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Wigner distribution function and phase-space formulation of quantum cosmology

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We present a phase-space formulation of quantum cosmology in terms of the Wigner distribution functions, which are useful for statistical descriptions of the quantum states of the Universe. Using a conformal formulation of field theory in curved space we derive a set of Wheeler-DeWitt-Vlasov equations and a generalized mass-shell condition satisfied by the Wigner functions up to the second adiabatic order. For minisuperspaces with diagonal metric and potentials of one variable, we find approximate solutions to these equations and exact solutions for the Bianchi type-II universe as an example. They all possess the correct classical limits.

I. INTRODUCTION

Work on the superspace formulation[1] and the Hamiltonian quantization[2] of general relativity in the 1960s unshered in the first wave of activity in quantum cosmology in the early 1970s, noted in particular by the work of Misner and co-workers.[3,4] Advances in the path-integral formulation of quantum gravity in the 1970s and the suggestion of Hawking[5] on constructing the path integral over compact positive-definite four-geometries provided the backdrop to the seminal work of Hartle and Hawking[6] on the wave function of the Universe. This, together with Vilenkin's[7] birth of the Universe scenario created the current second wave of activity in quantum cosmology. The Hartle-Hawking wave function $\psi[g_{ij},\phi] = N\int Dg_{ab}D\phi\, e^{-S[g_{ab},\phi]}$ is obtained as a path integral over compact Euclidean four-geometries M with metric ${}^{(4)}g_{ab}$, with compact boundary ∂M on which ${}^{(3)}g_{ij}$ is the induced three-metric, and over regular Euclidean matter-field configurations ϕ with values on ∂M. An unanswered question in this regard concerns the criterion one adopts in the choice of initial state for the wave function in superspace (or boundary conditions in Wheeler-DeWitt equation). A more general framework is to think in terms of the density matrix of the Universe $\rho[g_{ij},\phi;g'_{ij},\phi']$. Hawking and Page[8] have separately made such proposals. By virtue of its ability to convey statistical information, it is, in our view, conceptually more encompassing for the description of the quantum states of the Universe. As such, the Hartle-Hawking wave function $\psi[g_{ij},\phi]$ is considered as a pure state whose probability $P=|\psi|^2$ enters only as the diagonal element of ρ. The expectation value of any observable O is then calculated as $\mathrm{tr}(\rho O)$. Since we can only measure what happens in our specific Universe, the physically meaningful results are phrased in terms of conditional probabilities.[8,9]

Investigations into the statistical properties of the quantum states of the Universe may provide some useful clues to the questions of the origin and structure of our Universe. Based on the condition of our present observable Universe, there are only a handful of general guidelines one can follow in such an inquiry.[10,11] We know the present state of the Universe can be described by the Friedmann-Robertson-Walker (FRW) universe (large-scale homogeneity and isotropy, small-scale structures, near-flatness condition, vanishing of the cosmological constant, etc). We know that an inflationary stage described by the de Sitter model at some early time is desirable. We also know that whatever quantum states existed earlier should evolve to these classical conditions, and somewhat below the Planck scale there is a semiclassical regime.[12] (Above the Planck scale we may have to change to a stringy and foamy picture of spacetime.) For these reasons we view the emergence of the FRW or de Sitter solutions in any quantum cosmology calculation as desirable discriminating factors. We also rely heavily on the use of semiclassical approximations. It is against this rather scanty backdrop that one attempts to deduce the statistical properties of the quantum states of the Universe. One can address a few issues of physical interests. They are, for example, the following. (1) The *correlation* in the coordinate and momentum variables in the wave function, where the existence of a peak can be viewed as registering a prediction in the Everett interpretation of quantum mechanics.[13,14] (2) The *coherence* of the many-universe quantum state. In the Hartle-Hawking-Page picture the degree of coherence is related to the connectedness of four-geometries.[6,8] Other nontrivial topological properties of spacetime (e.g., wormholes, baby universes, spacetime foams[15]) may also register as mixed states in the superspace density matrix. (3) *Interaction* between our Universe (system) with other possible universes (bath) and *coarse graining* in these bath variables can lead to dissipative effects[16] observed in our Universe. (4) The *dynamics* of the correlation functions, loss of coherence, etc., as the system evolves (depicted by the von Neumann equation for the density matrix or the transport equations for the distribution functions). Problems of this nature have been studied for quantum fields in curved space before in connection with issues such as entropy generation in particle creation[17,18] and dissipation in quantum fields and semiclassical gravity.[16] Though formally similar, the physical meanings of these issues are different in quantum cosmology, where the states in superspace refer to the configurations of the Universe. The cause and effect of such notions and operations as interaction (between our Universe and other universes) and averaging (over ensembles of universes) need be further clarified. Although inquiries as such are unlikely to yield any direct cosmological implication, they will prove their usefulness if only we are led to asking a better set of questions, which in itself is an advancement in a field such as quantum cosmology.

A useful formalism with rather well-developed concepts and techniques towards analyzing the statistical properties of quantum systems is the use of phase-space representations. Phase-space techniques for quantum fields in curved space have been explored by a number of authors.[19] We have recently developed a covariant theory for nonequilibrium quantum fields in flat[20] and curved space[21] via the Wigner function[22] method. The Wigner function representation used often in kinetic theory is related to the density-matrix description in statistical mechanics and the coherent state representation found in quantum optics problems. Unlike the density matrix which contains full statistical information, the Wigner function describes only the occupation number but not the general correlation. However, if one is interested only in one-particle observables, one can say that it carries the same information as the relevant part (in the sense of subdynamics) of the density matrix.[18] The Wigner function is suitable for problems in quantum cosmology because it reduces to the probability distribution $f(x,p)$ in the classical limit and contains the semiclassical result as the first-order approximation,[23] which is the closest limit that observations in our classical Universe can be extrapolated to. Halliwell[23] has used the Wigner function to describe the correlation in the wave function of the Universe. Kodama[23] attempted at a reformulation of the Wheeler-DeWitt equation in terms of the Wigner functions using the ordinary flat-space form. But the straightforwardly extended operation of taking the average and difference of the coordinate variables of two points (in this case the ${}^{(3)}g_{ij}$) does not generally make sense in curved space. Minisuperspace is in general curved and a globally defined Fourier transform is not always available.[24] One needs to introduce a way of defining the covariant Wigner distribution, which in general is a nontrivial process. This was carried out in detail for curved spacetime in Ref. 21 by way of a Riemann normal coordinate (RNC) expansion.[25] Together with a derivative expansion[26] on the wave operators and potentials we succeeded in transforming the wave equation (Klein-Gordon for scalar fields) for quantum fields defined at two nearby points to two equations—one a transport equation of the Vlasov type and the other a generalized "mass-shell" condition. These expansions are carried out in successive adiabatic orders (in the derivatives of the wave operator which contains both kinetic and potential terms) arising from the variations of the field and the metrics. The lowest order gives the semiclassical WKB approximation, and with the second order nontrivial quantum correction begins to appear.

In this paper we seek a phase-space formulation of quantum cosmology by defining a covariant Wigner function on the phase space of the minisuperspace and deriving a set of kinetic equations associated with the Wheeler-DeWitt equation, the Vlasov transport equation and the generalized mass-shell condition. From these one can solve for the dynamics of the quantum distribution function and discuss issues such as the quantum corrections to the classical Einstein equations for the ensemble of universes, the change of coherence and correlation in the wave functions, the evolution of the density matrix (via the Wheeler–DeWitt–von Neumann equation) and other related issues on the statistical properties of quantum cosmology. We shall present the basic structure of the theory here and address these individual issues in later communications. The paper is organized as follows: In Sec. II we give the necessary background on quantum cosmology in the minisuperspace formulation and the Hartle-Hawking proposal. This is minimized to the extent that notations and definitions can be adequately understood. We will try to point out clearly at what steps in the development arbitrary choices or approximations are made and the motivation behind ours. In Sec. III we introduce the Wigner function formalism in minisuperspace models with diagonal three-metrics. Since in these cases the minisuperspaces are conformally flat, we do not need to invoke the general formalism of Ref. 21 involving RNC and derivative expansions. We will instead develop a conformally related formalism[27,28] which is a simpler generalization from flat space. We shall derive the Wheeler-DeWitt-Vlasov equation for this class of metric and give approximate solutions to those cases where the metric depends on only one parameter. For the Bianchi type-II universe[4] we can solve for the exact form of the Wigner function. (The Kantowski-Sachs universes are of the same structure and can be treated similarly.) In Sec. IV we construct the Wigner function of the Universe for the general (nonconformally flat) class of curved minisuperspaces and derive the set of Wheeler-DeWitt-Vlasov equations. Section V contains a few remarks and suggestions for possible future research.

II. QUANTUM COSMOLOGY AND MINISUPERSPACE FORMULATION

The classical action for pure gravity is

$$S=\frac{1}{16\pi G}\int_M d^4x\sqrt{-{}^4g}\;{}^4R+\frac{1}{8\pi G}\int_{\partial M}d^3x\sqrt{g}\,K_i^i\,, \tag{2.1}$$

where M is a four-dimensional spacetime with three-metric boundary ∂M, and the three metric ${}^4g$ on M induces the metric g on ∂M. ${}^4R$ is the scalar curvature, and K_i^i is the trace of the extrinsic curvature on ∂M. If n^i is the normal to ∂M, then $K_{ij}=\frac{1}{2}\pounds_{\mathbf{n}}(g_{ij})$, where $\pounds$ stands for

Lie derivative. (In this paper we use the sign convention and notation of Ref. 3: except that $a,b=0,1,2,3$ for spacetime indices; $i,j=1,2,3$ for spatial indices and μ,ν for minisuperspace tensor indices.)

In the standard 3+1 decomposition[2] based on constant-time spatial slicing the metric can be written in the form

$$ds^2=-(N^2-N_iN^i)dt^2+2N_i dt\, dx^i+g_{ij}dx^i dx^j\ , \qquad (2.2)$$

where N and N_i are the lapse and shift functions. The action (2.1) becomes

$$S=\frac{1}{16\pi G}\int d^4x\,\sqrt{-{}^4g}\,L$$
$$=\frac{1}{16\pi G}\int_M dt\, d^3x\sqrt{g}\,N[R-K_{ij}K^{ij}+(K_i^i)^2] \qquad (2.3)$$

(there is no boundary term), where R is the curvature of the three-metric g_{ij} with the Levi-Civita connection. Writing K_{ij} in the explicit form

$$K_{ij}=\frac{1}{N}\left[N_{(i|j)}-\frac{1}{2}\frac{\partial}{\partial t}g_{ij}\right] \qquad (2.4)$$

(where a vertical bar denotes covariant derivative with respect to g_{ij}), and introducing the canonical momenta

$$\pi^{ij}=\frac{\partial L}{\partial(\dot g_{ij})}=-\sqrt{g}\,(K^{ij}-g^{ij}K)\ . \qquad (2.5)$$

We can write (2.3) in the canonical form

$$S=\frac{1}{16\pi G}\int dt\, d^3x\left[\pi^{ij}\dot g_{ij}+2N_i\pi^{ij}{}_{|j}-\frac{N}{\sqrt{g}}[\pi^{ij}\pi_{ij}-\tfrac{1}{2}(\pi_i^i)^2-gR\,]\right]\ . \qquad (2.6)$$

In the classical theory, we may generate the field equations by extremizing (2.6) with respect to arbitrary variations of g_{ij}, π^{ij}, N_i, and N. Equivalently, we may extremize

$$S=\frac{1}{16\pi G}\int dt\, d^3x\ \pi^{ij}\dot g_{ij} \qquad (2.7)$$

(an overdot denotes d/dt) only with respect to those variations which respect the constraints

$$\pi^{ij}{}_{|j}=0\ , \qquad (2.8)$$

$$\pi^{ij}\pi_{ij}-\tfrac{1}{2}(\pi_i^i)^2-gR=0 \qquad (2.9)$$

with N and N^i acting as Lagrange multipliers in (2.6). On quantization, one may treat (g_{ij},π^{ij}) as a constrained system, or solve the constraints (2.8), (2.9) at the classical level, thus identifying the "true" dynamical degrees of freedom and then quantizing only those. We shall follow the second route.

Moreover, rather than considering arbitrary g_{ij}'s (superspace) we will consider only a restricted class (minisuperspace) of metrics of the form[3]

$$ds^2=-N^2(t)dt^2+e^{-2\Omega(t)}(e^{2\beta(t)})_{ij}\sigma^i\sigma^j\ . \qquad (2.10)$$

The spatial slices are compact and homogeneous, with σ's as the invariant basis one-forms. We first consider matrix β_{ij} as diagonal and traceless:

$$\beta_{ij}=\mathrm{diag}(\beta_++\sqrt{3}\beta_-,\beta_+-\sqrt{3}\beta_-,-2\beta_+)\ . \qquad (2.11)$$

Then

$$\dot g_{ij}=2g_{ik}(\dot\beta^k{}_j-\dot\Omega\delta^k{}_j),$$

where

$$\dot\beta^k{}_j=\mathrm{diag}(\dot\beta_++\sqrt{3}\dot\beta_-,\dot\beta_+-\sqrt{3}\dot\beta_-,-2\dot\beta_+)\ .$$

Correspondingly we parametrize

$$\pi^i{}_k=\frac{1}{2\pi}(p^i{}_k+\tfrac{1}{3}\delta^i{}_k\hat H)\ ,\quad \hat H=(2\pi)\pi^l{}_l \qquad (2.12)$$

$$6p^i{}_k=\mathrm{diag}(p_++\sqrt{3}p_-,p_+-\sqrt{3}p_-,-2p_+)\ , \qquad (2.13)$$

to find

$$\pi^{ij}\dot g_{ij}=\frac{1}{\pi}(p^i{}_k+\tfrac{1}{3}\delta^i{}_k\hat H)(\dot\beta^k{}_i-\dot\Omega\delta^k{}_i)$$
$$=\frac{1}{\pi}(p_+\dot\beta_++p_-\dot\beta_--\hat H\dot\Omega)\ .$$

So, writing $d\beta_+$ instead of $\dot\beta_+dt$ (similarly for $d\beta_-=\dot\beta_-dt$, $d\Omega=\dot\Omega\,dt$) and normalizing the space volume to $(4\pi)^2$, we recover the action (2.7) as

$$S=\int(p_+d\beta_++p_-d\beta_--\hat H\,d\Omega)\ . \qquad (2.14)$$

Equation (2.8) is trivially satisfied, and Eq. (2.9) gives

$$\hat H^2=p_+^2+p_-^2-24\pi^2 gR\ . \qquad (2.15)$$

One may recover to the explicit canonical form by introducing a "supertime" λ through[3]

$$\frac{d\Omega}{d\lambda}=-p_\Omega e^{-2\alpha}\ , \qquad (2.16)$$

where α is an arbitrary gauge [we will consider only the case in which $\alpha=\alpha(\Omega,\beta_\pm)$]. The resulting equations of motion giving $(\Omega,\beta_\pm)$ as functions of λ may be derived from the Hamiltonian

$$H=e^{-2\alpha}(-p_\Omega^2+p_+^2+p_-^2)-12\pi^2e^{-2\alpha}gR \qquad (2.17)$$

plus the constraint

$$H=0\ \ (\text{or } p_\Omega^2=\hat H^2)\ . \qquad (2.18)$$

We observe that H describes a point particle moving on a conformally flat space under the influence of an external potential. Conformal flatness follows from β_{ij} being diagonal which is not the case for the more general models.

If we want to find the full four-dimensional geometry, we must go back to Eqs. (2.6). From a variation with respect to π^{ij} we find

$$\frac{dg_{ij}}{dt}=\frac{2N}{\sqrt{g}}(\pi_{ij}-\tfrac{1}{2}g_{ij}\pi_i^i)\ . \qquad (2.19)$$

Tracing over (2.19) we get

$$-6\frac{d\Omega}{dt}=\frac{2N}{\sqrt{g}}\left[\frac{\hat H}{2\pi}-\frac{3}{2}\frac{\hat H}{2\pi}\right]=-\frac{N}{2\pi\sqrt{g}}\hat H\ . \qquad (2.20)$$

And, from (2.16) and (2.18),

$$\frac{dt}{d\lambda} = -\frac{12\pi\sqrt{g}}{N} e^{-2\alpha} . \tag{2.21}$$

Thus we may either fix N, solving (2.21) to find the flow of time with respect to "supertime," or fix $dt/d\lambda$, solving (2.21) for N.

As for quantization, one can quantize $(\Omega,\beta_\pm)$ as an unconstrained system described by the Hamiltonian H in (2.17), and then impose the constraint (2.18),

$$H\psi = 0 , \tag{2.22}$$

to project out the physical states. Equation (2.22) is the Wheeler-DeWitt equation. It seems natural to quantize by replacing p_Ω and $p_\pm$ by $-i(\partial/\partial\Omega)$, $-i(\partial/\partial\beta_\pm)$, respectively. However, we face an operator-ordering problem because α depends on $(\Omega,\beta_\pm)$. As α represents only a gauge choice, it seems natural to choose an ordering which makes the theory conformally invariant.[3,29] With this choice, the operator H reads

$$H = \Box - \frac{d-2}{4(d-1)}\mathbb{R} - Ve^{-2\alpha} , \tag{2.23}$$

where $\Box$ is the minisuperspace covariant D'Alembertian,

$$\Box = \frac{1}{\sqrt{-\mathscr{g}}}\partial_\mu(\mathscr{g}^{\mu\nu}\sqrt{i\mathscr{g}}\partial_\nu) , \tag{2.24}$$

$\mathscr{g}_{\mu\nu}$ is the minisuperspace metric [in our case $\mathscr{g}_{\mu\nu} = e^{2\alpha}\eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+)$; for more complicated examples, see Ref. 4], d is the minisuperspace dimension ($d=3$ here), $\mathbb{R}$ is the curvature derived from the Levi-Civita connection of the minisuperspace metric, and $V = 24\pi^2 gR$. Equation (2.22) is conformally invariant if we scale ψ as

$$\psi = e^{(1-d/2)\alpha}\phi . \tag{2.25}$$

There are two preferred choices of α: namely, $\alpha=0$ and the one that makes $V=\text{const}$ (Ref. 30). However, we shall keep α arbitrary, with an eye on more complex examples which are not conformally flat. To solve the Wheeler-DeWitt equation for the "wave function of the Universe" one needs to choose boundary conditions which are closest to the physical condition. The Hartle-Hawking prescription chooses,[5,6] in the vacuum case,

$$\psi(\Omega,\beta_\pm) = \int Dg_{ab} e^{-S[g_{ab}]} , \tag{2.26}$$

where the path integral is taken over all four-dimensional, Euclidean compact manifolds with metrics[4] g_{ab} with boundary ∂M, and g_{ij} given by $(\Omega,\beta_\pm)$ on ∂M. In the minisuperspace formulation, we restrict ourselves to Euclidean metrics of the form (2.10), having a (removable) singularity at some value of Euclidean time, when $\Omega = +\infty$ (Ref. 31).

An immediate problem is to define the measure with respect to which the integral (2.26) is taken.[31,32] There are no established mathematical methods to give a reliable evaluation of the integral (2.26) for general $(\Omega,\beta_\pm)$, but we may still aim to use (2.26) to find ψ on a particular region of minisuperspace, which then could be used as a boundary condition for (2.22). For example, one may pick this region to be the limit of "small metrics" $\Omega \to +\infty$, which are assumed to reach the singularity after short lapses.[3]

This program is not completely free of difficulties. We may compute (2.26) in the stationary phase approximation, that is, by first finding a classical solution satisfying Hartle-Hawking boundary conditions, and then integrating over linearized fluctuations around it. As the volume of the four-metric shrinks to zero, we expect the bulk integral in (2.3) to be negligible, and in the absence of surface terms (2.26) would reduce to just the Gaussian integration over fluctuations. This integral, however, is found to diverge as $\Omega \to \infty$ (Refs. 31 and 32). It is possible that the divergence is physical, but it is also possible that it simply signals the breakdown of the stationary phase method, in the same way that the WKB approximation to the wave function diverges near a classical turning point.

For our present concerns, the fact that one can in principle obtain boundary conditions for ψ from the Hartle-Hawking prescription is more important than the precise formulation of those conditions. Therefore, following Hartle and Hawking we shall adopt, as boundary conditions,[5,6]

$$\psi \to 1, \quad \partial_\Omega\psi \to 0 \quad \text{as } \Omega \to +\infty . \tag{2.27}$$

(For Bianchi type-I universes $\psi \equiv 1$ everywhere in the $\alpha=0$ gauge.)

Equations (2.22), (2.23), and (2.27) completely define our problem. In the next section we shall reformulate it in terms of Wigner functions in minisuperspace.

III. WIGNER FUNCTIONS IN CONFORMALLY FLAT (MINISUPER) SPACES

We now rephrase the minisuperspace formulation of quantum cosmology with the language of Wigner functions. Instead of the wave function of the Universe[6] $\psi(x)$ $[x = (\Omega,\beta_\pm)]$ it will prove convenient to work directly with the density matrix[8] ρ. For a pure state ρ factorizes into

$$\rho(x,x') \simeq \psi(x)\psi(x') \tag{3.1}$$

but more general ρ's can also be considered. ρ is real, symmetric, and satisfies the Wheeler-DeWitt equation

$$[\Box_x - \xi_d\mathbb{R}(x) - V_\alpha(x)]\rho(x,x') = 0 , \tag{3.2}$$

where

$$\xi_d = \frac{d-2}{4(d-1)} \quad \text{and} \quad V_\alpha(x) = e^{-2\alpha}V(x) \tag{3.3}$$

with boundary conditions

$$\rho \to 1, \quad (\partial_\Omega\rho, \partial_{\Omega'}\rho, \partial^2_{\Omega\Omega'}\rho) \to 0 \quad \text{as } \Omega,\Omega' \to +\infty . \tag{3.4}$$

In this section, we will consider only the conformally flat cases. Under a conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\alpha}g_{\mu\nu} \tag{3.5}$$

the density matrix transforms as

$$\rho \to \tilde{\rho} = \left\{ \exp\left[-\left[\frac{d-2}{2}\right][\alpha(x)+\alpha(x')] \right] \right\}\rho$$
$$= \Gamma(x,x')\rho \; . \tag{3.6}$$

Observe that this transformation law is constructed using the conformal properties of Eq. (3.2) and hence holds for more general density matrices than what is assumed in (3.1). If we define, in the gauge in which the metric of minisuperspace is $e^{2\alpha}\eta_{\mu\nu}$,

$$\rho = \Gamma(x,x')\rho_0 \; , \tag{3.7}$$

then ρ_0 is the density matrix corresponding to the $\alpha=0$ gauge obeying

$$\left[\eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} + V(x) \right]\rho_0(x,x') = 0 \; . \tag{3.8}$$

The Wigner function formalism is most useful when V is a slowly varying function of x. We start from the ansatz[21]

$$\rho_0\left[X+\frac{x}{2}, X-\frac{x}{2} \right] = \int \frac{d^dk}{(2\pi)^d} e^{ikx}F_0(X,k) \; . \tag{3.9}$$

F_0 is real and even in k, because of the symmetries of ρ. Developing $V(X+\frac{1}{2}x)$ in powers of x, and retaining only up to the second derivatives of V, we obtain, upon substituting back in (3.8) the Vlasov equation,

$$\left[k^\mu \frac{\partial}{\partial X^\mu} - \tfrac{1}{2}V_{,\mu}\frac{\partial}{\partial k_\mu} \right] F_0(X,k) = 0 \tag{3.10}$$

and the "mass-shell equation"

$$\left[k^2 - V(X) - \tfrac{1}{4}\Box_x - \tfrac{1}{8}V_{,\mu\nu}\frac{\partial^2}{\partial k_\mu \partial k_\nu} \right] F_0(X,k) = 0 \; . \tag{3.11}$$

Equations (3.10) and (3.11) become specially simple in the classical region. Since in this case the Wigner function is strongly peaked around the classical "mass shell" $H=k^2-V=0$, we may approximate $F_0(X,k)$ by[21]

$$F_0(X,k) = f_0(X,k)\delta(H) + \cdots \; ; \tag{3.12}$$

then Eq. (3.10) gives

$$\delta(H)\left[k^\mu \frac{\partial}{\partial X^\mu} + \tfrac{1}{2}V_{,\mu}\frac{\partial}{\partial k_\mu} \right] f_0(X,k) = 0 \; . \tag{3.13}$$

Here f_0 may be understood as the classical distribution function of an ensemble of universes, each obeying the Hamiltonian constraint and the classical equations of motion derived from H. Deviations from classical behavior are related to the derivative terms in Eq. (3.11). In order to analyze these terms further, we need more specific information on the form of V. As an example, consider the case where the potential $V=V(\beta)$ depends on only one spatial coordinate variable β in minisuperspace, which corresponds to the anisotropy parameter β in homogeneous cosmology.[3] Consideration of Eqs. (3.10) and (3.11) and the boundary conditions suggests the ansatz

$$F_0(X,k) = \delta(\kappa_\perp)f(\beta,\kappa) \; , \tag{3.14}$$

where we use κ instead of k to denote the momentum conjugate to β, and $\kappa_\perp$ stands for the other momentum components. Equations (3.10) and (3.11) reduce to a one-dimensional problem

$$\left[\kappa^2 - V(\beta) - \frac{1}{4}\frac{\partial^2}{\partial\beta^2} - \tfrac{1}{8}V''(\beta)\frac{\partial^2}{\partial\kappa^2} \right] f(\beta,\kappa) = 0 \; , \tag{3.15}$$

$$\left[\kappa\frac{\partial}{\partial\beta} + \tfrac{1}{2}V'(\beta)\frac{\partial}{\partial\kappa} \right] f(\beta,\kappa) = 0 \; . \tag{3.16}$$

Equation (3.16) implies that f depends upon κ and β only through the combination $H=\kappa^2-V(\beta)$. In effect (3.16) implies that $f(\beta,\kappa)=f(\beta',\kappa')$ whenever (β,κ) and (β',κ') are joined by a classical trajectory. Thus Eq. (3.15) reduces further to

$$\left[H - \frac{V''}{2}H\frac{d^2}{dH^2} - \tfrac{1}{4}[(V')^2+2V''V]\frac{d^2}{dH^2} \right] f(H) = 0 \tag{3.17}$$

which is consistent with (3.16), since up to the order of our approximation V'' and $[(V')^2+2V''V]$ are constants. However, in most problems the approximation that V'' and $(V')^2+2V''V$ are constant over the whole of minisuperspace is too crude to yield realistic results. We may still proceed by dividing the phase space (β,κ) in stripes of bounded β. The stripes are chosen small enough such that in each region we may approximate V'' and (V'^2+2VV'') by their value at some representative point. In this way one may solve Eq. (3.17), as we will show below. One may regain some global information on f by patching the slices together in a smooth way. This corresponds to assuming a quasiadiabatic evolution.

In terms of the new variables $\xi=H/V$, and coefficients $\gamma(\beta)=[1+(V')^2/2VV'']$, $\lambda(\beta)=(2V^2/V'')$, Equation (3.17) becomes

$$\left[[\xi+\gamma(\beta)]\frac{d^2}{d\xi^2} - \lambda(\beta)\xi \right] f(\xi) = 0 \; . \tag{3.18}$$

In the case in which V and $V''>0$, there are two classically forbidden regions for f, when $\xi<(-\gamma)<0$, and for $\xi>0$. If ξ has the form $\xi=(\kappa^2/V)-1$, then for real κ we shall find $\xi>(-1)$, while we always have $(-\gamma)<-1$. So the region $\xi<(-\gamma)$ has no physical relevance. Thus three different regions exist in Eq. (3.18): the classically allowed region $(-1)<\xi<0$, the classically forbidden region $\xi>0$, and the transition region $\xi\simeq 0$.

In the first two cases we may use the WKB solutions to Eq. (3.18). For $\xi>0$,

$$f(\xi) \simeq A(\beta)[(\lambda\xi)^{-1}(\xi+\gamma)]^{1/4} \times \exp\left[(-\lambda^{1/2})\left[\xi^{1/2}(\xi+\gamma)^{1/2} + \frac{\gamma}{2}\ln\frac{(\xi+\gamma)^{1/2}-\xi^{1/2}}{(\xi+\gamma)^{1/2}+\xi^{1/2}} \right] \right] . \tag{3.19}$$

For $\xi<0$,

$$f(\xi)\simeq A(\beta)[(\lambda|\xi|)^{-1}(\gamma-|\xi|)]^{1/4}\times\sin\left[(\lambda^{1/2})\left[|\xi|^{1/2}(\gamma-|\xi|)^{1/2}-\gamma\arctan\frac{|\xi|^{1/2}}{(\gamma-|\xi|)^{1/2}}\right]\right] . \quad (3.20)$$

In the transition region, (3.18) may be approximated by

$$\left[\frac{d^2}{d\xi^2}-\left[\frac{\lambda}{\gamma}\right]\xi\right]f(\xi)=0 \quad (3.21)$$

yielding

$$f(\xi)\simeq A(\beta)\left[\left[\frac{\lambda}{\gamma}\right]^{1/3}\xi\right]^{1/2}\times K_{1/3}\left[\frac{2}{3}\left[\left[\frac{\lambda}{\gamma}\right]^{1/3}\xi\right]^{3/2}\right] , \quad (3.22)$$

where K is the usual modified Bessel function. In Eqs. (3.19), (3.20), and (3.22) we have introduced an arbitrary constant A, which as the different stripes are matched will become an arbitrary function of β. Although f cannot be determined further from the second-order adiabatic approximation, we may still get some useful information out of it: Recall that $V\simeq{}^{(3)}g\,{}^{(3)}R$, where ${}^{(3)}g$ is the determinant of the metric and ${}^{(3)}R$ is the scalar curvature of a three-dimensional slice of the physical spacetime. As we approach the singularity, V will go to zero in most (velocity dominated) models.[3,33] We assume that this is the case for $\beta\to\infty$, and that as $V\to 0$, $\lambda=O(V)$ and γ remain bounded.

Let us consider first the approach to the singularity. If $\kappa^2>0$, as $\beta\to\infty$, $\xi=H/V\sim\kappa^2/V\to\infty$, and from Eq. (3.19) we find that

$$f(\xi)\simeq e^{-\lambda^{1/2}\xi}\simeq e^{-(\kappa^2/V^{1/2})}\to 0 . \quad (3.23)$$

So the interesting region is $\kappa^2\sim 0$, where, from Eq. (3.22), $f\sim A(\beta)\sim$const. To find the asymptotic value of A as $\beta\to\infty$, we recall that from our boundary conditions, we must have

$$\int_{-\infty}^{\infty}d\kappa\, f(\beta,\kappa)=\rho(\beta,\beta)\to 1 \quad \text{as } \beta\to\infty . \quad (3.24)$$

The integral is dominated by the $\kappa\sim 0$ region, where f is given by (3.22), and we find

$$A(\beta)\sim 2CV^{-1/2}(\beta)\left[\frac{\lambda}{\gamma}\beta\right]^{1/3} \quad \text{as } \beta\to\infty , \quad (3.25)$$

where $C^{-1}=\int d\xi\, K_{1/3}(\xi)$.

In the opposite limit $\beta\to-\infty$, $V\to\infty$, $\lambda=O(V)$, γ is bounded. For fixed κ^2, we have $\xi\to-1$ as $\beta\to-\infty$, and from (3.20), f will oscillate with a frequency proportional to $\lambda^{1/2}$, which increases to ∞. On the other hand, f is always exponentially suppressed for high enough ξ. So again f will be dominated by the values of κ for which ξ remains close to 0 as $\beta\to-\infty$. The fact that the Wigner function is appreciably different from zero only for values of momentum satisfying the classical Hamiltonian constraint $H=0$ has been identified by Halliwell[23] as signaling a quantum to classical transition in quantum gravity.

We shall conclude this section by showing a concrete example to which the above analysis applies. Our example is a Bianchi type-II universe,[4] in which [cf. Eqs. (2.10) and (2.11)]

$$V=Ce^{-4\Omega}e^{4(\beta_++\sqrt{3}\beta_-)} , \quad (3.26)$$

where C is a constant. After a boost and an inversion in minisuperspace, we get $V=Ce^{-a\beta}$, where $a=4\sqrt{3}$. Clearly $\gamma=\frac{3}{2}$ and $\lambda(\beta)=2a^{-2}V(\beta)$. Observe that V,λ go to 0 or ∞ as $\beta\to\infty$ and $-\infty$, respectively, always with $\lambda=O(V)$. On the other hand, γ, being a constant, remains bounded. In this simple case, we may actually solve the Klein-Gordon equation subject to the boundary conditions (2.27). The wave function of the Universe is

$$\psi(\beta)=J_0(2C^{1/2}a^{-1}e^{-a\beta/2}) , \quad (3.27)$$

where J_0 is the usual Bessel function. The corresponding Wigner function of the Universe is

$$f(\beta,\kappa)=\int d\epsilon\, e^{-ik\epsilon}\times J_0\left[2C^{1/2}a^{-1}\exp\left[-\frac{a}{2}\left[\beta+\frac{\epsilon}{2}\right]\right]\right]\times J_0\left[2C^{1/2}a^{-1}\exp\left[-\frac{a}{2}\left[\beta-\frac{\epsilon}{2}\right]\right]\right] . \quad (3.28)$$

It is easy to show that as $\beta\to-\infty$, $f\sim\pi a\delta(\kappa^2-V(\beta))$ manifests classical behavior, which is in agreement with our previous analysis.

IV. WHEELER-DEWITT-VLASOV EQUATION IN MINISUPERSPACE

Our construction of Wigner functions in minisuperspace in Sec. III made use in an essential way of conformal flatness, and therefore cannot be extended to more general models which are not conformally flat (e.g., those with nondiagonal β_{ij}) (Refs. 3 and 4). In this section we shall show how Wigner functions can be constructed in an arbitrary curved minisuperspace, by extending the results of Refs. 21 and 27. In these general models the constraints (2.8) and (2.9) are also more complex to deal with than in the diagonal case, but we shall not consider that side of the problem here.

As in Sec. III, the main object of interest is the density matrix ρ [Eq. (3.1)] obeying the conformal Klein-Gordon equation (3.2). We would like to give a representation of ρ in the form (3.9) but we cannot do that because the original minisuperspace does not allow for a globally defined Fourier transform, nor is there a preferred conformally related manifold (e.g., flat space) one can use as reference. In particular, expressions such as $X\pm x/2$ in Eq. (3.9) are undefined. We get around this difficulty by employing particular coordinates in which we can actually add and subtract "points." Concretely, we choose an arbitrary point Q in minisuperspace and set up a Riemann normal coordinate (RNC) system with origin at

Q [which entails also choosing a d-bein $e^{\mu}_{\gamma}(Q)$, $\gamma=0,\dots d-1$]. Then we can make the ansatz[21]

$$\rho\left[X+\frac{x}{2},X-\frac{x}{2}\right]\simeq\Delta_{VM}^{1/2}(P,P')\times\int\frac{d^dk}{(2\pi)^d}\frac{e^{ikx}}{\sqrt{-g(X)}}G^{(Q)}(X,k)\ , \tag{4.1}$$

where P and P' are the points whose RNC around Q are $X\pm x/2$, $\Delta_{\rm VM}$ is the van Vleck–Morett determinant, and $k\cdot x=k_\mu x^\mu$. The price we have to pay is that $G^{(Q)}$ will in general depend on the arbitrary point Q. This dependence has been analyzed in Ref. 21. It turns out that the dependence of $G^{(Q)}$ on Q has the same order of magnitude as the fourth derivatives of the minisuperspace metric. When the metric and external potential are slowly varying, these high-order derivatives will be negligible. In what follows we shall neglect terms higher than second order in the derivatives of the metric (or potential). Defining G on non-RNC frames through the relation

$$G'(X^{\mu'},k_{\mu'})=G\left[X^\mu,\frac{\partial X^{\mu'}}{\partial X^\mu}k_{\mu'}\right]\ , \tag{4.2}$$

we may consider G as a covariantly defined object. In other words, both ρ (which is a biscalar) and G are defined in arbitrary systems, but they are related to each other in RNC through Eq. (4.1).

The ansatz (4.1) is, however, not particularly suitable for the consideration of conformally related spacetimes, since one must impose a highly nontrivial transformation law on G in order to maintain the conformal invariance of the theory, while one would expect a physical distribution function to have simple conformal properties. Instead, here we shall derive an alternative form of the conformal Wigner function which is particularly adapted to conformally related spacetime.[27,28] Let us perform a conformal transformation on a curved minisuperspace with metric $g_{\mu\nu}$ to $\tilde g_{\mu\nu}=e^{2\alpha}g_{\mu\nu}$. Correspondingly we shall have a new connection

$$\tilde\Gamma^{\mu}_{\nu\rho}=\Gamma^{\mu}_{\nu\rho}+\delta\Gamma^{\mu}_{\nu\rho} \tag{4.3}$$

whose variation is

$$\delta\Gamma^{\mu}_{\nu\rho}=\delta^{\mu}_{\nu}\alpha_{,\rho}+\delta^{\nu}_{\rho}\alpha_{,\nu}-g_{\nu\rho}g^{\mu\sigma}\alpha_{,\sigma}\ . \tag{4.4}$$

The new curvature tensors are

$$\tilde R^{\ \mu}_{\nu\rho\sigma}=R^{\ \mu}_{\nu\rho\sigma}+\delta R^{\ \mu}_{\nu\rho\sigma}\ , \tag{4.5}$$

$$\tilde R_{\nu\sigma}=R_{\nu\sigma}+\delta R_{\nu\sigma}\ , \tag{4.6}$$

$$\tilde R=e^{-2\alpha}(R+\delta R) \tag{4.7}$$

with variations

$$\delta R^{\ \mu}_{\nu\rho\sigma}=-\delta^{\mu}_{\rho}\alpha_{;\nu\sigma}+g_{\nu\rho}\alpha^{\mu}_{;\sigma}+\delta^{\mu}_{\rho}\alpha_{,\nu}\alpha_{,\sigma}-g_{\nu\rho}\alpha^{\mu}_{,}\alpha_{,\sigma}-\delta^{\mu}_{\rho}g_{\nu\sigma}(\partial\alpha)^2-(\rho\leftrightarrow\sigma)\ , \tag{4.8}$$

$$\delta R_{\nu\sigma}=-(d-2)\alpha_{;\nu\sigma}-g_{\nu\sigma}\Box\alpha+(d-2)\alpha_{,\nu}\alpha_{,\sigma}-(d-2)g_{\nu\sigma}(\partial\alpha)^2\ , \tag{4.9}$$

$$\delta R=-2(d-1)\Box\alpha-(d-1)(d-2)(\partial\alpha)^2\ . \tag{4.10}$$

In Eqs. (4.8)–(4.10) covariant derivatives are computed with the original (untransformed) connection. Observe that the Weyl tensor

$$C^{\mu}_{\nu\rho\sigma}=R^{\ \mu}_{\nu\rho\sigma}-(d-2)^{-1}(\delta^{\mu}_{\rho}R_{\nu\sigma}-\delta_{\nu\rho}R^{\ \mu}_{\sigma}-\delta^{\mu}_{\sigma}R_{\nu\rho}+\delta_{\nu\sigma}R^{\ \mu}_{\rho}) +(d-2)^{-1}(d-1)^{-1}(\delta^{\mu}_{\rho}g_{\nu\sigma}-\delta^{\mu}_{\sigma}g_{\nu\rho})R \tag{4.11}$$

remains invariant. The d-bein e^{μ}_{γ} transforms as

$$e^{\mu}_{\gamma}\to\tilde e^{\mu}_{\gamma}=e^{-\alpha}e^{\mu}_{\gamma}\ . \tag{4.12}$$

The density matrix of the Universe $\rho(x,x')$ transforms as

$$\rho\to\tilde\rho(x,x')=\exp\left[-\left[\frac{d-2}{2}\right][\alpha(x)+\alpha(x')]\right]\times\rho(x,x')\ . \tag{4.13}$$

On dimensional grounds, we expect the Wigner function f to transform as

$$f\to\tilde f=e^{2\alpha}f\ . \tag{4.14}$$

We shall assume that the function $G(X,k)$ from Eq. (4.1) is related to the Wigner function f in the functional form

$$G(X,k)=\exp\left[\mathbb{A}\left[X,k,\frac{\partial}{\partial X},\frac{\partial}{\partial k}\right]\right]f(X,k)\ . \tag{4.15}$$

$\mathbb{A}$ must be built out of elements containing no more than two derivatives of the metric (or of f), and goes to zero in the flat-spacetime limit. The possibilities are

$$\mathbb{A}=D^{\mu}_{\rho\sigma}(X)\frac{\partial^2}{\partial k_\rho\partial k_\sigma}\nabla_\mu+E^{\mu}_{\rho\sigma\lambda}(X)\frac{\partial^3}{\partial k_\rho\partial k_\sigma\partial k_\lambda}k_\mu+F_{\rho\sigma}(X)\frac{\partial^2}{\partial k_\rho\partial k_\sigma}\ , \tag{4.16}$$

where ∇_μ is the covariant derivative in phase space:

$$\nabla_\mu G=\partial_\mu G+\Gamma^{\rho}_{\mu\sigma}k_\rho\frac{\partial}{\partial k_\sigma}G\ . \tag{4.17}$$

It will prove unnecessary to consider more general forms for $\mathbb{A}$. Under a conformal transformation [Eq. (4.1) under Eqs. (4.13)–(4.15)], D goes to $\tilde D=D+\delta D$. Similarly, E goes to $\tilde E=E+\delta E$ and F goes to $\tilde F=F+\delta F$. The variations δD, δE, δF are given by

$$\delta D^{\mu}_{(\rho\sigma)}=\tfrac{1}{8}\delta\Gamma^{\mu}_{\rho\sigma}\ , \tag{4.18}$$

$$\{\delta E^{\mu}_{(\rho\sigma\lambda)}+D^{\nu}_{(\rho\sigma}\delta\Gamma^{\mu}_{\lambda)\nu}+\tfrac{1}{24}[(\delta\Gamma^{\mu}_{(\rho\lambda})_{;\sigma)}+\delta\Gamma^{\mu}_{\nu(\lambda}\delta\Gamma^{\nu}_{\rho\sigma)}]\}=0\ , \tag{4.19}$$

$$\left[\delta F_{(\rho\sigma)}-(d-2)\alpha_{;\nu}D^{\nu}_{(\rho\sigma)}-\frac{d-2}{8}\alpha_{,(\rho\sigma)}+\tfrac{1}{12}\delta R_{(\rho\sigma)}\right]=0\ , \tag{4.20}$$

where parentheses around indices denote symmetrization. In order to find a realization of Eqs. (4.18)–(4.20) it is convenient to introduce the vector

$$U_\rho=(d-1)^{-1}e^{\gamma}_{\rho}e^{\nu}_{\gamma;\nu} \tag{4.21}$$

which under a conformal transformation goes into $\tilde U_\rho=U_\rho+\alpha_{,\rho}$. Then a solution of Eqs. (4.18)–(4.20) is[27]

$$D^{\mu}_{\rho\sigma}=\tfrac{1}{8}(\delta^{\mu}_{\rho}U_\sigma+\delta^{\mu}_{\sigma}U_\rho-g_{\rho\sigma}U^{\mu})\ , \tag{4.22}$$

$$E^{\mu}_{\rho\sigma\lambda}=-\frac{1}{24(d-2)}(\delta^{\mu}_{\lambda}W_{\rho\sigma}+\delta^{\mu}_{\rho}W_{\lambda\sigma}-g_{\rho\lambda}W^{\mu}_{\sigma})\ , \tag{4.23}$$

$$F_{\rho\sigma}=\tfrac{1}{24}W_{\rho\sigma}-\frac{1}{24(d-1)}g_{\rho\sigma}R\ , \tag{4.24}$$

where

$$W_{\rho\sigma}=R_{\rho\sigma}-\frac{1}{2(d-1)}\delta_{\rho\sigma}R+3(d-2)U_\rho U_\sigma-\tfrac{3}{2}(d-2)g_{\rho\sigma}U^2\ . \tag{4.25}$$

Obviously, all metric and curvature tensors in Eqs. (4.21)–(4.25) are those of the minisuperspace, not those of the physical spacetime.

The last step is to find the Vlasov equation governing the Wigner function f by applying (4.1), (4.15), (4.16), and (4.22)–(4.24) to the Wheeler-DeWitt equation for the wave function of the Universe. These equations can be reduced to an explicitly conformally invariant form. The Wheeler-DeWitt-Vlasov transport equation reads

$$\left[k^{\mu}\bar\nabla_\mu+\tfrac{1}{2}\bar V_\mu\frac{\partial}{\partial k_\mu}\right]f=0\ , \tag{4.26}$$

where $\bar V_\mu=V_{\alpha,\mu}+2U_\mu V_\alpha$. The potential

$$V_\alpha=24\pi^2e^{-2\alpha}\,{}^{(3)}g\,{}^{(3)}R\ , \tag{4.27}$$

where ${}^{(3)}g$ and ${}^{(3)}R$ refer to a three-dimensional slice of the physical spacetime [cf. Eqs. (2.16)–(2.23)] and

$$\bar\nabla_\mu f=\nabla_\mu f-8D^{\rho}_{\mu\sigma}k_\rho\frac{\partial f}{\partial k_\sigma}-2U_\mu f\ . \tag{4.28}$$

Observe that if the d-bein e^{a}_{ρ} becomes a coordinate basis, then U_ρ would be zero and $\bar\nabla_\mu$ would reduce to a phase-space covariant derivative Eq. (4.17).

For the generalized mass-shell condition we find

$$\left[k^2-V_\alpha-\tfrac{1}{4}\bar\Box+\tfrac{1}{8}\bar V_{\mu\nu}\frac{\partial^2}{\partial k_\mu\partial k_\nu}-\tfrac{1}{12}C^{\mu\nu}_{\rho\sigma}k_\mu k_\nu\frac{\partial^2}{\partial k_\rho\partial k_\sigma}-\tfrac{1}{4}(U_{\mu;\nu}-U_{\nu;\mu})k^{\mu}\frac{\partial}{\partial k_\nu}-\frac{1}{4(d-2)}\left[\mathbb{R}_{\rho\sigma}V_\alpha\frac{\partial^2}{\partial k_\rho\partial k_\sigma}-\mathbb{R}k^{\rho}\frac{\partial}{\partial k_\rho}\right]-\frac{1}{2(d-2)}\mathbb{R}\right]f=0\ , \tag{4.29}$$

where C is the Weyl tensor and

$$\bar\Box=g^{\mu\nu}(\bar\nabla_\mu\bar\nabla_\nu+8D^{\rho}_{\mu\nu}\bar\nabla_\rho)\ , \tag{4.30}$$

$$\bar V_{\mu\nu}=\nabla_\nu\bar V_\mu+2U_\nu\bar V_\mu+8D^{\rho}_{\mu\nu}\bar V_\rho\ , \tag{4.31}$$

$$\mathbb{R}_{\rho\sigma}=R_{\rho\sigma}-\frac{1}{2(d-1)}\delta_{\rho\sigma}R+(d-2)(U_{\rho;\sigma}+U_\rho U_\sigma-\tfrac{1}{2}g_{\rho\sigma}U^2)\ , \tag{4.32}$$

$$\mathbb{R}=\mathbb{R}^{\rho}_{\rho}\ . \tag{4.33}$$

$\mathbb{R}_{\rho\sigma}$ is identically zero in a conformally flat minisuperspace, if the d-bein is chosen to be $e^{\mu}_{\gamma}=e^{-\alpha}\delta^{\mu}_{\gamma}$, and the metric is $g_{\mu\nu}=e^{2\alpha}\eta_{\mu\nu}$.

The coupled Eqs. (4.26) and (4.29) describe the dynamics of the Wigner function in the conformally invariant formalism. It should be noted that the choice of conformal coupling is arbitrary. Since we do not know of any good reason to advocate for any particular type of coupling (see Ref. 29), other options such as minimal coupling deserve equal attention. For such more general cases the covariant formulation of Ref. 21, rather than the conformal formulation presented here, is useful.

Even within the framework of the conformal approach, sometimes a particular representative of a conformally related class of metrics stands out. Such is the case, e.g., for the class of conformally flat metrics. In this situation, a specially adapted formalism may well be simpler than the fully conformally covariant approach. The formulation presented here is devised for the general case with emphasis on conformal and coordinate covariance, without singling out a particular representative metric or frame. It is in this context we believe that Eqs. (4.26) and (4.29) should prove their usefulness.

V. REMARKS

In this paper we have given a phase-space formulation of quantum cosmology in terms of the quantum distribution (Wigner) function of the Universe. The dynamics of the Wigner function is given by a Wheeler-DeWitt-Vlasov transport equation supplemented by a generalized mass-shell condition. The boundary conditions can be derived from Hartle and Hawking's proposal for the wave function of the Universe.

We have investigated the Wigner function dynamics up to the second order in a derivative expansion of the minisuperspace metric. While a first-order expression reproduces the classical evolution, the second-order expansion displays nontrivial quantum behavior. We have applied these techniques to the class of conformally flat minisuperspaces with diagonal metrics and as an example found specific solutions for the Bianchi type-II universe. We also derived the transport equations for the more general nonconformally flat models with nondiagonal metrics.

We believe the Wigner function approach has great potential in the analysis of the statistical properties of quantum states of the Universe. As for future work, in addition to the physical problems mentioned in the Introduction where our results can be usefully applied, a few theoretical problems need also be tended to. They are (i) a more accurate determination of the Hartle-Hawking or alternative boundary conditions in the density matrix-distribution function framework, (ii) a more thorough understanding of the meaning of statistical distributions in superspace, (iii) a more extended discussion of the properties of the Wheeler-DeWitt-Vlasov equation derived here, including its application to the simple but important classes of cosmological models such as the Friedmann, de Sitter, and mixmaster universes, and (iv) a formal definition of Wigner functions and their physical meaning in the full superspace. Since superspaces possesses both a metric and a connection, this is possible in principle.

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