
$$=L_0^2\left|\int_0^\infty dE\frac{E^{3/4}\sqrt{\pi}F(E)e^{-iEv}}{\ln\left[\frac{6\pi}{E}\right]-2u}\right|^2 , \quad (4.12)$$

where we have used the large-positive-$u$ forms for $\omega$ and $H_0^{(2)}[(6\pi/E)e^{-2u}]$. Here $\rho$ falls off for large positive $u$, which means that a $\rho$=const contour of a wave packet closes somewhere at large positive $u$.

We will now construct a representative wave packet and show that its behavior is very different from that of the microsuperspace packet. We will take a packet peaked around $u=u_0$ ($u_0$ large and negative) at $v=v_0$, and investigate its behavior for $y=0$ and large negative $u$ as $v$ advances. We will then show that this is the region of interest. In (4.9) we take

$$F(E)=\begin{cases}\tilde{N}e^{-iEv_0}\exp\left[-\frac{3\pi i}{E}e^{-2u_0}\right], & E_0-\delta\le E\le E_0+\delta,\\ 0 \text{ otherwise,}\end{cases} \quad (4.13)$$

where $E_0$ is small and $\delta<E_0$. Expanding $1/E$ about $E_0$ and keeping the first term, we can construct the wave packet

$$\rho(y=0)=N^2e^{-3u}\frac{\sin^2\left[\frac{3\pi}{E_0^2}(e^{-2u}-e^{-2u_0})+v-v_0\right]}{\left[\frac{3\pi}{E_0^2}(e^{-2u}-e^{-2u_0})+v-v_0\right]^2} . \quad (4.14)$$

This form shows the necessity of using $H^{(2)}$ as the solution to (2.32), since it gives the proper classical trajectory. We can see that the position of the peak of this packet follows roughly the trajectory given by (2.18). In fact, if one calculates

$$\langle\hat{y}^2\rangle=\frac{\int_{-\infty}^{\infty}y^2e^{-24\pi e^{-2u}y^2}dy}{\int_{-\infty}^{\infty}e^{-24\pi e^{-2u}y^2}dy} \quad (4.15)$$

and equates $\sqrt{\langle\hat{y}^2\rangle}$ with the rms value of $y$ found from (2.16) one can obtain a value for $\tilde{A}$ which, when plugged into (2.18) (with $p_v=E_0$), gives exactly the position of the center of the packet as a function of $v$.

We may take the classical trajectories shown by the $u=u_0$ line and the dashed line in Fig. 1 as representing in sketch the motion of the microsuperspace wave packet and the large-negative-$u$ peak of the minisuperspace packet, respectively. Notice that for small $E_0$ the minisuperspace packet bounces (when $du/dv=-1$) at a value of $u$ that is still large and negative. We can see that the behavior of the microsuperspace and minisuperspace packets is qualitatively different, since the minisuperspace packet will always bounce at some value of $x$ and exit from the $y=0$ channel. Notice that the smaller we make $E_0$ the longer the minisuperspace packet will stay near the $u=u_0$ line before its inevitable bounce; this is the only sense in which the microsuperspace behavior of our model stays "close" to its minisuperspace evolution.

In the next section we will use these wave packets to discuss various ideas presented in the Introduction about the meaning of minisuperspace quantization.

## V. CONCLUSIONS AND DISCUSSION

Given the two wave packets (3.35) and (4.14), we can see that their overall behavior is not at all similar. Is there any way in which we can say that the microsuperspace packet tells us something about the behavior of the full minisuperspace?

The first possibility of approximation mentioned in the Introduction was that wave packets peaked around the minisuperspace sector and initially moving along the minisuperspace trajectory would tend to follow the minisuperspace behavior at all times. The present set of solutions is a counterexample to such a scenario. Refining our notion of approximation through the density operator approach does not help. The microsuperspace position variables necessarily follow different statistics when their distribution is derived from the minisuperspace wave function than when they are determined from the microsuperspace state. This does not show that the corresponding concept of approximation is wrong; it simply tells us that the conditions under which it is valid are not satisfied in the models studied, and hence we have no right to expect that they would be satisfied in general. There are, however, situations in ordinary quantum mechanics where such an approximation works,[8] and there may be situations in gravity where it is equally appropriate.

The sense of approximation defined by Misner[9] of energy flowing into the minisuperspace mode is more difficult to apply to our particular microsuperspace-minisuperspace example, since this model is not couched in simple mode-sum terms. This sense of approximation makes the most sense if the system has a Hamiltonian (such as the ADM Hamiltonian) that is in the form of a sum over terms, each of which can be interpreted as a partial "Hamiltonian" corresponding to one mode, and there is some mechanism that causes the expectation value of the "Hamiltonian" corresponding to the minisuperspace mode to become large, while the "energy" of the other modes decreases. The mechanism that causes the change could be an interaction term or (perhaps more likely) a Hamiltonian such as that for a two-dimensional harmonic oscillator where the frequencies of the partial oscillators depend on time, each time dependence being different. That is, $\hat{H}=\frac{1}{2}[\hat{p}_x^{\,2}+\omega_1^2(t)\hat{x}^{\,2}+\hat{p}_y^{\,2}+\omega_2^2(t)\hat{y}^{\,2}]\equiv\hat{H}_x+\hat{H}_y$, and the "energy" in each direction is defined as $\langle\hat{H}_x\rangle(t)$ and $\langle\hat{H}_y\rangle(t)$. If we consider the Taub model in the mixmaster model, we can see the problem that will arise in general relativity. The super-Hamiltonian cannot be interpreted as an energy since it is zero. The ADM Hamiltonian depends directly on the choice of internal time used. If we use $\tau$ as time, $-p_\tau$ is the ADM Hamiltonian and

$$H_{\rm ADM}(\tau)\simeq[p_x^2+p_y^2+(24\pi)^2e^{-4(x-\tau)}y^2]^{1/2} \tag{5.1}$$

for small $y$. Notice that this form is the relativistic analogue of the $\hat{H}_x+\hat{H}_y$ given above, but the square root form and the fact that the $y$ frequency depends on $x$ as well as $\tau$ make it more difficult to apply the simple idea of distinguishing partial energies. If one were to choose $v$ as time, the operator $\hat{E}$ of Sec. II is the operator form of the ADM Hamiltonian, and

$$H_{\rm ADM}(v)\simeq\frac{-1}{4p_u}[p_y^2+(24\pi)^2e^{4u}y^2]\ , \tag{5.2}$$

which has the disadvantage of being even farther from a simple sum of two "energies" and of being "time independent," which will create more difficulties in applying the idea of energy flowing into the minisuperspace (here microsuperspace) mode. As we have said, the fact that this model is not of the form of a sum over space-dependent modes makes it unsuitable as a testing ground for the energy conjecture, but some of the problems we have mentioned in relation to it will appear in other models that will be the subject of future work.

The last conjecture, that the microsuperspace gives some manner of qualitative idea about the minisuperspace, seems, in light of the totally divergent behavior of the wave packets, not to be applicable to this model.

Perhaps the only notion of approximation that works for our models is that of the minisuperspace staying near the microsuperspace for some time. Here we can use the results of Appendix B, where we show that Eq. (2.25) has the true form

$$-4\frac{\partial^2\Psi}{\partial u\,\partial v}=-\frac{\partial^2\Psi}{\partial y^2}+(24\pi)^2\left[\frac{R_0}{R_P}\right]^4e^{-4u}y^2\Psi\ . \tag{5.3}$$

where $R_0$ is the scale length of the Universe (the radius of the Universe when $u=0$) and $R_P$ is the Planck length. The only change in the solution of Sec. III would be to replace $e^{-u}$ by $(R_0/R_P)e^{-u}$. From Eq. (4.14) we can see that if we consider $v$ to be "time" then the peak of the wave packet is where

$$v-v_0=\frac{3\pi}{E_0^2}\left[\frac{R_0}{R_P}\right]^2(e^{-2u_0}-e^{-2u}) \tag{5.4}$$

or

$$u(v)=-\tfrac{1}{2}\ln\left[e^{-2u_0}-\left[\frac{R_P}{R_0}\right]^2\frac{E_0^2}{3\pi}(v-v_0)\right]\ . \tag{5.5}$$

Very roughly this represents a path that stays near $u=u_0$

until $v - v_0 = (3\pi/E_0^2)(R_0/R_P)^2 e^{-2u_0}$, at which time the distance from $u = u_0$ blows up. The value (5.4) of $v - v_0$ is thus a good estimate of the "time" interval that the microsuperspace solution gives a useful prediction of the behavior of the minisuperspace. If we call the value of $v$ where the bounce occurs $v_B$, and assume that $u \approx u_0$ until that time, we can use the fact that $u_0 = \tau_0 - x_0$, $v_0 = \tau_0 + x_0$, $v_B = \tau_B + x_B$ to show that

$$\tau_B - \tau_0 = \frac{3\pi}{2E_0^2}\left[\frac{R_0}{R_P}\right]^2 e^{-2u_0} . \tag{5.6}$$

Since $R_0 e^{-\tau}$ is proportional to the cube root of the volume of a $t$ = const hypersurface, we may call it the "radius of the Universe," and $\tau_B - \tau_0$ represents the number of $e$-foldings of the radius between the initial moment and the bounce. If we take the scale length $R_0$ to be $R_P$, then

$$\tau_B - \tau_0 = \frac{3\pi}{3E_0^2} e^{-2u_0} , \tag{5.7}$$

and remembering that $u_0$ had to be large and negative for our approximation to be valid and $E_0$ was assumed to be small, $\tau_B - \tau_0$ can be a respectable number. Notice that if the radius of the Universe today were a reasonable $10^{28}$ cm, that the total change in $\tau$ from the moment when $R = R_P$ until now would be $\sim 140$. One can see that for moderate values of $u_0$ and $E_0$ the microsuperspace approximation can easily be valid for a number of $e$-foldings that far exceeds this number.

The simple model we have chosen is not meant to give a realistic picture of the manner in which a typical quantum minisuperspace would be embedded in superspace, but to serve as an easily soluble example that can shed some light on the process of constructing such embeddings, since it *is* a true gravitational problem, and one can expect many of the problems encountered here to be found in more realistic minisuperspace-microsuperspace combinations. In the future, as we mentioned in the Introduction, we plan to investigate increasingly complicated and more realistic models.

## ACKNOWLEDGMENTS

One of us (M.R.) would like to thank the Physics Department of the University of Utah for hospitality throughout the 1987–1988 academic year. This work was partially supported by NSF Grant No. PHY 8503653 to the University of Utah.

## APPENDIX A

In Sec. IV it was necessary to sum the series $S$ from Eq. (3.45) in different approximations. The first of these only involved showing that

$$\sum_{l=0}^{\infty} \frac{(2l+\frac{1}{2})^{1/2}}{2^{2l} l!^2}\left[\frac{1}{2}\right]^l$$

converges, and the second was to sum the slowly convergent series for large positive $\omega$. Both of these problems can be solved by considering the exact sum

$$e^{-(a+1/2)y^2} = \frac{1}{\sqrt{2(a+1)}} \sum_{l=0}^{\infty} \frac{1}{l! 2^{2l}} \left[\frac{1}{2(a+1)} - 1\right] \times e^{-y^2/2} H_{2l}(y) , \tag{A1}$$

which comes from the tabulated integral for $\int_{-\infty}^{\infty} e^{-by^2} H_{2l}(y) dy$. Taking $y = 0$ in (A1) one finds

$$\sqrt{2(a+1)} = \sum_{l=0}^{\infty} \frac{(2l)!}{(l!)^2 2^{2l}} \left[1 - \frac{1}{2(a+1)}\right]^l . \tag{A2}$$

We want to compare this sum with the following approximation. Replace the factorials by Stirling's formula and the sum by an integral. One arrives at

$$\int_0^{\infty} \frac{1}{\sqrt{\pi}\sqrt{l}} \exp\left\{ l \left[ \ln \left[ 1 - \frac{1}{2(a+1)} \right] \right] \right\} dl . \tag{A3}$$

Now consider two possibilities, $a = 0$ and $a$ large. For $a = 0$ we find

$$\sqrt{2} = \sum_{l=0}^{\infty} \frac{(2l)!}{(l!)^2 2^{2l}} \left[\frac{1}{2}\right]^l , \tag{A4}$$

and the integral gives $1/\sqrt{\ln 2}$, only about a 20% error. Applying the same procedure to (4.4) we see that

$$S = e^u \sum_{l=0}^{\infty} \frac{\sqrt{2l+1/2}(2l)!}{2^{2l}(l!)^2} \left[\frac{1}{2}\right]^l \simeq e^u \int_0^{\infty} \sqrt{2/\pi} e^{-l \ln 2} = e^u \sqrt{2/\pi} \frac{1}{\ln 2} , \tag{A5}$$

so $S$ has the form given in (4.5) with $Q$ a number close to the numerical factor in the last expression in (A5).

For large $a$ (A3) reduces to

$$\int_0^{\infty} \frac{1}{\sqrt{\pi}\sqrt{l}} \exp(-l/2a) dl , \tag{A6}$$

which is exactly $\sqrt{2a}$, the large-$a$ value of the left-hand side of (A2). Now apply the operator $(-\partial_y^2 + y^2)$ to both sides of (A1) and take $y = 0$ to get

$$(a+\tfrac{1}{2})\sqrt{2(a+1)} = \sum_{l=0}^{\infty} \frac{(2l+\frac{1}{2})(2l)!}{(l!)^2 2^{2l}} \left[1 - \frac{1}{2(a+1)}\right]^l . \tag{A7}$$

Again taking the large-$l$ values for the terms of the sum and converting to an integral, the sum becomes

$$\int_0^{\infty} \frac{2l}{\sqrt{\pi}\sqrt{l}} e^{-l/2a} dl = 2\sqrt{2} a^{3/2} , \tag{A8}$$

which is twice the exact value for large $a$. Applying the same idea to

$$\sum_{l=0}^{\infty} \sqrt{2l+\tfrac{1}{2}} \frac{(2l)!}{(l!)^2 2^{2l}} \left[1 - \frac{1}{2(a+1)}\right]^l , \tag{A9}$$

we find that this expression (for large $a$) is approximately equal to

$$\int_0^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}} e^{-l/2a} dl = \frac{2^{3/2}}{\sqrt{\pi}} a . \tag{A10}$$

The sum (4.10) is of this form with $a \sim \omega^2 e^{2u}/24\pi$. If we absorb any inaccuracy in the numerical factors by replacing these factors by a constant $L_0$, we arrive at (4.11).

## APPENDIX B

To put in units explicitly we scale the dimensionless one-forms (1.3) by multiplying them by a factor $R_0$ with dimensions of length. The time coordinate ($x^0 = ct$) can also be scaled by $R_0$. This makes the four-dimensional scalar curvature into a dimensionless quantity multiplied by $1/R_0^2$. Since the factor $1/16\pi$ (1.5) is $c^3/16\pi G$ in usual units, the true action (2.10) is

$$I = \frac{c^3 R_0^2}{G} \int (p_u du + p_y dy + p_v dv) \ .$$

We can now construct "physical coordinates" $p_u^{(p)} \equiv (c^3 R_0 / G) p_u$, $u^{(p)} \equiv R_0 u$, etc. The commutation relations become $[\hat{u}^{(p)}, \hat{p}_u^{(p)}] = \hbar$, which means that the operator $\hat{p}_u$ is to be realized as $-i(G\hbar/R_0^2 c^3)\partial/\partial u = (R_P/R_0)^2 \partial/\partial u$ and similarly for $\hat{p}_v$ and $\hat{p}_y$. Inserting these true relations into (2.29) and multiplying by $(R_0/R_P)^4$ one arrives at Eq. (5.3).


[1]B. DeWitt, Phys. Rev. **160**, 1113 (1967).

[2]C. Misner, in *Magic without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972).

[3]J. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[4]See the reviews, M. Ryan, *Hamiltonian Cosmology* (Springer, Berlin, 1972); M. MacCallum, in *General Relativity. An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[5]The literature is too extensive to mention it all. See the list of papers by J. Halliwell [Institute for Theoretical Physics Report No. NSF-ITP-88-132 (unpublished)]. The seminal works are S. Hawking, Pontif. Accad. Sci. Varia **48**, 563 (1982); J. Hartle and S. Hawking, Phys. Rev. D **28**, 2690 (1983).

[6]A. Taub, Ann. Math. **53**, 472 (1951).

[7]C. Misner, Phys. Rev. Lett. **22**, 1071 (1969).

[8]K. V. Kuchař and M. P. Ryan, in *Gravitational Collapse and Relativity*, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986).

[9]C. Misner, Phys. Rev. **186**, 1328 (1969).

[10]C. Misner, Phys. Rev. **186**, 1319 (1969).

[11]D. Chitre, Phys. Rev. D **6**, 3390 (1972).

[12]E. Salusti and F. Zirilli, Lett. Nuovo Cimento **4**, 999 (1970).

[13]B. Berger, Phys. Rev. D **11**, 2770 (1975).

[14]R. Gowdy, Phys. Rev. Lett. **27**, 826 (1971).

[15]S. W. Hawking, Nucl. Phys. **B239**, 257 (1984); S. W. Hawking and D. N. Page, *ibid.* **B264**, 185 (1986); D. N. Page, Phys. Rev. D **34**, 2267 (1986).

[16]K. V. Kuchař, in Einstein Studies II, edited by J. Stachel (unpublished).

[17]K. V. Kuchař, J. Math. Phys. **22**, 2640 (1981).

[18]See, e.g., S. B. Giddings and A. Strominger, Nucl. Phys. **B321**, 481 (1989), and references therein.

[19]A. Vilenkin, Tufts report, 1988 (unpublished); J. B. Hartle, Phys. Rev. D **37**, 2818 (1988); **38**, 2985 (1988).

[20]T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949); see also S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), for the basic references.

[21]A. Magnon-Ashtekar, thesis, Département de Mathématiques, Pures, Université de Clermont, 1976.

[22]A. Ashtekar and A. Magnon, Proc. R. Soc. London **A346**, 375 (1975).

[23]B. S. Kay, Commun. Math. Phys. **62**, 55 (1978).

[24]B. S. Kay and R. M. Wald, University of Chicago report, 1989 (unpublished).

[25]K. V. Kuchař (in preparation).