Fusions of operators in the minimal and $N=1$ superconformal theories

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Starting from the fusion rules of primary operators, all the fusion coefficients of the minimal conformal theories are explicitly obtained in closed formulas using only the second-order Fuchsian-type differential equations with monodromy and crossing symmetry. These are explicitly demonstrated in the case of conformal theory with a central charge $c=\frac{7}{10}$.

I. INTRODUCTION

All the physics of the conformal theories is encoded into the operator-product expansions (OPE's) of primary operators. In free theories with central charges $\frac{1}{2}$ and 1, their OPE can be explicitly evaluated using the correlation functions of spinors and scalars. For example, the primary operators in the bosonic string theory are the vertex operators which are polynomials of ∂X multiplied by e^{ikX}. The fusion of the two simplest primary operators $e^{ik_1X(z)}$ and $e^{ik_2X(0)}$ is computed from the correlation function

$$\langle X(z)X(0)\rangle=-\ln z \tag{1}$$

as

$$e^{ik_1X(z)}e^{ik_2X(0)}=z^{k_1k_2}\{1+zik_1{}^{\mu}\partial X_{\mu}(0)+\tfrac{1}{2}z^2[ik_1{}^{\mu}ik_2{}^{\nu}\partial X_{\mu}(z)\partial X_{\nu}(0)+k_1{}^{\mu}\partial\partial X_{\mu}(0)]+\cdots\}e^{i(k_1+k_2)X(0)} \tag{2}$$

with analogous expressions in the antiholomorphic parts. Two tachyons couple to higher-spin particles in the string theories. But the OPE for general conformal theories is very difficult to compute. These OPE's can be examined at two different levels. One can study what primary operators are present in the expansion of two primary operators. At this level, one examines the fusion rules

$$\phi_i\phi_j=\sum_k A_{ij}{}^k\phi_k\ . \tag{3}$$

The next step in a more detailed understanding of the OPE requires quite laborious analyses to determine the fusion coefficients:[1]

$$\phi_i(z)\phi_j(0)=\sum_k\sum_{\{k_i\}}C_{ij}{}^k z^{-\Delta_i-\Delta_j+\Delta_k+\sum k_i}\beta_{ij}^{k\{k_i\}}\times L_{-k_1}\cdots L_{-k_n}\phi_k(0)\ . \tag{4}$$

As computed for some conformal field theories in Refs. 1 and 2, the fusion coefficients $C_{ij}{}^k$ are generally complicated expressions involving gamma functions.

Recently, in some elegant papers,[3,4] the role of mapping class groups in understanding conformal theories is emphasized. The modular transformations of characters are intimately connected to the fusion rules in the sense of Eq. (3).

In this paper we obtain a closed formula for fusion rules in the sense of Eq. (3) in all the minimal conformal and superconformal theories. Starting from the characters of primary operators, we calculate the modular transformation of the characters and determine the fusion rules. We show that the fusion coefficients in the sense of Eq. (4) for all the minimal conformal theories can be determined only from the second-order Fuchsian-type differential equations with monodromy and crossing symmetry. In Sec. II we fix the notation by computing the fusion rules of minimal unitary as well as nonunitary conformal theories. Section III discusses the fusion rules of $N=1$ superconformal theories. The operator-product expansion of all the conformal theories in the sense of Eq. (4) is shown in Sec. IV to be calculated using only the second-order Fuchsian differential equation with monodromy and crossing symmetry. The main results of this paper are Eqs. (45), (49)–(51), (53), and (54), which give all the fusion coefficients explicitly in closed formulas. In Sec. V we illustrate the general discussion of Secs. II and IV with a specific example of $c=\frac{7}{10}$ theory.

II. MINIMAL UNITARY AND NONUNITARY CONFORMAL THEORIES

Minimal conformal theories with the central charge[1,5]

$$c=1-\frac{6(p-p')}{pp'} \tag{5}$$

have a finite number of primary operators with conformal dimensions:

$$h_{(r,s)}=h_{(p'-r,p-s)}=\frac{(rp-sp')^2-(p-p')^2}{4pp'}\ ,\quad 1\le r\le p'-1,\ \ 1\le s\le p-1\ . \tag{6}$$

The modular transformation of $\chi_{(r,s)}$ is given as

$$\chi_{(r,s)}\left[-\frac{1}{\tau}\right] = \sum_{1\le r'\le p'-1,1\le s'\le p-1,s'p'\le r'p} S_{(r,s)}{}^{(r',s')}\chi_{(r',s')}(\tau) \quad (7)$$

with

$$S_{(r,s)}{}^{(r',s')}=2\left[\frac{2}{pp'}\right]^{1/2}(-)^{rs'+r's+1}\sin\frac{\pi prr'}{p'}\sin\frac{\pi p'ss'}{p} . \quad (8)$$

S is symmetric and real, satisfying

$$\sum_{r',s'} S_{(r,s)}{}^{(r',s')}S_{(r',s')}{}^{(r'',s'')}=\delta_{r,r''}\delta_{s,s''} . \quad (9)$$

We review the essential formula to determine the fusion rules and the reader is directed to Ref. 3 for details (see also Ref. 5). The operators $\phi_i(b)$ insert the primary operators ϕ_i at both the initial time $t=0$ and final time $t=\tau$, while the trace χ_j is taken over the descendent states of the primary operators ϕ_j. According to the fusion rules of Eq. (3), the trace is taken over the states k as

$$\phi_i(b)\chi_j = A_{ij}{}^k\chi_k . \quad (10)$$

The operators $\phi_i(a)$ insert the primary operators ϕ_i at $\sigma=0$ and $\sigma=\pi$. Since the trace is taken over the descendent states of the primary operators ϕ_j, we have characters χ_j on the right-hand side as

$$\phi_i(a)\chi_j=\lambda_i{}^{(j)}\chi_j . \quad (11)$$

The modular transformation matrix S determines the fusion rules A by

$$A_{ij}{}^k=\sum_n S_j{}^n\lambda_i{}^{(n)}(S_k{}^n)^* , \quad (12)$$

where the condition $A_{i0}{}^k=\delta_i^k$ determines

$$\lambda_i{}^{(n)}=\frac{S_i{}^n}{S_0{}^n} . \quad (13)$$

The similarity transformation of λ by S gives the fusion rules as

$$A_{(r_i,s_i)(r_j,s_j)}{}^{(r_k,s_k)}=\sum_{r,s} S_{(r_j,s_j)}{}^{(r,s)}\lambda_{(r_i,s_i)}{}^{(r,s)}(S_{(r_k,s_k)}{}^{(r,s)})^* . \quad (14)$$

The fusion rule A is symmetric in i and j. The summation over r and s gives

$$\phi_{(r_i,s_i)}\phi_{(r_j,s_j)}=\sum_{r=|r_i-r_j|+1,r_i+r_j-r=1\,\mathrm{mod}2}^{\min(r_i+r_j-1,2p'-r-r'-1)}\ \sum_{s=|s_i-s_j|+1,s_i+s_j-s=1\,\mathrm{mod}2}^{\min(s_i+s_j-1,2p-s-s'-1)}\phi_{(r,s)} . \quad (15)$$

The result of Eq. (15) agrees with the results of the indicial equations of the differential equations of the four-point correlators.[1]

III. FUSION RULES FOR $N=1$ SUPERCONFORMAL THEORIES

The $N=1$ superconformal algebras have two conserved currents: spin 2 (energy-momentum tensor) and spin $\frac{3}{2}$ (its superpartner). A discrete series of unitary models exists with the central charge

$$c=\frac{3}{2}\left[1-\frac{8}{m(m+2)}\right],\quad m=3,4,5,\ldots . \quad (16)$$

The coset construction for this model by Ref. 6 is

$$\frac{\mathrm{SU}(2)_{m-2}\otimes \mathrm{SU}(2)_2}{\mathrm{SU}(2)_m} , \quad (17)$$

where the subscript denotes the level of SU(2) Kac-Moody algebra. The coset space decomposition of characters $\chi_{l,m}(z,\tau)$ of representation l (twice of spin) in the level-m SU(2) Kac-Moody algebra is

$$\chi_{p-1,m-2}(z,\tau)\chi_{r,2}(z,\tau)=\sum_{q=1}^{m+1} b_{p,q}{}^{(r)}(\tau)\chi_{q-1,m}(z,\tau) , \quad (18)$$

where the branching coefficient $b_{p,q}{}^{(r)}(\tau)$ has the symmetries

$$b_{p,q}{}^{(r)}=b_{m-p,m-q+2}^{(2-r)} ,$$
$$b_{p,q}{}^{(r)}=0 \quad \text{if } p-q\in 2Z+r+1 , \quad (19)$$

and has the modular transformation

$$b_{p,q}{}^{(r)}\left[-\frac{1}{\tau}\right]=\left[\frac{2}{m(m+2)}\right]^{1/2}\sum_{p'=1}^{m-1}\sum_{q'=1}^{m+1}\sum_{r'=0}^{2}\sin\frac{\pi pp'}{m}\sin\frac{\pi qq'}{m+2}\sin\frac{\pi(r+1)(r'+1)}{4}b_{p',q'}{}^{(r')}(\tau) . \quad (20)$$

From the symmetry of Eq. (19), we can restrict the (p,q) to be

$$1\le p\le m-1,\quad p\le q\le m+1,\quad 0\le r\le 2 . \quad (21)$$

The fusion rules for these branching coefficients are

$$A_{(p_iq_ir_i)(p_jq_jr_j)}{}^{(p_kq_kr_k)}=\sum_{p=1}^{m-1}\sum_{q=1}^{m+1}\sum_{r=0}^{2}S_{(p_jq_jr_j)}{}^{(pqr)}\frac{S_{(p_iq_ir_i)}{}^{(pqr)}}{S_{(110)}{}^{(pqr)}}(S_{(p_kq_kr_k)}{}^{(pqr)})^* \ . \tag{22}$$

The summation over p, q, and r gives

$$b_{p_iq_i}{}^{(r_i)}b_{p_jq_j}{}^{(r_j)}=\sum_{p_k=|p_i-p_j|+1}^{\min(2m-p_i-p_j-1,p_i+p_j-1)}\ \sum_{q_k=|q_i-q_j|+1}^{\min(2m-q_i-q_j+3,q_i+q_j-1)} b_{p_kq_k}{}^{(r_k)} \ , \tag{23}$$

where $r_k=r_i+r_j \bmod 2$,

$$\begin{bmatrix} p_i+p_j-p_k \\ q_i+q_j-q_k \end{bmatrix}=1 \bmod 2 \ .$$

When there are several terms allowed in the summation which are equivalent by symmetry of Eq. (19), only a single term is allowed. The superconformal characters are given in terms of b as

$$\chi^{\rm NS}_{pq}=\tfrac{1}{2}(b^{(0)}_{pq}+b^{(2)}_{pq}),\quad p-q\in\text{even} \ ,$$
$$\chi^{\widetilde{\rm NS}}_{pq}=\tfrac{1}{2}(b^{(0)}_{pq}-b^{(2)}_{pq}),\quad p-q\in\text{even} \ , \tag{24}$$
$$\chi^{\rm R}_{pq}=\frac{1}{\sqrt{2}}b^{(1)}_{pq},\quad p-q\in\text{odd} \ ,$$

where NS (for Neveu-Schwarz), $\widetilde{\rm NS}$, and R (for Ramond) denote the boundary condition $(--)$, $(+-)$, and $(-+)$ along the σ and τ directions. Generally the fusion rules are

$$\chi^{\rm NS}\otimes\chi^{\rm NS}=\sum\chi^{\rm NS},\quad \chi^{\rm NS}\otimes\chi^{\widetilde{\rm NS}}=0 \ ,$$
$$\chi^{\rm NS}\otimes\chi^{\rm R}=\sum\chi^{\rm R},\quad \chi^{\widetilde{\rm NS}}\otimes\chi^{\widetilde{\rm NS}}=\sum\chi^{\rm NS} \ , \tag{25}$$
$$\chi^{\widetilde{\rm NS}}\otimes\chi^{\rm R}=0,\quad \chi^{\rm R}\otimes\chi^{\rm R}=\sum\chi^{\rm NS} \ .$$

All the nonvanishing fusion coefficients for $m=3$ superconformal theories (which are related to $m=4$ conformal theories with conformal dimension-$\frac{3}{2}$ operators identified with the supercharge) are

$$\begin{aligned}
&\chi^{\rm NS}_{11}\otimes\chi^{\rm NS}_{11}=\chi^{\rm NS}_{11},\ \chi^{\rm NS}_{11}\otimes\chi^{\rm NS}_{13}=\chi^{\rm NS}_{13}\ ,\ \chi^{\rm NS}_{11}\otimes\chi^{\rm R}_{12}=\chi^{\rm R}_{12},\ \chi^{\rm NS}_{11}\otimes\chi^{\rm R}_{14}=\chi^{\rm R}_{14},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm NS}_{13}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13}\ ,\\
&\chi^{\rm NS}_{13}\otimes\chi^{\rm R}_{12}=\chi^{\rm R}_{12}+\chi^{\rm R}_{14},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm R}_{14}=\chi^{\rm R}_{12},\ \chi^{\widetilde{\rm NS}}_{11}\otimes\chi^{\widetilde{\rm NS}}_{11}=\chi^{\rm NS}_{11},\ \chi^{\widetilde{\rm NS}}_{11}\otimes\chi^{\widetilde{\rm NS}}_{13}=\chi^{\rm NS}_{13}\ ,\\
&\chi^{\widetilde{\rm NS}}_{13}\otimes\chi^{\widetilde{\rm NS}}_{13}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13},\ \chi^{\rm R}_{12}\otimes\chi^{\rm R}_{12}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13},\ \chi^{\rm R}_{12}\otimes\chi^{\rm R}_{14}=\chi^{\rm NS}_{13}\ .
\end{aligned} \tag{26}$$

The fusion rules for $m=4$ superconformal theories are also listed:

$$\begin{aligned}
&\chi^{\rm NS}_{11}\otimes\chi^{\rm NS}_{11}=\chi^{\rm NS}_{11},\ \chi^{\rm NS}_{11}\otimes\chi^{\rm NS}_{13}=\chi^{\rm NS}_{13},\ \chi^{\rm NS}_{11}\otimes\chi^{\rm NS}_{15}=\chi^{\rm NS}_{15},\ \chi^{\rm NS}_{11}\otimes\chi^{\rm R}_{12}=\chi^{\rm R}_{12},\ \chi^{\rm NS}_{11}\otimes\chi^{\rm R}_{14}=\chi^{\rm R}_{14}\ ,\chi^{\rm NS}_{11}\otimes\chi^{\rm R}_{21}=\chi^{\rm R}_{21}\ ,\\
&\chi^{\rm NS}_{11}\otimes\chi^{\rm R}_{23}=\chi^{\rm R}_{23},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm NS}_{13}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13}+\chi^{\rm NS}_{15},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm NS}_{15}=\chi^{\rm NS}_{13},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm R}_{12}=\chi^{\rm R}_{12}+\chi^{\rm R}_{14}\ ,\\
&\chi^{\rm NS}_{13}\otimes\chi^{\rm R}_{14}=\chi^{\rm R}_{12}+\chi^{\rm R}_{14},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm R}_{21}=\chi^{\rm R}_{23},\ \chi^{\rm NS}_{13}\otimes\chi^{\rm R}_{23}=\chi^{\rm R}_{21},\ \chi^{\rm NS}_{15}\otimes\chi^{\rm NS}_{15}=\chi^{\rm NS}_{11},\ \chi^{\rm NS}_{15}\otimes\chi^{\rm R}_{12}=\chi^{\rm R}_{14}\ ,\\
&\chi^{\rm NS}_{15}\otimes\chi^{\rm R}_{14}=\chi^{\rm R}_{12},\ \chi^{\rm NS}_{15}\otimes\chi^{\rm R}_{21}=\chi^{\rm R}_{21},\ \chi^{\rm NS}_{15}\otimes\chi^{\rm R}_{23}=\chi^{\rm R}_{23},\ \chi^{\rm R}_{12}\otimes\chi^{\rm R}_{12}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13},\ \chi^{\rm R}_{12}\otimes\chi^{\rm R}_{14}=\chi^{\rm NS}_{13}+\chi^{\rm NS}_{15}\ ,\\
&\chi^{\rm R}_{12}\otimes\chi^{\rm R}_{21}=\chi^{\rm NS}_{22},\ \chi^{\rm R}_{12}\otimes\chi^{\rm R}_{23}=\chi^{\rm NS}_{22},\ \chi^{\rm R}_{14}\otimes\chi^{\rm R}_{14}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13},\ \chi^{\rm R}_{14}\otimes\chi^{\rm R}_{21}=\chi^{\rm NS}_{22}\ ,\\
&\chi^{\rm R}_{14}\otimes\chi^{\rm R}_{23}=\chi^{\rm NS}_{22},\ \chi^{\rm R}_{21}\otimes\chi^{\rm R}_{21}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{15},\ \chi^{\rm R}_{21}\otimes\chi^{\rm R}_{23}=\chi^{\rm NS}_{13},\ \chi^{\rm R}_{23}\otimes\chi^{\rm R}_{23}=\chi^{\rm NS}_{11}+\chi^{\rm NS}_{13}+\chi^{\rm NS}_{15}\ .
\end{aligned}$$

IV. FUSION COEFFICIENTS IN THE ARBITRARY MINIMAL UNITARY CONFORMAL THEORIES

To obtain more detailed information of operator-product expansions of primary operators, we use the four-point correlators

$$\langle\phi_k(z_1,\bar z_1)\phi_l(z_2,\bar z_2)\phi_n(z_3,\bar z_3)\phi_m(z_4,\bar z_4)\rangle$$

satisfying the differential equation given by the decoupling of the spurious states. One can move three of four z_i's at arbitrary points due to the SL(2,C) invariance, and we choose these points $z_1=\infty,z_2=1,z_4=0$, respectively. Also the same reasoning applies to the antiholomorphic parts. From now on, we will drop the antiholomorphic coordinates for typographical simplicity. If the primary operator $\phi_n(z)$ generates the null state at grade d, then the four-point correlator satisfies the dth-order differential equation.[1] One should solve generally the differential equation of order d to fix all the conformal blocks in Eq. (4). However if we want to obtain the fusion coefficients C_{ij}^k only, there is enormous simplification using the crossing symmetry

$$\langle\phi_k(\infty)\phi_l(1)\phi_n(x)\phi_m(0)\rangle =\langle\phi_k(\infty)\phi_m(1)\phi_n(1-x)\phi_l(0)\rangle \ . \tag{28}$$

Therefore, one can determine all of the fusion coefficients without solving the higher-order ($d \geq 3$) differential equations. We choose ϕ_n which generates the null state at grade 2, so the differential equations become second order. The differential equation for the four-point correlator is

$$\left[\frac{3}{2(2\Delta_n+1)}\frac{d^2}{dx^2}+\frac{1-2x}{x(1-x)}\frac{d}{dx} + \frac{(\Delta_n-\Delta_k)x^2+(\Delta_m+\Delta_k-\Delta_l-\Delta_n)x-\Delta_m}{x^2(1-x)^2}\right] \times \langle \phi_k \phi_l \phi_n(x) \phi_m \rangle = 0 , \quad (29)$$

where Δ_i is the conformal dimension of primary operator ϕ_i. The solution is

$$A_1 x^{\alpha}(1-x)^{\beta} F(a,b,c,x) + A_2 x^{\alpha}(1-x)^{\beta} x^{1-c} \times F(a-c+1,b-c+1,2-c,x) , \quad (30)$$

where

$$\alpha_{\pm}=\tfrac{1}{2}\{1-\Delta^{-1}\pm[(1-\Delta^{-1})^2+4\Delta_m\Delta^{-1}]^{1/2}\} ,$$

$$\beta_{\pm}=\tfrac{1}{2}\{1-\Delta^{-1}\pm[(1-\Delta^{-1})^2+4\Delta_l\Delta^{-1}]^{1/2}\} ,$$

$$\gamma_{\pm}=\tfrac{1}{2}\{2\Delta^{-1}-1\pm[(2\Delta^{-1}-1)^2+4(\Delta_k-\Delta_n)\Delta^{-1}]^{1/2}\} ,$$

$$\Delta^{-1}=\tfrac{2}{3}(2\Delta_n+1) .$$

Here, α,β take one of the values of $\alpha_{\pm},\beta_{\pm}$, respectively, and the α is chosen to be $\Delta_p-\Delta_n-\Delta_m$, where Δ_p is the conformal dimension of the intermediate operator. The arguments of the hypergeometric function are given by

$$a=\alpha_{\pm}+\beta_{\pm}+\gamma_{\pm}, \quad b=\alpha_{\pm}+\beta_{\pm}+\gamma_{\mp}, \quad c=1+\alpha_{\pm}-\alpha_{\mp} .$$

The correlator should be a single-valued function under rotation of x and $\bar{x}$ around $x=0$ or 1 as well as $\bar{x}$. This monodromy requirement fixes the four unknown coefficients apart from overall normalization in the correlator, where each term is multiplied by the antiholomorphic part. The monodromy condition about the point $x=0$ allows only the following;

$$A_1 x^{\alpha}(1-x)^{\beta} F(a,b,c,x)(\mathrm{AH}) + A_2 x^{\alpha}(1-x)^{\beta} x^{1-c} \times F(a-c+1,b-c+1,2-c,x)(\mathrm{AH}) , \quad (31)$$

where (AH) means the antiholomorphic part.

To investigate the behavior under the transformation $(1-x)\to(1-x)e^{2\pi i}$, let us change the x dependence of the solution into $1-x$ via hypergeometric function identities:

$$F(a,b,c,x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-x) + (1-x)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \times F(c-a,c-b,c-a-b+1,1-x) . \quad (32)$$

The monodromy requirement of x around 1 fixes the relative coefficients as

$$\frac{A_2}{A_1}=-\frac{\Gamma(c)\Gamma(c)\Gamma(1-a)\Gamma(1-b)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)\Gamma(2-c)\Gamma(2-c)} \quad (33)$$

using Eq. (32) and

$$F(a,b,c,x)=(1-x)^{c-a-b}F(c-a,c-b,c,x) . \quad (34)$$

Here we present an argument that all the fusion coefficients in Eq. (4) of the minimal conformal theories can be determined from the second-order Fuchsian differential equations with the monodromy and the crossing symmetry.

We restrict the case on unitary theory, so the central charge becomes

$$c=1-\frac{6}{m(m+1)}, \quad m=3,4,\ldots$$

and the conformal dimension of the primary operator

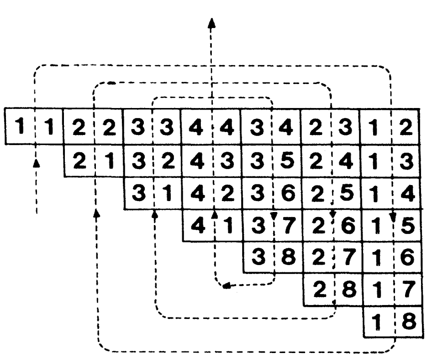

FIG. 1. A path for the determination of fusion coefficients $C_{ij}{}^{k}$ in Eq. (4) of the $m=8$ minimal conformal theory. The C_i matrix is calculated in the order of $C_{(1,1)}, C_{(1,2)}, C_{(1,3)}, \ldots$ as shown by arrows.

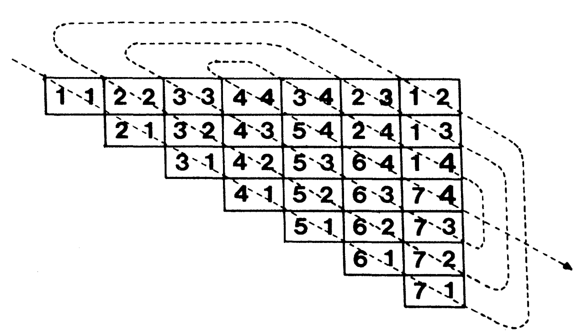

FIG. 2. Alternative path for the determination of $C_{ij}{}^{k}$.

$\phi_{(p,q)}$ is

$$h_{(p,q)}=h_{(m-p,m-q+1)}=\frac{[(m+1)p-mq]^2-1}{4m(m+1)}\ ,$$
$$1\le p\le[m/2],\quad 1\le q\le m\ . \qquad (35)$$

The fusion coefficients $C_{ij}{}^{k}$ can be regarded as a (j,k) element of the C_i matrix where i, j, and k can take any pair of (p,q) as in Eq. (35). All these fusion coefficient matrices can be determined using the second-order differential equation with the monodromy condition and the crossing symmetries. There are two paths where the new fusion coefficient can be expressed in terms of the previously determined ones. Of course, the results are independent of path. Here we present the general results following the path in Fig. 1.

1. Fusion matrix $C_{(1,1)}$

$C_{(1,1)}$ is a unit diagonal matrix since the primary operator $\phi_{(1,1)}$ is an identity operator.

2. Fusion matrix $C_{(1,2)}$

The $C_{(1,2)}$ matrix can be determined using the monodromy and crossing symmetry of the four-point correlator.

(i) Monodromy condition. As was discussed before, the monodromy requirements can fix the relative ratio of fusion coefficients. Since there are only one or two nonvanishing elements in each row of the $C_{(1,2)}$ matrix, and since $C_{ij}{}^{k}$ is totally symmetric, we can determine some parts of the $C_{(1,2)}$ matrix.

One can start from

$$C_{(1,2)(1,1)(1,2)}=1\ . \qquad (36)$$

Using Eq. (33), $C_{(1,2)(1,2)(1,3)}$ can be determined via the monodromy condition of the correlator

$$\langle(1,2)(1,2)(1,2)(1,2)\rangle_x \equiv \langle \phi_{(1,2)}(\infty)\phi_{(1,2)}(1)\phi_{(1,2)}(x,\bar{x})\phi_{(1,2)}(0)\rangle$$

in shorthand notation. The result is

$$[C_{(1,2)(1,2)(1,3)}]^2=-\frac{\Gamma\left[\frac{2m-1}{m+1}\right]\Gamma\left[\frac{m}{m+1}\right]\left[\Gamma\left[\frac{2}{m+1}\right]\right]^2}{\Gamma\left[\frac{1}{m+1}\right]\Gamma\left[\frac{2-m}{m+1}\right]\left[\Gamma\left[\frac{2m}{m+1}\right]\right]^2}\ . \qquad (37)$$

By the same method, one can fix the next row of $C_{(1,2)}$ successively until it terminates at $C_{(1,2)(1,m-1)(1,m)}$. The explicit form of the fusion coefficients corresponding to this series is

$$[C_{(1,2)(1,k)(1,k+1)}]^2=(-)^{k-1}\prod_{l=2}^{k}\frac{\Gamma\left[\frac{lm-1}{m+1}\right]\Gamma\left[\frac{(l-1)m}{m+1}\right]\left[\Gamma\left[\frac{(2-l)m+2}{m+1}\right]\right]^2}{\Gamma\left[\frac{(1-l)m+2}{m+1}\right]\Gamma\left[\frac{(2-l)m+1}{m+1}\right]\left[\Gamma\left[\frac{lm}{m+1}\right]\right]^2}\ , \qquad (38)$$

which is the result of the monodromy requirement on the correlator $\langle(1,2)(1,k)(1,2)(1,k)\rangle$ with $2\le k\le m-1$.

(ii) Crossing symmetry. The remaining row of the $C_{(1,2)}$ matrix can be fixed by crossing symmetry of the four-point correlator in addition to the monodromy conditions. First, let us consider the relation

$$\langle(2,1)(2,1)(1,2)(1,2)\rangle_x=\langle(2,1)(1,2)(1,2)(2,1)\rangle_{1-x}\ . \qquad (39)$$

According to Eq. (31), the expression of the above correlators is obtained as

$$x^{\alpha}(1-x)^{\beta}F(a,b,c,x)(\mathrm{AH})=[C_{(1,2)(2,1)(2,2)}]^2x^{\alpha}(1-x)^{\beta}F(a,b,a+b-c+1,1-x)(\mathrm{AH}) \qquad (40)$$

with

$$a=\frac{1}{m+1},\quad b=-1,\quad c=\frac{2}{m+1},\quad \alpha=\frac{2-m}{2(m+1)},\quad \beta=-\tfrac{1}{2}\ .$$

Thus the fusion coefficient is

$$[C_{(1,2)(2,1)(2,2)}]^2=\left[\frac{\Gamma\left[\frac{2}{m+1}\right]\Gamma\left[\frac{m+2}{m+1}\right]}{\Gamma\left[\frac{1}{m+1}\right]\Gamma\left[\frac{m+3}{m+1}\right]}\right]^2=(\tfrac{1}{2})^2\ . \qquad (41)$$

Now, $((2,k),(2,k+1))$ elements of the $C_{(1,2)}$ matrix are successively determined by the monodromy condition as

$$[C_{(1,2)(2,k)(2,k+1)}]^2=(-)^{k-1}\frac{1}{4}\prod_{l=2}^{k}\frac{\Gamma\left[\frac{(l-1)m-2}{m+1}\right]\Gamma\left[\frac{(l-2)m-1}{m+1}\right]\left[\Gamma\left[\frac{(3-l)m+3}{m+1}\right]\right]^2}{\Gamma\left[\frac{(2-l)m+3}{m+1}\right]\Gamma\left[\frac{(3-l)m+2}{m+1}\right]\left[\Gamma\left[\frac{(l-1)m-1}{m+1}\right]\right]^2}\quad(2\le k\le m-1)\ . \tag{42}$$

To evaluate the $((s,1),(s,2))$ matrix elements of the $C_{(1,2)}$ matrix, let us consider the crossing symmetry

$$\langle (s,1)(s,1)(1,2)(1,2)\rangle_x=\langle (s,1)(1,2)(1,2)(s,1)\rangle_{1-x}\ . \tag{43}$$

By repeating the same steps, one obtains the fusion coefficient

$$[C_{(1,2)(s,1)(s,2)}]^2=\left[\frac{\Gamma\left[\frac{2}{m+1}\right]\Gamma\left[\frac{(s-1)m+s}{m+1}\right]}{\Gamma\left[\frac{1}{m+1}\right]\Gamma\left[\frac{(s-1)m+s+1}{m+1}\right]}\right]^2 \tag{44}$$

which is a simple rational number in general. The remaining part of $C_{(1,2)}$ matrix can be determined from monodromy of the correlator $\langle (1,2)(s,k)(1,2)(s,k)\rangle_x$ as

$$[C_{(1,2)(s,k)(s,k+1)}]^2=(-)^{k-1}\left[\frac{\Gamma\left[\frac{2}{m+1}\right]\Gamma\left[\frac{(s-1)m+s}{m+1}\right]}{\Gamma\left[\frac{1}{m+1}\right]\Gamma\left[\frac{(s-1)m+s+1}{m+1}\right]}\right]^2 \times\prod_{l=2}^{k}\frac{\Gamma\left[\frac{(1-s+l)m-s}{m+1}\right]\Gamma\left[\frac{(l-s)m+1-s}{m+1}\right]\left[\Gamma\left[\frac{(s-l+1)m+s+1}{m+1}\right]\right]^2}{\Gamma\left[\frac{(s-l)m+s+1}{m+1}\right]\Gamma\left[\frac{(s-l+1)m+s}{m+1}\right]\left[\Gamma\left[\frac{(1-s+l)m+1-s}{m+1}\right]\right]^2}\ . \tag{45}$$

This formula includes Eqs. (38), (42), and (44) with $s=1$, $s=2$, and $k=1$, respectively.

3. *Fusion matrix* $C_{(p,q)}$ *($q\neq 1$)*

Generally, $C_{(p,q)(\alpha,\beta)(\gamma,\delta)}$ can be obtained from the crossing symmetry

$$\langle (\alpha,\beta)(\gamma,\delta)(1,2)(p,q-1)\rangle_x=\langle (\alpha,\beta)(p,q-1)(1,2)(\gamma,\delta)\rangle_{1-x}\ . \tag{46}$$

Note that $q\neq 1$ and we will use another crossing symmetry later for the case $q=1$. The left-hand side of Eq. (46) becomes

$$x^{\bar\alpha}(1-x)^{\bar\beta}(\mathrm{AH})[C_{(\alpha,\beta)(\gamma,\delta)(p,q)}C_{(1,2)(p,q-1)(p,q)}F_{(a,b,c,x)}(\mathrm{AH}) +C_{(\alpha,\beta)(\gamma,\delta)(p,q-2)}C_{(1,2)(p,q-1)(p,q-2)}x^{1-c}F(a-c+1,b-c+1,2-c,x)(\mathrm{AH})]\ , \tag{47}$$

where

$$a=\frac{1}{2(m+1)}[(q-p+\alpha-\beta-\gamma+\delta)m+(1-p+\alpha-\gamma)]\ ,$$

$$b=\frac{1}{2(m+1)}[(q-p-\alpha+\beta-\gamma+\delta)m+(1-p-\alpha-\gamma)]\ ,$$

$$c=\frac{1}{m+1}[(q-p)m+(1-p)]\ .$$

The right-hand side also gives the similar form in $1-x$ dependence as

$$x^{\bar\alpha}(1-x)^{\bar\beta}(\mathrm{AH})[C_{(\alpha,\beta)(p,q-1)(\gamma,\delta+1)}C_{(1,2)(\gamma,\delta)(\gamma,\delta+1)}F(a,b,a+b-c+1,1-x)(\mathrm{AH}) +C_{(\alpha,\beta)(p,q-1)(\gamma,\delta-1)}C_{(1,2)(\gamma,\delta)(\gamma,\delta-1)}(1-x)^{c-a-b}F(c-b,c-a,c-a-b+1,1-x)(\mathrm{AH})]\ .$$

Since Eqs. (47) and (48) are equal via the crossing symmetry, one can compare directly through Eqs. (32) and (34), and finally obtain the following recursion relations for the fusion coefficients $C_{(p,q)(\alpha,\beta)(\gamma,\delta)}$:

$$C_{(\alpha,\beta)(\gamma,\delta)(p,q)}C_{(1,2)(p,q-1)(p,q)}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right]^2 + C_{(\alpha,\beta)(\gamma,\delta)(p,q-2)}C_{(1,2)(p,q-1)(p,q-2)}\left[\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\right]^2 = C_{(\alpha,\beta)(p,q-1)(\gamma,\delta+1)}C_{(1,2)(\gamma,\delta)(\gamma,\delta+1)} \quad (49)$$

and

$$C_{(\alpha,\beta)(\gamma,\delta)(p,q)}C_{(1,2)(p,q-1)(p,q)}\left[\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right]^2 + C_{(\alpha,\beta)(\gamma,\delta)(p,q-2)}C_{(1,2)(p,q-1)(p,q-2)}\left[\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}\right]^2 = C_{(\alpha,\beta)(p,q-1)(\gamma,\delta-1)}C_{(1,2)(\gamma,\delta)(\gamma,\delta-1)} . \quad (50)$$

Note that when $\delta=1$ or m, Eqs. (50) or (49) drop out, respectively, and when $2\le\delta\le m-1$, two equations give consistent results. Furthermore, the second terms of the left-hand side of Eqs. (49) and (50) disappear when $q=2$.

4. Fusion matrix $C_{(p,1)}$

The remaining matrices are $C_{(p,1)}$ $(2\le p\le[m/2])$. Fusion matrices $C_{(p,1)}$ correspond to the initial entries in the left half of the boxes describing the path in Fig. 1. First, $C_{(2,1)}$ can be obtained similarly to that of $C_{(1,2)}$ using monodromy and crossing symmetry as

$$[C_{(2,1)(s,k)(s+1,k)}]^2=(-)^{s-1}\left[\frac{\Gamma\left[-\frac{2}{m}\right]\Gamma\left[\frac{(k-1)m-1}{m}\right]}{\Gamma\left[-\frac{1}{m}\right]\Gamma\left[\frac{(k-1)m-2}{m}\right]}\right]^2 \times\prod_{l=2}^{s}\frac{\Gamma\left[\frac{(l-k)m+l-1}{m}\right]\Gamma\left[\frac{(l-k+1)m+l+1}{m}\right]\left[\Gamma\left[\frac{(k-l+1)m-l}{m}\right]\right]^2}{\Gamma\left[\frac{(k-l+1)m+1-l}{m}\right]\Gamma\left[\frac{(k-l)m-l-1}{m}\right]\left[\Gamma\left[\frac{(l-k+1)m+l}{m}\right]\right]^2} . \quad (51)$$

The $C_{(p,1)}$ $(p\ge 3)$ matrix can be also determined from the relation

$$\langle(\alpha,\beta)(\gamma,\beta)(2,1)(p-1,1)\rangle_x=\langle(\alpha,\beta)(p-1,1)(2,1)(\gamma,\beta)\rangle_{1-x} \quad (52)$$

which gives the recursion relation for $C_{(p,1)(\alpha,\beta)(\gamma,\beta)}$ as

$$C_{(\alpha,\beta)(\gamma,\beta)(p,1)}C_{(2,1)(p-1,1)(p,1)}\left[\frac{\Gamma(c')\Gamma(c'-a'-b')}{\Gamma(c'-a')\Gamma(c'-b')}\right]^2 + C_{(\alpha,\beta)(\gamma,\beta)(p-2,1)}C_{(2,1)(p-1,1)(p-2,1)}\left[\frac{\Gamma(2-c')\Gamma(c'-a'-b')}{\Gamma(1-a')\Gamma(1-b')}\right]^2 = C_{(\alpha,\beta)(p-1,1)(\gamma+1,\beta)}C_{(2,1)(\gamma,\beta)(\gamma+1,\beta)} \quad (53)$$

and

$$C_{(\alpha,\beta)(\gamma,\beta)(p,1)}C_{(2,1)(p-1,1)(p,1)}\left[\frac{\Gamma(c')\Gamma(a'+b'-c')}{\Gamma(a')\Gamma(b')}\right]^2 + C_{(\alpha,\beta)(\gamma,\beta)(p-2,1)}C_{(2,1)(p-1,1)(p-2,1)}\left[\frac{\Gamma(2-c')\Gamma(a'+b'-c')}{\Gamma(a'-c'+1)\Gamma(b'-c'+1)}\right]^2 = C_{(\alpha,\beta)(p-1,1)(\gamma-1,\beta)}C_{(2,1)(\gamma,\beta)(\gamma-1,\beta)} , \quad (54)$$

where

$$a'=-\frac{m+1}{m}b,\quad b'=-\frac{m+1}{m}a,\quad c'=-\frac{m+1}{m}c ,$$

and a,b,c are in Eq. (47). Through steps (1)–(4) we have computed all fusion coefficients for the arbitrary minimal uni-

TABLE I. Fusion coefficient matrices A_{ij}^{k} in Eq. (3) of the tricritical Ising model are listed. The primary operators are labeled by (r,s) as in Eq. (6), where the primary operators are ordered in increasing conformal dimensions.

$$A_{(1,1)}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\qquad A_{(2,1)}=\begin{bmatrix}0&0&0&1&0&0\\0&0&1&0&1&0\\0&1&0&0&0&0\\1&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}$$

$$A_{(2,2)}=\begin{bmatrix}0&1&0&0&0&0\\1&0&1&0&1&1\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&1&0&1&0&0\\0&1&0&0&0&0\end{bmatrix}\qquad A_{(3,2)}=\begin{bmatrix}0&0&0&0&1&0\\0&1&0&1&0&0\\0&0&1&0&0&1\\0&1&0&0&0&0\\1&0&0&0&1&0\\0&0&1&0&0&0\end{bmatrix}$$

$$A_{(3,3)}=\begin{bmatrix}0&0&1&0&0&0\\0&1&0&1&0&0\\1&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&1\\0&0&0&0&1&0\end{bmatrix}\qquad A_{(3,1)}=\begin{bmatrix}0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\end{bmatrix}$$

tary conformal theories. In fact, there is another path for the complete determination of $C_{ij}{}^{k}$, which is shown in Fig. 2, and the parallel procedure can be done to obtain the similar recursion relation as in Eqs. (49), (50), (53), and (54). As was shown before, the roles of $C_{(1,2)}$ and $C_{(2,1)}$ were crucial in those steps. The basic reason for this is that the primary operators which generate the null state at grade 2 are $\phi_{(1,2)}$ and $\phi_{(2,1)}$. For these reasons, there are two kinds of determining paths as in Figs. 1 and 2. In Fig. 2 the roles of $C_{(1,2)}$ and $C_{(2,1)}$ are interchanged from those in Fig. 1, but the final results are, of course, consistent.

V. A SPECIFIC EXAMPLE OF FUSION COEFFICIENTS

We will demonstrate the procedures of Secs. II and IV explicitly with $c=\frac{7}{10}$ conformal theories. The six primary operators are ordered according to their conformal dimensions as $(p,q)=(1,1)$, (2,2), (3,3), (2,1), (3,2), and (3,1) in the increasing order of conformal dimensions 0, $\frac{3}{80}$, $\frac{1}{10}$, $\frac{7}{16}$, $\frac{3}{5}$, and $\frac{3}{2}$, respectively. The real symmetric modular transformation matrix [Eq. (8)] for $m=4$ is

$$S=\frac{1}{\sqrt{5}}\begin{bmatrix}\sin\frac{\pi}{5}&\sqrt{2}\sin\frac{2\pi}{5}&\sin\frac{2\pi}{5}&\sqrt{2}\sin\frac{\pi}{5}&\sin\frac{2\pi}{5}&\sin\frac{\pi}{5}\\ \sqrt{2}\sin\frac{2\pi}{5}&0&\sqrt{2}\sin\frac{\pi}{5}&0&-\sqrt{2}\sin\frac{\pi}{5}&-\sqrt{2}\sin\frac{2\pi}{5}\\ \sin\frac{2\pi}{5}&\sqrt{2}\sin\frac{\pi}{5}&-\sin\frac{\pi}{5}&-\sqrt{2}\sin\frac{2\pi}{5}&-\sin\frac{\pi}{5}&\sin\frac{2\pi}{5}\\ \sqrt{2}\sin\frac{\pi}{5}&0&-\sqrt{2}\sin\frac{2\pi}{5}&0&\sqrt{2}\sin\frac{2\pi}{5}&-\sqrt{2}\sin\frac{\pi}{5}\\ \sin\frac{2\pi}{5}&-\sqrt{2}\sin\frac{\pi}{5}&-\sin\frac{\pi}{5}&\sqrt{2}\sin\frac{2\pi}{5}&-\sin\frac{\pi}{5}&\sin\frac{2\pi}{5}\\ \sin\frac{\pi}{5}&-\sqrt{2}\sin\frac{2\pi}{5}&\sin\frac{2\pi}{5}&-\sqrt{2}\sin\frac{\pi}{5}&\sin\frac{2\pi}{5}&\sin\frac{\pi}{5}\end{bmatrix}. \tag{55}$$

The fusion rules $A_{ij}{}^{k}$ in Eq. (3) are obtained as in Table I, and fusion coefficients $C_{ij}{}^{k}$ in Eq. (4) are determined according to the procedure discussed in the previous section. Their explicit values are listed below and the correlators (or their relation) which we have used are given inside the curly brackets.

(1) $C_{(1,1)}=I$ (unit diagonal 6×6 matrix).

(2) $C_{(1,2)}$ matrix:

$$C_{(1,2)(1,1)(1,2)}=1,\quad C_{(1,2)(1,2)(1,3)}=\frac{2}{3}\left[\frac{\Gamma(\frac{4}{5})[\Gamma(\frac{2}{5})]^3}{\Gamma(\frac{1}{5})[\Gamma(\frac{3}{5})]^3}\right]^{1/2}\{\langle(1,2)(1,2)(1,2)(1,2)\rangle_x\}\ ,$$

$$C_{(1,2)(1,3)(1,4)}=\tfrac{3}{7}\{\langle(1,2)(1,3)(1,2)(1,3)\rangle_x\}\ ,$$

$$C_{(1,2)(2,1)(2,2)}=\tfrac{1}{2}\{\langle(2,1)(2,1)(1,2)(1,2)\rangle_x=\langle(2,1)(1,2)(1,2)(2,1)\rangle_{1-x}\}\ ,$$

$$C_{(1,2)(2,2)(2,2)}=\left[\frac{\Gamma(\frac{4}{5})[\Gamma(\frac{2}{5})]^3}{\Gamma(\frac{1}{5})[\Gamma(\frac{3}{5})]^3}\right]^{1/2}\{\langle(1,2)(2,2)(1,2)(2,2)\rangle_x\}\ .$$

(3) $C_{(1,3)}$ matrix:

$$C_{(1,3)(1,1)(1,3)}=1\ ,$$

$$C_{(1,3)(2,2)(2,2)}=\frac{1}{6}\left[\frac{\Gamma(\frac{4}{5})[\Gamma(\frac{2}{5})]^3}{\Gamma(\frac{1}{5})[\Gamma(\frac{3}{5})]^3}\right]^{1/2}\{\langle(2,2)(2,2)(1,2)(1,2)\rangle_x=\langle(2,2)(1,2)(1,2)(2,2)\rangle_{1-x}\}\ ,$$

$$C_{(1,3)(2,2)(2,1)}=\tfrac{3}{4}\{\langle(2,2)(2,1)(1,2)(1,2)\rangle_x=\langle(2,2)(1,2)(1,2)(2,1)\rangle_{1-x}\}\ ,$$

$$C_{(1,3)(1,3)(1,3)}=\tfrac{2}{3}\left[\frac{\Gamma(\frac{4}{5})[\Gamma(\frac{2}{5})]^3}{\Gamma(\frac{1}{5})[\Gamma(\frac{3}{5})]^3}\right]^{1/2}\{\langle(1,3)(1,3)(1,2)(1,2)\rangle_x=\langle(1,3)(1,2)(1,2)(1,3)\rangle_{1-x}\}\ .$$

(4) $C_{(1,4)}$ matrix:

$$C_{(14,)(1,1)(1,4)}=1\ ,$$

$$C_{(1,4)(2,2)(2,2)}=\tfrac{1}{56}\{\langle(2,2)(2,2)(1,2)(1,3)\rangle_x=\langle(2,2)(1,3)(1,2)(2,2)\rangle_{1-x}\}\ ,$$

$$C_{(1,4)(2,1)(2,1)}=\tfrac{7}{8}\{\langle(2,1)(2,1)(1,2)(1,3)\rangle_x=\langle(2,1)(1,3)(1,2)(2,1)\rangle_{1-x}\}\ .$$

The remaining coefficients are equivalent to one of the above via the symmetry property of fusion matrices. Of course, one can proceed further to obtain $C_{(2,1)}$ and $C_{(2,2)}$ but it will reproduce the previous values related by symmetry of fusion coefficients. Note that the path to compute the fusion coefficients for the tricritical Ising model is not unique, but the final results are independent of path.

Note added. After we finished the manuscript, Zamolodchikov kindly informed us that Pogossian and he have used a recursion algorithm on the superconformal theory [Yad. Fiz. (to be published)]. Also Kitazawa *et al.*[7,8] have studied the operator-product expansion in the $N=1$ superconformal theory using the Dotsenko and Fateev method.

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APPENDIX

For references, we compute $\beta_{ij}{}^k$ coefficients in Eq. (4) and multipoint correlator in terms of fusion coefficients and $\beta_{ij}{}^k$. The descendent states of primary operators ϕ_k at grade N in the OPE in Eq. (4) are

$$|N,\Delta_k\rangle=\sum_{k_1,\ldots,k_n\ \sum k_i=N}\beta_{ij}^{k\{k_i\}}L_{-k_1}\cdots L_{-k_n}\phi_k(0)|0\rangle\ . \tag{A1}$$

From the conformal symmetries, the descendent states satisfy the recursion formulas

$$L_n|N+n,\Delta_k\rangle=\{N+\Delta_k-\Delta_i-\Delta_j+(n+1)\Delta_i\}|N,\Delta_k\rangle\ . \tag{A2}$$

The lowest few terms are listed below:

$$|1,\Delta_k\rangle=\beta_{ij}^{k\{1\}}L_{-1}|\Delta_k\rangle,\quad |2,\Delta_k\rangle=\beta_{ij}^{k\{2\}}L_{-2}|\Delta_k\rangle+\beta_{ij}^{k\{11\}}L_{-1}L_{-1}|\Delta_k\rangle\ ,$$

$$|3,\Delta_k\rangle=\beta_{ij}^{k\{3\}}L_{-3}|\Delta_k\rangle+\beta_{ij}^{k\{21\}}L_{-2}L_{-1}|\Delta_k\rangle+\beta_{ij}^{k\{111\}}L_{-1}L_{-1}L_{-1}|\Delta_k\rangle$$

with

$$\beta_{ij}^{k\{1\}}=(\Delta_k+\Delta_i-\Delta_j)/2\Delta_k\ ,$$
$$\beta_{ij}^{k\{11\}}=\frac{(4\Delta_k+c/2)(\Delta_k+\Delta_i-\Delta_j)(1+\Delta_i-\Delta_j+\Delta_k)/2\Delta_k-3(2\Delta_i-\Delta_j+\Delta_k)}{16\Delta_k^2+(2c-10)\Delta_k+c}\ , \tag{A3}$$
$$\beta_{ij}^{k\{2\}}=\frac{(4\Delta_k+c/2)(\Delta_k+2\Delta_i-\Delta_j)-3(\Delta_k+\Delta_i-\Delta_j)(1+\Delta_i-\Delta_j+\Delta_k)}{16\Delta_k^2+(2c-10)\Delta_k+c}\ .$$

The next terms of β are presented for the $\Delta_i=\Delta_j$ case only for simplicity as

$$\beta_{ii}^{k\{3\}}=\frac{\Delta_k^2+(4\Delta_i-1)\Delta_k+2\Delta_i}{32\Delta_k^2+(4c-20)\Delta_k+2c},\quad \beta_{ii}^{k\{21\}}=\beta_{ii}^{k\{3\}},\quad \beta_{ii}^{k\{111\}}=\frac{8\Delta_k^2+(c-2)\Delta_k+36\Delta_i+2c}{384\Delta_k^2+(48c-20)\Delta_k+24c}\ . \tag{A4}$$

The expression of n-point correlators ($n\geq 5$) can be obtained using the conformal symmetries and standard OPE. Thus one can obtain the full expression of multipoint correlators once the fusion coefficients are known. The five-point correlator, for example, is obtained as

$$\langle \phi_k(\infty)\phi_l(1)\phi_i(y)\phi_n(x)\phi_m(0)\rangle=\sum_{rs} C_{nm}^r C_{ir}^s C_{kls}\, x^{\Delta_r-\Delta_n-\Delta_m}\sum_{\{k_1\}}\beta_{nm}^{r\{k_1\}}x^{\sum k_1}D_{\{k_1\}}(y,z)\,y^{\Delta_s-\Delta_i-\Delta_r}$$
$$\times\sum_{\{k_2\}}\beta_{ir}^{s\{k_2\}}y^{\sum k_2}\left.\frac{\langle k|\phi_l(z)L_{\{-k_2\}}|s\rangle}{\langle k|\phi_l(1)|s\rangle}\right|_{z=1}\ , \tag{A5}$$

where $D_{\{k_1\}}(y,z)$ is

$$D_{\{0\}}=1,\quad D_{\{1\}}=-\left[\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right],\quad D_{\{11\}}=\left[\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]^2,\quad D_{\{2\}}=-y^{-1}\frac{\partial}{\partial y}-z^{-1}\frac{\partial}{\partial z}+\Delta_i y^{-2}+\Delta_l z^{-2} \tag{A6}$$

with its antiholomorphic part.

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