PHYSICAL REVIEW D VOLUME 40, NUMBER 11 1 DECEMBER 1989

Comparison between two strictly local QCD sum rules

N. A. Papadopoulos and H. Vogel
Institut für Physik, Johannes-Gutenberg Universität, Staudinger Weg 7, Postfach 3980, D-6500 Mainz, West Germany
(Received 1 June 1989)

Two strictly local QCD sum rules, analytic extrapolation by conformal mapping and analytic continuation by duality, are developed and presented in full detail. They allow the extrapolation of the QCD amplitude to a single point near zero in the complex q^2 plane. Being orthogonal to the usual QCD sum rules, they present a drastic enlargement of phenomenological applications. In addition, the stability of both methods is shown explicitly, a fact which makes them particularly reliable. The difference between the two methods is illustrated in connection with the determination of the hadronic $(g-2)$ factor of the muon. Their effectiveness is demonstrated in the calculation of the topological susceptibility where both methods lead to $\chi_t^{1/4}=171\pm4$ MeV.

I. INTRODUCTION

QCD, as the theory of strong interactions, is intimately related to hadrons. In spite of this, the connection of QCD with hadrons is a very difficult task. Consequently, the connection between QCD and experiment is also extremely difficult to establish. The reason is that QCD is formulated at the level of quarks and gluons and that at present the solution of the confinement problem is still lacking. However, after the pioneering work of Shifman, Vainshtein, and Zahkarov[1] (SVZ) it was possible to deal with this problem in a satisfactory way from the phenomenological point of view, with a method which is very close to the field-theoretical framework. There are two crucial steps in this framework. The first is to include in the usual perturbative amplitude a nonperturbative contribution, corresponding to the operator-product expansion in the presence of a nonperturbative vacuum state and involving terms with inverse powers in the momentum transfer. Such a QCD amplitude continuing the perturbative and nonperturbative part is more appropriate when we start from asymptotic freedom to approach the confinement region (the hadrons). The second crucial step is to use additional mathematical methods outside of the domain of field theory to derive properties of hadrons from the asymptotic amplitude and in this way to connect the test QCD with experiment. These methods are usually called QCD sum rules, even if sometimes one uses this expression for the whole approach.

In the present paper we shall deal mainly with the second step. We develop and present a new QCD sum rule which we call, for obvious reasons, analytic extrapolation by conformal mapping (AEC). It has been applied in connection with the solution of the U(1) problem in Ref. 2, where only the result was given. However, a detailed presentation is necessary in order to put further research in a position to make use of this extremely useful sum rule. There are at least three reasons why we think this worthwhile.

First, the above sum rule AEC, together with another one, analytic continuation by duality[3] (ACD), belongs to a class of QCD sum rules which we call strictly local QCD sum rules and which differ completely from the usual QCD sum rules. They allow one to extrapolate the asymptotic QCD amplitude (Π^{QCD} valid in $1\ll|t|$) to the single point, i.e., $t=0$, where, using the usual (semiglobal or global) methods, the QCD amplitude is connected with expressions such as

$$\int_{t_1}^{t_2}\mathrm{Im}\Pi(t)w(t)dt\ ,$$

with $w(t)$ a weight function. For this reason we call the two methods AEC and ACD strictly local sum rules. Therefore, it is also obvious that the domain of applications of the strictly local does not overlap the domain of applications of the usual semiglobal or global QCD sum rules.

Second, and surprisingly enough, as we are going to show in this paper, *the strictly local QCD sum rules are stable.* That means that they do not suffer from the Hadamard instability.[4] (In the usual terminology, this is called an ill-posed problem in the Hadamard sense.) We may realize that this result is not at all trivial from the fact that of most of the semiglobal or global sum rules, also the ones proposed by SVZ (the moments and the inverse Laplace-type sum rules) it is not known if they are stable. Even more, there are examples where the Hadamard instability really appears here. [The usual stability criteria (the existence of a plateau) do not guarantee the absence of the Hadamard instability.] This may happen in the following way. If we use a Breit-Wigner resonance parametrization in order to derive the position of a physical pole and after that try again to do the same, but with a second, different parametrization with, say, one free parameter more, it may turn out that the pole position is completely differenι from the first one. In our opinion, this does not mean that all the results of semiglobal or global QCD sum rules are incorrect, but this shows that, being aware of this problem, one should be more cautious with every distinct application of the semiglobal or global sum rules.

Third, as we shall show, semilocal QCD sum rules do not require the use of resonance assumptions or parametrizations and in this sense the so obtained results are model independent. Some of our results concerning the stability are recent and therefore were not incorporated in previous physical applications of the ACD method.

The plan of this paper is as follows. In the next section we discuss the problem of low-energy extrapolation of the asymptotic amplitude on general grounds. We shall see why this is not so trivial and where the Hadamard instability comes from. In Sec. III we give a solution to this problem in the form of the AEC method and we give the proof that this strictly local QCD sum rule is stable. Our treatment will be very detailed and as we hope clear enough to be useful to those who want to apply it on various QCD problems. Some more technical points are left for Appendixes A, B, and C. In Sec. IV we treat the ACD method also from the point of view taken in Sec. II in order to facilitate a comparison of the two methods, and we also show its stability in an analogous way as for the AEC method.

In Sec. V we test the applicability and the practical use of the two methods taking a realistic model where everything is exactly known. In particular, the assumptions about the various errors are tested very carefully. We show that both methods work extremely well. In Sec. VI, using only the QCD expression, the determination of the hadronic part of the $(g-2)$ factor of the muon is obtained by the ACD method. In this case, the AEC method does not apply and the reason is explained. In Sec. VII we apply both methods in connection with the solution of the U(1) problem. We calculate the topological susceptibility χ_t and we compare it with some recent results of lattice gauge calculations. In Sec. VIII we present some conclusions. In Appendix A, we give explicitly the conformal transformation which is necessary in the AEC method and, in Appendix B, we determine the weight function C which is used in Sec. III. Appendix C concerns some useful numerical details of the calculation.

II. THE PROBLEM OF LOW-ENERGY EXTRAPOLATION IN QCD

The QCD amplitude we would like to discuss is the two-point function $\Pi(q^2)$, without loss of generality, for the electromagnetic currents:

$$(q^\mu q^\nu - g^{\mu\nu}q^2)\Pi(q^2)=\int d^4x\, e^{iqx}\langle 0|[J^\mu(x)J^\nu(0)]|0\rangle . \tag{1}$$

$\Pi(t=q^2)$ is expected to be an analytic function in the complex cut plane. The corresponding QCD amplitude $\Pi^{\rm QCD}$, as given, for example, by SVZ (Ref. 1), is an asymptotic expression valid in the complex cut plane outside of a disk with a large radius R (for $|t|\geq R$), see Fig. 1. We shall therefore denote it more generally, with $\Pi^{\rm asy}(t)$ $[\Pi^{\rm QCD}(t)=\Pi^{\rm asy}(t)]$.

What we would like to obtain from $\Pi^{\rm asy}(t)$ is its extrapolation at low energies. There are two possibilities. The extrapolation in the resonance region leads to the semi-global or global QCD sum rules.[1,5] (How the Hadamard

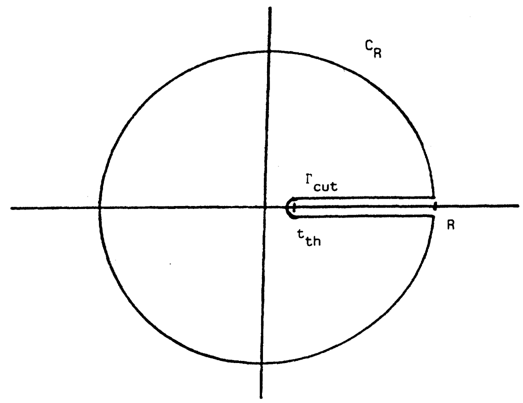

FIG. 1. The contour of integration in the complex t plane.

instability arises in the extrapolation to the resonance region and further applications beyond Refs. 1 and 5 have recently been discussed in Ref. 6.) The extrapolation *to a single point* t_0, below the cut near zero (or, e.g., $t_0=0$), leads to the strictly local QCD sum rules we are dealing with in the present paper. We shall denote the so obtained theoretical expression at t_0 by $\Pi^{\rm theor}(t_0)$. This is the expression which we have to compare with experiment in the end.

As already mentioned, our task is to obtain $\Pi^{\rm theory}(t_0)$ from given information $\Pi^{\rm asy}(t)$ on the large circle C_R, Fig. 1. Here, as in all QCD sum rules, the starting point is the Cauchy theorem or the Cauchy integral formula:

$$\Pi(t_0)=\frac{1}{\pi}\int_{t_{\rm th}}^{R}\frac{1}{t-t_0}{\rm Im}\Pi(t)dt + \frac{1}{2\pi i}\int_{C_R}\frac{1}{t-t_0}\Pi^{\rm asy}(t)dt\ . \tag{2}$$

Here we have tacitly assumed that $\Pi=\Pi^{\rm asy}$ on C_R. In order to be more general, as it will be useful in the next section, we allow the possibility of having also kernels different from $1/(t-t_0)$, we take a known auxiliary function g analytic in the cut disk. So we have, assuming $g(t_0)\neq 0$, similarly as above,

$$\Pi(t_0)=\frac{1}{g}(t_0)\left[\frac{1}{\pi}\int_{t_{\rm th}}^{R}\frac{1}{t-t_0}{\rm Im}[g(t)\Pi(t)]dt + \frac{1}{2\pi i}\int_{C_R}\frac{g(t)}{t-t_0}\Pi^{\rm asy}(t)dt\right]. \tag{3}$$

Now we are confronted with two problems: (i) Π is only known on one part C_R of the boundary $\Gamma=\Gamma_{\rm cut}+C_R$; (ii) even on C_R we have only the asymptotic expression of Π [with Π (without any index) we denote systematically the true, physical amplitude] $\Pi^{\rm asy}$ with

$$|\Pi^{\rm asy}(t)-\Pi(t)|<\epsilon\quad [\epsilon \ll 1 \text{ but } \Pi^{\rm asy}(t)\neq\Pi(t)]\ .$$

If there was only problem (i), the treatment would be straightforward: We choose a new function $C(t)$ analytic in the disk (see, e.g., Secs. III and IV for the explicit form) which allows us to obtain

$$\left| \frac{1}{\pi} \int_{t_{\rm th}}^{R} \frac{1}{t-t_0} \mathrm{Im}[C(t)g(t)\Pi(t)] \right| < \epsilon \tag{4}$$

and we have, from Eq. (3), assuming $C(t_0) \neq 0$,

$$\Pi^{\rm theor}(t_0) = \frac{1}{2\pi i} \frac{1}{C(t_0)g(t_0)} \times \int_{C_R} dt \frac{g(t)}{t-t_0} [C(t)\Pi^{\rm asy}(t)] + O(\epsilon) \ . \tag{5}$$

So it is possible to calculate $\Pi^{\rm theor}(t_0)$ with any precision $|\Pi^{\rm theor}(t_0) - \Pi(t_0)| < \epsilon$ by a good choice of $C(t)$. Different choices of the kernel $g(t)$ will lead to the same result.

The situation changes dramatically if in addition we have problem (ii). Proceeding similarly as before, we may obtain

$$\Pi^{\rm theory}(t_0) = \frac{1}{2\pi i} \frac{1}{C(t_0)g(t_0)} \times \int_{C_R} dt \frac{g(t)}{t-t_0} C(t)\Pi^{\rm asy}(t) + O(E(\epsilon)) \ . \tag{6}$$

This result is unstable in the sense of Hadamard.[4] Small changes in the input $\Pi^{\rm asy}(t)$ may give huge changes in the output $\Pi^{\rm theor}(t_0)$. [$E(\epsilon)$ may be of order much larger than 1.] Without further information on Π, no solution to this problem is possible. However, if we add some very weak phenomenological information, as the position of the threshold and some very rough estimate of Π on the cut, we can stabilize the problem. This additional information acts in the present case (to our knowledge the stability condition in the case of semiglobal or global sum rules is not known) as a filter to eliminate the unwanted and unphysical set of functions which possess similar asymptotic behavior as Π. Now the different integral formulas [for different $g(t)$] are no longer equivalent since they are sensitive to different kinds of additional information and have different efficiencies in using this information. A concrete and satisfactory solution to the present problem is given in the next sections.

III. AEC (ANALYTIC EXTRAPOLATION BY CONFORMAL MAPPING)

A. The method

This method was found by Ciulli, Pomponiu, and Sabba-Stefanescu[7] in order to extrapolate experimental scattering amplitudes into regions of momentum space which cannot be reached by experiment. Here we want to extrapolate theoretical amplitudes to regions which cannot be described by theory. Therefore, we have to bring it in a form which is appropriate for applications in QCD.

The method applies first on the unit disk (whereas our physical amplitude is given on the cut disk), where many powerful tools of complex analysis are at our disposal. It corresponds to the choice of a specific kernel, the Poisson kernel, which has the great advantage of being positive definite.

The stability of AEC can be explicitly proven if we use, in addition, the position of the threshold and an upper bound of Π on the cut. In order to apply the original method[7] we have to perform a conformal mapping from the cut disk (t plane) to the unit disk (w plane), see Fig. 2.

This can be done by five standard conformal mappings which are given explicitly in Appendix A. We shall denote the mapping of the t plane by K^{-1} since it is its inverse K which we have to use mostly [$w = K^{-1}(t)$].

We can now formulate the information we have of $\Pi(t)$ in terms of the w plane. Since we expect that $\Pi^{\rm asy}(t)$ will be a good approximation for the true amplitude $\Pi(t)$ on C_R, we have, with $\Gamma_1 = K^{-1}(C_R)$,

$$|\Pi(K(w)) - \Pi^{\rm asy}(K(w))| < \epsilon e(w) \quad \text{on } \Gamma_1 \tag{7}$$

and from the upper bound of Π on the cut we have, with $\Gamma_2 = K^{-1}(\Gamma_{\rm cut})$,

$$|\Pi(K(w))| < \mu m(w) \quad \text{on } \Gamma_2 \ . \tag{8}$$

The functions $e(w)$ and $m(w)$ should not have essential singularities and are normalized to 1 on the boundary of Γ_1 and Γ_2 (at the end points of Γ_1 and Γ_2). The constants ϵ and μ determine the scale. μ is allowed to be a large number.

After this preparation we now come to the essential part of the construction: the determination of a weight function C which allows us to "minimize our ignorance" about the function Π on the cut and the derivation of the AEC integral formula. For the function C we require the following properties: (i) C has no zeros on the whole unit disk (since it should be possible to extrapolate to every point on the unit disk); (ii) C is analytic in the unit disk; (iii) $|C(w)| = 1/e(w)$ on Γ_1, $|C(w)| = \epsilon/\mu m(w)$ on Γ_2. $C(w)$ is the so-called outer function and can be explicitly constructed from (i)–(iii) (see Appendix B):

$$C(w) \equiv \exp\left[\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\alpha \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{1}{e(e^{i\alpha})} + \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} d\alpha \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{\epsilon}{\mu m(e^{i\alpha})} \right] . \tag{9}$$

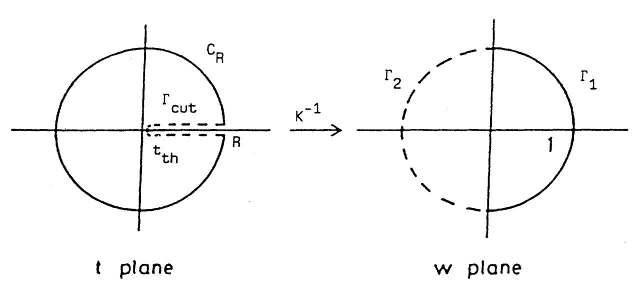

FIG. 2. The conformal mapping from the cut disk (t plane) to the unit disk (w plane).

For the product $C\Pi$ we have, from (7) and (8),

$$|C(w)\Pi(K(w))-C(w)\Pi^{\text{asy}}(K(w))|<\epsilon \quad \text{on } \Gamma_1 , \qquad |C(w)\Pi(K(w))|<\epsilon \quad \text{on } \Gamma_2 . \tag{10}$$

We see now that the product $C\Pi$ is small (of order ϵ) on the cut. The final result is obtained (cf. also Sec. II) by the use of the Poisson integral formula $[f(w_0)=\int_{\|w\|=1} dw\, P(w,w_0)f(w)]$ for the product $C\Pi$. The Poisson kernel is given by

$$P(w,w_0)=\frac{1}{2\pi}\text{Re}\frac{w+w_0}{w-w_0} \tag{11}$$

with $w\in\Gamma_1+\Gamma_2$ and $w_0=K^{-1}(t_0)$. So we have for $\Pi^{\text{theor}}(t_0)=\Pi^{\text{theor}}(K(w_0))$ the AEC integral formula

$$\Pi^{\text{theor}}(t_0)=\frac{1}{C(w_0)}\int_{\Gamma_1} dw\, P(w,w_0)C(w)\Pi^{\text{asy}}(K(w)) . \tag{12}$$

The not appropriate information on Γ_2 corresponds formally to $\Pi^{\text{asy}}|\Gamma_2=0$.

B. The stability of the method

The fact that the Poisson kernel is positive definite allows a very simple proof to show that the result is indeed stable.[7] To check the stability we consider the difference to the true value:

$$\begin{aligned}\Delta&=|\Pi(K(w_0))-\Pi^{\text{theor}}(K(w_0))|\\&=\left|\frac{1}{C(w_0)}\left[\int_{\Gamma_1+\Gamma_2} dw\, P(w,w_0)C(w)\Pi(K(w))\right.\right.\\&\qquad\left.\left.-\int_{\Gamma_1} dw\, P(w,w_0)C(w)\Pi^{\text{asy}}(K(w))\right]\right| .\end{aligned} \tag{13}$$

This can also be written as

$$\begin{aligned}\Delta=&\left|\frac{1}{C(w_0)}\right|\\&\times\left|\int_{\Gamma_1} dw\, P(w,w_0)[C(w)\Pi(K(w))-C(w)\Pi^{\text{asy}}(K(w))]\right.\\&\left.+\int_{\Gamma_2} dw\, P(w,w_0)C(w)\Pi(K(w))\right| .\end{aligned} \tag{14}$$

So we have

$$\begin{aligned}\Delta\le&\left|\frac{1}{C(w_0)}\right|\\&\times\left[\left|\int_{\Gamma_1} dw\, P(w,w_0)[C(w)\Pi(K(w))-C(w)\Pi^{\text{asy}}(K(w))]\right|\right.\\&\left.+\left|\int_{\Gamma_2} dw\, P(w,w_0)C(w)\Pi(K(w))\right|\right] .\end{aligned} \tag{15}$$

Since P is real and positive, we can write

$$\begin{aligned}\Delta\le&\left|\frac{1}{C(w_0)}\right|\\&\times\left[\int_{\Gamma_1} dw\, P(w,w_0)|C(w)\Pi(K(w))-C(w)\Pi^{\text{asy}}(K(w))|\right.\\&\left.+\int_{\Gamma_2} P(w,w_0)|C(w)\Pi(K(w))|\right] .\end{aligned} \tag{16}$$

Using the limits in (10) we obtain

$$\begin{aligned}\Delta&\le\left|\frac{1}{C(w_0)}\right|\left[\int_{\Gamma_1} dw\, P(w,w_0)\epsilon+\int_{\Gamma_2} dw\, P(w,w_0)\epsilon\right]\\&=\left|\frac{\epsilon}{C(w_0)}\right|\int_{\Gamma_1+\Gamma_2} dw\, P(w,w_0) .\end{aligned} \tag{17}$$

So we have, from $\int_{\Gamma_1+\Gamma_2} dw\, P(w,w_0)=1$,

$$\Delta\le\frac{\epsilon}{|C(w_0)|} . \tag{18}$$

This shows that the method is stable. Small changes in ϵ give rise only to small changes in the predicted $\Pi^{\text{theor}}(t_0)$.

Ciulli, Pomponiu, and Sabba-Stefanescu,[7] in addition, have proven that this sum rule is the best method in the class of integral formulas, with the addition of an upper bound on Π on the cut. Any other sum rule of the form

$$\Pi^{\text{theor}}(t_0)=\frac{1}{C(w_0)g(w_0)}\times\int_{\Gamma_1} dw\, P(w,w_0)g(w)C(w)\Pi^{\text{asy}}(K(w)) \tag{19}$$

with g analytic in the unit disk will give larger errors.

IV. ACD (ANALYTIC CONTINUATION BY DUALITY)

The second strictly local QCD sum rule we have to discuss is the ACD method. In order to facilitate the comparison between the two methods, we give a derivation of ACD which is similar in spirit to the derivation of the AEC method and the general considerations of Sec. II.

A. The method

ACD, as opposed to AEC, is formulated and derived already in the physical t plane. Additional information needed to stabilize the problem is again the position of a threshold and a rough estimate of an upper bound of $\text{Im}\Pi$ on the cut. This method is, in general, more flexible than AEC. Additional information can easily be implemented in order either to take into account specific physical requirements, as, e.g., the separate treatment of a pole part in an amplitude, if present,[3,8] or to lower the error. However, the error is expected to be in both methods of the same order. The main purpose of this and the next subsection is neither to demonstrate the flexibility of the method nor the smallness of the error, but to show the stability of the ACD method. Therefore we try to proceed as we did in the previous section. For the function $g(t)$ we take here $g(t)=1$ (which corresponds to the Cauchy kernel) and we denote the weight function by $D(t)$ instead of $C(t)$ for distinction. The method is characterized by the determination of the weight function D, chosen in such a way as to "minimize our ignorance" on the cut

$$D(t)\equiv 1-(t-t_0)\sum_{n=0}^{N} a_n t^n , \tag{20}$$

where the sum $\sum a_n t^n$ is chosen to approximate the Cauchy kernel on the cut

$$\frac{1}{t-t_0}\approx \sum_{n=0}^{N} a_n t^n, \quad t\in[t_{\rm th},R] . \tag{21}$$

This can be obtained by a least-squares fit (or equivalently by orthogonal polynomials), where the coefficients a_n are determined by the following conditions:

$$\int_{t_{\rm th}}^{R} dt \left[\frac{1}{t-t_0}-\sum_{n=0} a_n t^n\right] t^m=0 \quad \text{for } m=0,1,\ldots,N . \tag{22}$$

The function D has these properties: (i) D is analytic on the whole complex plane; (ii) $[1/(t-t_0)]D(t)$ is approximately zero on the cut; (iii) $D(t_0)=1$. The choice of the weight function D is the main difference in the AEC method. D contrary to C (in the previous section) does not contain any information on the function Π. This, although it may seem to be a disadvantage, allows more flexibility of the method. Beyond that, a new parameter N which determines the order of approximation of the kernel $1/(t-t_0)$ has entered into the game. N will play the role of a stabilizing parameter. Now, proceeding similarly as in Secs. II and III A, we obtain the ACD integral formula

$$\Pi^{\rm theor}(t_0)=\frac{1}{2\pi i}\int_{C_R} dt\frac{D(t)}{t-t_0}\Pi^{\rm asy}(t) \tag{23}$$

or explicitly

$$\Pi^{\rm theor}(t_0)=\frac{1}{2\pi i}\int_{C_R} dt\left[\frac{1}{t-t_0}-\sum_{n=0}^{N} a_n t^n\right]\Pi^{\rm asy}(t) . \tag{24}$$

B. The stability

In order to show the stability of the ACD method, we shall follow closely the steps taken in Sec. III B. We may similarly assume to have the following information on Π:

$$|\Pi(t)-\Pi^{\rm asy}(t)|<\epsilon e(t) \quad \text{on } C_R , \tag{25}$$

$$|{\rm Im}\Pi(t)|<\mu m(t) \quad \text{on the cut} . \tag{26}$$

The functions $e(t)$ and $m(t)$ and the constants ϵ and μ correspond to the functions and constants of the AEC method. We have now to determine the error which is given by

$$|\Pi(t_0)-\Pi^{\rm theor}(t_0)| . \tag{27}$$

For the amplitude $\Pi(t_0)$ we have

$$\Pi(t_0)=\frac{1}{2\pi i}\int_{C_R} dt\frac{D(t)}{t-t_0}\Pi(t)+\frac{1}{\pi}\int_{t_{\rm th}}^{R} dt\frac{D(t)}{t-t_0}{\rm Im}\Pi(t) . \tag{28}$$

Putting (23) and (28) in (27), we easily obtain the inequality

$$|\Pi(t_0)-\Pi^{\rm theor}(t_0)|<\frac{1}{2\pi}\int d\phi\, R\left|\frac{D(Re^{i\phi})}{Re^{i\phi}-t_0}\right||\Pi(Re^{i\phi})-\Pi^{\rm asy}(Re^{i\phi})|+\frac{1}{\pi}\left|\int_{t_{\rm th}}^{R} dt\frac{D(t)}{t-t_0}{\rm Im}\Pi(t)\right| \le \Delta^{\rm asy}+\Delta^{\rm fit} . \tag{29}$$

There are two different contributions to the total error: One, which we call the asymptotic error ($\Delta^{\rm asy}$), comes from the circle C_R and the other, which we call the fit error ($\Delta^{\rm fit}$), from the integral on the cut. Taking into account the limits in Eqs. (25) and (26), we have, for the two errors,

$$\Delta^{\rm asy}(N)\equiv\epsilon\frac{1}{2\pi}\int_0^{2\pi} R\, d\phi\left|\frac{1}{Re^{i\phi}-t_0}-\sum_{n=0}^{N} a_n R^n e^{in\phi}\right| \times e(Re^{i\phi}) , \tag{30}$$

TABLE I. Fit coefficients for $1/t=\sum_{n=0}^{N} a_n t^n$ in the interval $0.16\le t\le 2$.

N	a_0	a_1	a_2	a_3	a_4	a_5
1	3.219 651	−1.710 159	0	0	0	0
2	5.077 822	−6.249 119	2.101 370	0	0	0
3	6.905 627	−13.936 518	10.427 741	−2.569 868	0	0
4	8.710 920	−24.823 696	30.098 908	−16.116 911	3.135 890	0
5	10.502 856	−38.896 917	66.598 818	−57.101 783	23.774 409	−3.821 948

$$\Delta^{\mathrm{fit}}(N) \equiv \mu \frac{1}{\pi} \int_{t_{\mathrm{th}}}^{R} dt \left| \frac{1}{t-t_0} - \sum_{n=0}^{N} a_n t^n \right| m(t) . \quad (31)$$

In order to show the stability, we have to study the role of the stabilizing parameter N. Particularly we have to show (i) that the total error $\Delta(N) = \Delta^{\mathrm{asy}}(N) + \Delta^{\mathrm{fit}}(N)$ tends to zero if we use the exact function Π, i.e., for $\epsilon = 0$, (ii) and that $\Delta(N)$ reaches a minimum of a specific $N = N_{\mathrm{opt}}$.

The first requirement is easy to see. If $\epsilon = 0$, then $\Delta^{\mathrm{asy}}(N) = 0$ and since we can approximate $1/(t-t_0)$ in $[t_{\mathrm{th}}, R]$ to any accuracy by a polynomial, $\Delta^{\mathrm{fit}}(N)$ tends to zero, too.

The second point requires lengthy calculations, since the coefficient a_n cannot be written easily in terms of t_{th}, R, and t_0. So we prefer to illustrate the situation on a simple but realistic example. We take $t_0 = 0$, $e(t) = 1$, $m(t) = 1$, and $\epsilon/\mu = 10^{-4}$. These choices are normally satisfied in QCD, even if for Π^{asy} we take only the first few terms in the operator-product expansion, as it is usually done.

For $t_{\mathrm{th}} = 0.16$ and $R = 2$ the coefficients a_n are given in Table I. The corresponding fit errors, asymptotic errors, and total errors are plotted in Fig. 3. As we see, the total error reaches its minimum at $N = N_{\mathrm{opt}} = 4$. This N corresponds nearly to the number of known terms in Π^{asy} which was also confirmed by our experience. We regard the above results, summarized in Fig. 3, as a manifestation of the stability in ACD.

In practice, the total error can be lowered by making use of more phenomenological information. For every slowly varying $\mathrm{Im}\Pi(t)$, the quantity $\Delta^{\mathrm{fit}}(N_{\mathrm{opt}})$ is tiny; in fact, as visible in Eq. (22),

$$\frac{1}{\pi} \int_{t_{\mathrm{th}}}^{R} dt \left[\frac{1}{t-t_0} - \sum a_n t^n \right] \mathrm{Im}\Pi(t)$$

vanishes exactly to the extent that $\mathrm{Im}\Pi(t)$ can be represented by polynomials of degree $M < N$.

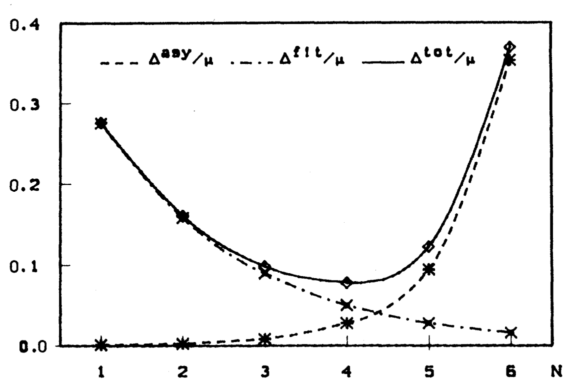

FIG. 3. The errors of the ACD method are given in dependence of the degree N of the fit polynomial. This behavior is in general assuming the scale of the amplitude on the cut (μ) is 10^4 times larger than the accuracy of the asymptotic expansion (ϵ). The curves have no meaning, they should only guide the eye.

V. COMPARISON BETWEEN AEC AND ACD ON AN EXPLICITLY KNOWN MODEL AMPLITUDE

In this section we would like to treat our problem starting from the asymptotic expression taken from an explicitly known function. This allows us to study the quality of both methods since the result is already known, the various errors are precisely calculable, and it is possible to test precisely any further assumption made in the previous section.

The amplitude we want to discuss is given by

$$f(t) = (1-v^2)\frac{1}{2v} \ln \left[\frac{v-1}{v+1} \right]$$
$$\text{with } v = \left[1 - \frac{4m^2}{t} \right]^{1/2} . \quad (32)$$

It is taken from a vertex graph in QED (Ref. 9). It has a cut on the complex t plane at $t > t_{\mathrm{th}} = 4m^2$ and represents very well, as a model, the physical amplitude Π we are actually interested in. The starting point for our investigation is its asymptotic expression for $1 \ll |t|$. We can easily obtain from (32) the first p terms

$$f^{\mathrm{asy}}(t) = \sum_{k=0}^{p} (m^2)^{k+1} \frac{1}{t^{k+1}} (f_k \ln - t/m^2 + g_k) . \quad (33)$$

The coefficients f_k and g_k are given for $k = 0,1,2,3,4$ correspondingly by

f_k	-2	-4	-12	-40	-140
g_k	0	4	14	$\frac{148}{3}$	$\frac{533}{3}$

In the realistic situation, only the terms with $p = 2$ or 3 are known. We have taken here one more term in order to test some assumptions about the error estimate.

Our aim is to determine the value of f at $t = 0$ from the asymptotic expression (33):

$$f^{\mathrm{theor}}(0) . \quad (34)$$

We know of course from (32) that

$$f(0) = 1 . \quad (35)$$

We fix $m^2 = 0.04$ and we take the radius R, where f^{asy} should be valid, of order $10t_{\mathrm{th}}$ or more.

We are coming now to the explicit calculations with AEC and with ACD.

A. $f^{\mathrm{theor}}(0)$ from AEC

In order to apply the results of Sec. II A, we first have to know the upper limit corresponding to Eqs. (7) and (8):

$$|f(K(w)) - f^{\mathrm{asy}}(K(w))| < \epsilon e(w) \text{ on } \Gamma_1 , \quad (36)$$

$$|f(K(w))| < \mu m(w) \text{ on } \Gamma_2 . \quad (37)$$

For the first condition, since we want to test the method, we do not take $f - f^{\mathrm{asy}}$ as it would be possible from Eqs. (32) and (33), but we make similar approximations as in the practical situation. From the asymptotic expression

(33) with $p=2$ or 3, taking $e(t)=1$ in order to facilitate the numerical computation, we have, for ϵ from the f^{asy} $(k=p+1)$ term $(p=3$ or 4) with coefficients, e.g., a factor 2 larger than the order of magnitude of the known terms f_k and g_k (for $k \le p$), for $R=1.5$ $(R/t_{\text{th}} \approx 10)$,

$$\epsilon = 1.0\times 10^{-4} \quad (\text{for } p=2)$$

and (38)

$$\epsilon = 1.0\times 10^{-5} \quad (\text{for } p=3)\ .$$

We can infer from $f - f^{\text{asy}}$, as Fig. 4 shows, that this estimate of the asymptotic error is a very good choice.

In this particular example, as opposed to the physical situation, we have to take into account for the upper bound of the amplitude of f on the cut [corresponding to Eq. (37)], the presence of a pole of f at the threshold $t=4m^2$. We therefore have

$$\mu m(t) = \frac{K_1}{t\sqrt{1-4m^2/t}} + K_2\ . \tag{39}$$

The first term represents the imaginary part of f on the cut, K_1 and K_2 are constants. K_2 corresponds to the real part of f. With this information using (9), (12), and (18), we obtain, for $R=1.5$,

$$\begin{aligned} p=2,\quad & f^{\text{theor}}(0) = 1.004 \pm 0.042\ , \\ p=3,\quad & f^{\text{theor}}(0) = 1.004 \pm 0.016\ . \end{aligned} \tag{40}$$

The corresponding results for various values of R are given in Fig. 5.

B. $f^{\text{theor}}(0)$ from ACD and comparison

For the calculation of the numerical value of $f^{\text{theor}}(0)$, it is sufficient to use f^{asy} only [see (24)]. Here too, we

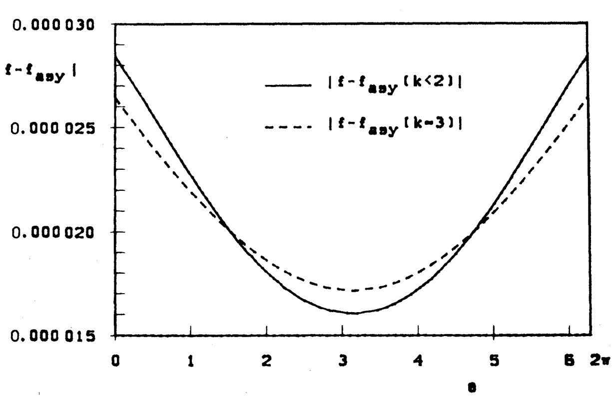

FIG. 4. The difference of the real function and the asymptotic expansion $(k \le 2)$ as a function of the angle which parametrizes the circle in the cut plane is compared with the next-order term of the asymptotic expansion.

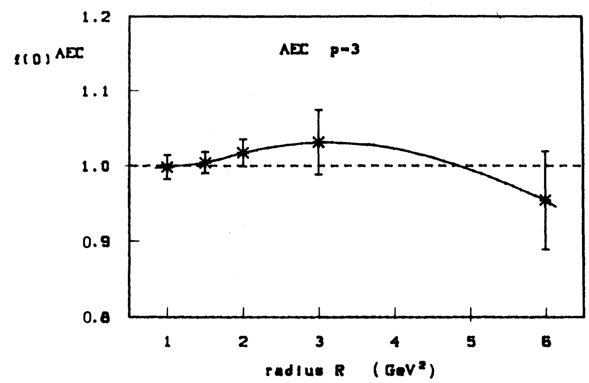

FIG. 5. AEC results as a function of the radius R.

need, of course, additional information on f in order to give an estimate of the error. The determination of the error itself is not less important. We therefore start with the discussion of the various errors appearing in the ACD method. For the determination of the fit error we may use f^{asy} $(R,k=p+1)$ (see the discussion in Sec. V A and Fig. 4) and we have

$$\Delta_N^{\text{asy}}(R,p) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, Re^{i\phi} |f^{\text{asy}}(Re^{i\phi}, k=p+1)|\ . \tag{41}$$

For the fit errors we have two possibilities. We may first use similar information of f corresponding to an upper limit of f on the cut

$$\Delta_{N,1}^{\text{fit}}(R) = \frac{1}{\pi}\int_{t_{\text{th}}}^{R} dt \left| \left\{ \frac{1}{t} - \sum_{n=0}^{N} a_n t^n \right\} \right| |\text{Im} f(t)|\ , \tag{42}$$

or we may use some information coming from the specific form of the ACD method. In this case we have

$$\Delta_{N,2}^{\text{fit}}(R) \equiv \left| \frac{1}{\pi}\int_{t_{\text{th}}}^{R} dt \left\{ \frac{1}{t} - \sum_{n=0}^{N} a_n t^n \right\} \text{Im} f(t) \right|\ . \tag{43}$$

For $\text{Im} f$ we take

$$\mu m(t) = |\text{Im} f(t)| = K_1 \frac{1}{t(1-4m^2/t)^{1/2}} \tag{44}$$

as in (39). The fit error $\Delta_{N,1}^{\text{fit}}(R)$ (the AEC-like error) leads to a total error which is larger than the error in the AEC method, as expected (since AEC leads to the minimal possible error). The second fit error $\Delta_{N,2}^{\text{fit}}(R)$ leads to a total error which is of the same order as the one in the AEC method. It turns out that even with less information on $\text{Im} f$, putting $\text{Im} f = \text{const}$, the corresponding fit error is of the same order of magnitude as above. A comparison of the two (total) errors Δ_1 and Δ_2 is given in the following $(p=3)$:

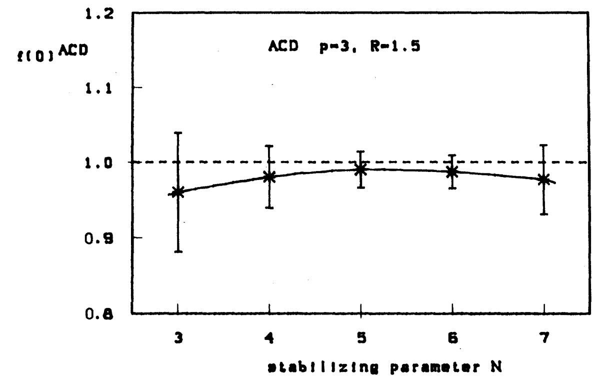

FIG. 6. ACD results as a function of the degree N of the fit polynomial.

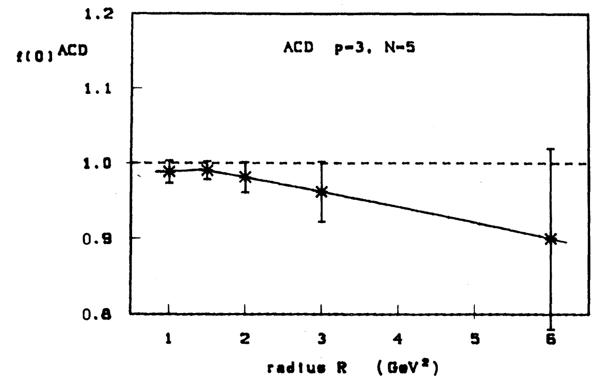

FIG. 7. The same as in Fig. 6, but as a function of R.

N	2	3	4	5	6	7	8	
Δ_1	0.15	0.079	0.042	0.024	0.022	0.046	0.20	$R=1.5$
Δ_2	0.092	0.042	0.020	0.012	0.015	0.042	0.15	
Δ_1	0.34	0.23	0.16	0.11	0.073	0.041	0.042	$R=3.0$
Δ_2	0.25	0.15	0.095	0.060	0.038	0.025	0.030	

(45)

Calculating now $f^{\text{theor}}(0)$ with the ACD integral formula (24) and the error $\Delta=\Delta^{\text{fit}}+\Delta^{\text{asy}}$, we obtain, for $p=2$, $R=1.5$, and $N_{\text{opt}}=4$ (see Fig. 6),

$$f^{\text{theor}}(0)=0.990\pm0.030(\pm0.051) \tag{46a}$$

and for $p=3$, $R=1.5$, and $N_{\text{opt}}=5$,

$$f^{\text{theor}}(0)=0.990\pm0.012(\pm0.024)\ . \tag{46b}$$

The error Δ_1 is given in the parentheses. For various values of R the result is given in Fig. 7.

To compare it with AEC, the result of the previous subsection is again displayed:

$$p=2,\quad f^{\text{theor}}(0)=1.004\pm0.042,$$

$$p=3,\quad f^{\text{theor}}(0)=1.004\pm0.016\ .$$

As we see, the two methods lead to completely similar results; the same is also valid for the error estimates.

VI. ON THE HADRONIC PART OF THE MUON'S $(g-2)$ FACTOR

The anomalous magnetic moment of the muon is an example where the strictly local sum rules in its flexible form ACD, allow one to test the full $U(1)\times SU(2)\times SU(3)$ theory, without using the e^+e^- experimental data. This is indeed possible since starting from the QCD amplitude Π^{QCD} given in Refs. 1 and 10 and using ACD, the crucial low-energy contribution to the integral for the hadronic part of the muon anomaly a_H,

$$a_H=\frac{4\alpha^2}{\pi}\int_{4m^2}^{\infty}dt\frac{g(t)}{t}\text{Im}\Pi\ , \tag{47}$$

can be calculated without use of any e^+e^- data.[10] This has been discussed in short in Ref. 11 in connection with the W and Z mass shift where by the same reasoning also the full $U(1)\times SU(2)\times SU(3)$ theory can be tested without use of e^+e^- data.

Here we would like to discuss the muon anomaly from the point of view of comparison between AEC and ACD. It turns out that the AEC method cannot be applied to this problem whereas the ACD method is flexible enough to succeed. We shall first show how this can be achieved within the ACD framework. The essential point is the calculation of the low-energy integral

$$\int_{t_{\text{th}}}^{R}dt\frac{g(t)}{t}\text{Im}\Pi(t) \tag{48}$$

using the asymptotic amplitudes $\Pi^{\text{asy}}=\Pi^{\text{QCD}}$, valid outside of a disk of radius R. The function $g(t)$ is given by

$$g(t)=\int_0^1 dx\frac{x^2(1-x)}{x^2+t/(1-x)}\ . \tag{49}$$

For $4m_\mu^2<t$ (m_μ is the muon mass) and $t<0$, this can be written as

$$g(t)=\tfrac{1}{2}v^2(2-v^2)+\frac{1}{v^2}(1+v)^2(1+v^2)[\ln(1+v)-v+\tfrac{1}{2}v^2]+\frac{1+v}{1-v}v^2\ln v \tag{50}$$

with

$$v=\frac{1-(1-4m_\mu^2/t)^{1/2}}{1+(1-4m_\mu^2/t)^{1/2}} .$$

So we have

$$\lim_{|t|\to\infty} g(t)\sim\frac{1}{3}\frac{m_\mu^2}{t}+O\left[\frac{m_\mu^4}{t^2}\ln\left[\frac{m_\mu^2}{t}\right]\right] . \tag{51}$$

For $\mathrm{Im}t<0$, $g(t)$ develops a cut on the real negative axis and we may think that essentially

$$g(t)\sim\ln(t) \tag{52}$$

is valid. For this reason we can apply neither the AEC nor the ACD integral formulas on the original form. That means that we cannot write

$$\Pi^{\text{theor}}(0)g(0)=\frac{1}{2\pi i}\int_{C_R}\frac{1}{t}\Pi^{\text{asy}}(t)g(t) . \tag{53}$$

In the following we shall show that with QCD it is possible to overcome this difficulty but not with the AEC method. Within ACD we approximate not $1/t$, but the function $g(t)/t$ in the interval $[t_{\text{th}},R]$ by a polynomial

$$\frac{1}{t}g(t)=\sum_{n=0}^{N}a_n t^n \quad \text{for } t\in[t_{\text{th}},R] . \tag{54}$$

So we have first

$$\int_{t_{\text{th}}}^{R}dt\frac{1}{t}g(t)\mathrm{Im}\Pi(t)=\int_{t_{\text{th}}}^{R}dt\left[\sum_{n=0}^{N}a_n t^n\right]\mathrm{Im}\Pi(t) \tag{55}$$

and using the Cauchy theorem for the expression on the right-hand side we obtain

$$\frac{1}{\pi}\int_{t_{\text{th}}}^{R}dt\frac{1}{t}g(t)\mathrm{Im}\Pi(t)=\frac{1}{2\pi i}\int_{C_R}\left[\sum_{n=0}^{N}a_n t^n\right]\Pi^{\text{asy}}(t) . \tag{56}$$

This constitutes the solution of the problem within ACD (Ref. 10). The predicted value for the anomaly magnetic moment is, within an error of order 20%, in excellent agreement with experiment.[12]

In connection with the AEC method such a solution is not possible. The reason is the cut in the function $g(t)\sim\ln(t)$. Here we would have to use expressions such as

$$\int_{C_R}dt\frac{1}{t}g(t)\Pi^{\text{asy}}(t)\sim\int_{C_R}dt\frac{1}{t}\ln(t)\Pi^{\text{asy}}(t) . \tag{57}$$

This means that the integral

$$\int_{\text{th}}^{R}dt\frac{1}{t}g(t)\Pi(t) \tag{58}$$

corresponds to an infinite sum of all derivatives of Π at zero $[\Pi^{(K)}(0)]$. Since the knowledge of only finite terms is not sufficient, we understand why the AEC method does not work in this case.

VII. THE TOPOLOGICAL SUSCEPTIBILITY AND THE U(1) PROBLEM

In the last few years it was generally accepted that the solution of the U(1) problem is directly connected with a nonvanishing topological susceptibility.[13,14] In the limit $N_c\to\infty$ the unexpected large mass of the η' meson is proportional to the topological susceptibility of a pure Yang-Mills theory, i.e., a theory without quarks.

The topological susceptibility is defined as

$$\chi_t=U(q^2=0)=-i\int d^4x\, e^{iqx}\langle 0|T(Q(x),Q(0))|0\rangle_{q^2=0} , \tag{59}$$

where

$$Q(x)=\frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^{a}(x)F^a_{\rho\sigma}(x)$$

is the topological charge density of the pure Yang-Mills theory. Using chiral Ward identities one can obtain the Witten-Veneziano formula in the limit $N_c\to\infty$ (Ref. 14),

$$\chi_t=\frac{f_\pi^2}{2N_f}(m_\eta^2+m_{\eta'}^2-2m_K^2) , \tag{60}$$

taking into account meson mixing as well. For $N_f=3$ we have the phenomenological value

$$\chi_t=(180\ \text{MeV})^4 .$$

A theoretical calculation of χ_t has been lacking for a long time. From (59) we see that we need the value of the two-point function $U(q^2)$ at zero momentum transfer. Apart from lattice theory the only methods for performing calculations in this momentum region are strictly local QCD sum rules.

We found, using AEC and ACD QCD sum rules,

$$\chi_t^{1/4}=171\pm4\ \text{MeV},\quad \text{AEC method} ,$$

and

$$\chi_t^{1/4}=170\pm6\ \text{MeV},\quad \text{ACD method} ,$$

in good agreement with the phenomenological value. To perform the calculation we take the asymptotic expansion of $U(q^2)$ for $q^2\to\infty$ from Ref. 15 and use the factorization assumption. Furthermore, we drop the perturbative part, since the topology susceptibility is a pure nonperturbative effect[13] and explicit computation gives zero contribution. Then

$$U_{\text{asy}}(q^2)=\left[-\frac{9}{11}+\frac{1}{q^2}\rho_c^{-2}\frac{216}{55}\right]\frac{1}{\ln(-q^2/\Lambda_{\text{YM}}^2)}\times\left\langle\frac{\alpha_s}{\pi}FF\right\rangle_{\text{YM}}-\frac{1}{q^4}135\frac{\pi^2}{88}\frac{1}{\ln(-q^2/\Lambda_{\text{YM}}^2)}\left\langle\frac{\alpha_s}{\pi}FF\right\rangle_{\text{YM}} \tag{61}$$

with $\rho_c = 200$ MeV the instanton radius.

In (61) we need the values of the scale parameter $\Lambda_{\rm YM}$ and the gluon condensate $\langle (\alpha_s/\pi)FF\rangle_{\rm YM}$ of the pure Yang-Mills theory. The corresponding amplitude of real QCD, i.e., Yang-Mills theory with quarks, differs only in the values of these parameters.

$\langle (\alpha/\pi)FF\rangle_{\rm YM}$ is expected to be quite different from the usual gluon condensate. It has been estimated by Novikov, Shifman, Vainshtein, and Zakharov[16] using the relation $\langle (\alpha/\pi)FF\rangle_{\rm YM} = \langle (\alpha/\pi)FF\rangle_h$ with $\langle (\alpha/\pi)FF\rangle_h$ corresponding to the QCD theory where all quarks are taken heavy. They propose

$$\left\langle \frac{\alpha_s}{\pi}FF\right\rangle_{\rm YM} \approx 3\left\langle \frac{\alpha_s}{\pi}FF\right\rangle_{\rm QCD} = 0.035\ \mathrm{GeV}^4 .$$

For $\Lambda_{\rm YM}$ it is reasonable to take $\Lambda_{\rm QCD} \approx 150$ MeV. Additionally, we need for the application of strictly local QCD sum rules the position of the threshold, an upper bound for $U(q^2)$ on the cut of the radius R. For the threshold we take the mass of the first pseudoscalar glueball candidate $\iota(1460)$ (Refs. 17 and 18):

$$t_{\rm th} = m_\iota^2 = 2.13\ \mathrm{GeV}^2 .$$

As an upper bound for $U(q^2)$ on the cut we use the very large constant

$$\mu m(q^2) = 0.1\ \mathrm{GeV}^4 .$$

The dependence on the upper bound is very weak. For $R = 20\ \mathrm{GeV}^2$ it is now an easy task to get from (19) to (24) within the computational error

$$\chi_t^{1/4} = 171 \pm 4\ \mathrm{MeV},\quad \mathrm{AEC} ,$$

$$\chi_t^{1/4} = 170 \pm 6\ \mathrm{MeV},\quad \mathrm{ACD} .$$

The dependence on the threshold as given in Fig. 8 is quite weak. The dependence on Λ is exhibited in Fig. 9. For $\Lambda = 0.15 \pm 0.05$ GeV this dependence also is relatively weak. The gluon condensate dependence is expected to be linear Eq. (61). In Fig. 10 the topology susceptibility is sketched against the QCD gluon condensate assuming

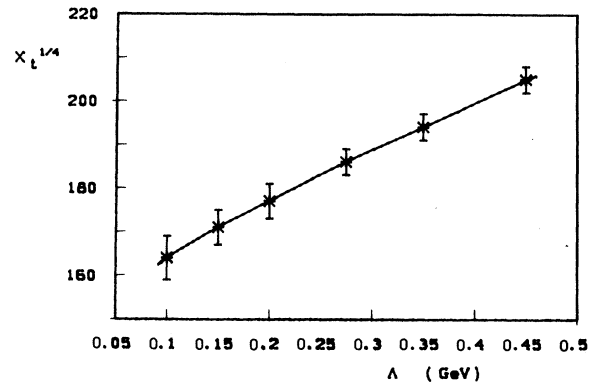

FIG. 9. The topological susceptibility as a function of Λ.

the relation $\langle (\alpha/\pi)FF\rangle_{\rm YM} = 3\langle (\alpha/\pi)FF\rangle$. The topology susceptibility increases with a larger gluon condensate. In spite of this we prefer to keep the smaller value, since the topology susceptibility of real QCD, i.e., $\langle (\alpha/\pi)FF\rangle_{\rm YM}$ substituted by $\langle (\alpha/\pi)FF\rangle_{\rm QCD}$ in the calculation, should be very close to zero.[13]

At this point it may be interesting to give also a comparison with some lattice-gauge-theory results.[19–24] The computation of the topology susceptibility may be one of the most reliable results of lattice gauge theory since no quenched approximation is necessary.

The first calculation gave[19]

$$\chi_t^{1/2} = 55 \pm 10\ \mathrm{MeV} ,$$

but more recent ones lead to larger values: e.g.,

$$\chi_t^{1/4} = 247 \pm 28\ \mathrm{MeV}\ \ (\mathrm{Ref.}\ 19) ,$$

$$\chi_t^{1/4} = 221 \pm 13\ \mathrm{MeV}\ \ (\mathrm{Ref.}\ 20) .$$

On the other hand, calculations which use cooling yield, e.g.,

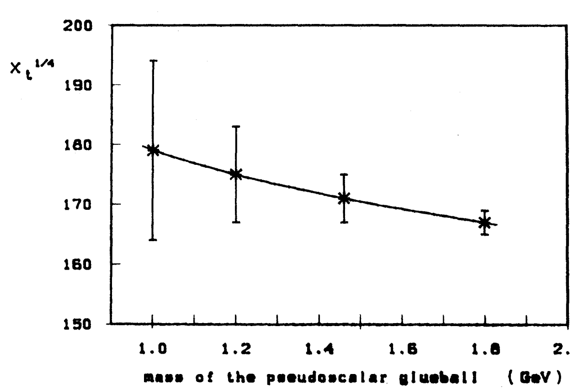

FIG. 8. The dependence of the topological susceptibility on the mass of the lightest pseudoscalar glueball corresponding to the threshold.

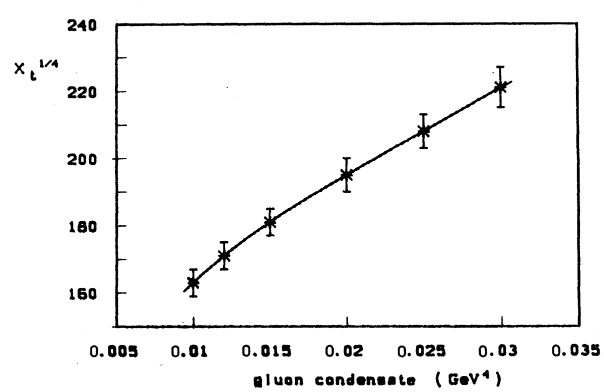

FIG. 10. The topological susceptibility as a function of the QCD gluon condensate assuming the value of the pure gauge theory's gluon condensate is three times the value of the QCD gluon condensate.

$$\chi_t^{1/4}=190\pm 7 \text{ MeV} \quad (\text{Ref. } 21) ,$$

$$\chi_t^{1/4}=146\pm 11 \text{ MeV} \quad (\text{Ref. } 22) .$$

So strictly local QCD sum rules enable at least as good results as lattice gauge theory.

Another interesting bit of insight into the nature of U(1)-symmetry breaking is only possible using strictly local QCD sum rules. Taking Eq. (61) up to order $1/q^2$ and neglecting the contribution of the light quarks in the Witten-Veneziano formula, we obtain for the η' mass the symmetry-breaking expression

$$m_{\eta'}^2 f_{\eta'}^2 = c\frac{1}{N_c}\left\langle \frac{\alpha_s}{\pi}GG\right\rangle \quad \text{with } c=0.27\pm 0.06$$

which is in close analogy to the famous symmetry-breaking expression for the other pseudoscalar mesons: e.g.,

$$m_\pi^2 f_\pi^2 = -2m_q\langle \bar{q}q\rangle .$$

VIII. CONCLUSIONS

The strictly local sum rules AEC and ACD have been presented. They allow one to extrapolate the asymptotic QCD amplitude to a single point below the cut in the q^2 plane, at $q^2=0$ or $q^2<0$. This by itself presents a drastic enlargement of the domain of applications within the framework of QCD sum rules. In addition, two salient properties of the strictly local QCD sum rules have appeared till now which may prove to be extremely useful for future phenomology applications: First, to obtain from the asymptotic QCD amplitude its value at $q^2=0$, very weak phenomenological information is needed, less than in the usual (global or semiglobal) QCD sum rules. Second, as we have shown in this paper, the inverse problem which is connected with every extrapolation of the asymptotic QCD amplitude to low energies, does not cause any difficulties in our case. That means that Hadamard instability does not appear in the strictly local QCD sum rules and therefore these sum rules have a reliable mathematical basis and the calculations can be trusted.

Both methods, AEC and ACD, have been discussed in a very detailed way, generally and in concrete examples. They have been tested by the help of a generic function which is exactly known and, of course, its asymptotic expression. A physical example which also shows the difference between AEC and ACD is the determination with ACD from QCD of the hadronic part of the $(g-2)$ factor of the muon without the use of any e^+e^- data. A second physical example which shows the effectiveness and reliability of the strictly local QCD sum rules is the determination from QCD by analytic methods of the topological susceptibility χ_t, an observable which is usually believed to be only accessible to lattice gauge calculations.

ACKNOWLEDGMENTS

We wish to thank S. Ciulli, G. Menessier, N. F. Nasrallah, F. Scheck, and K. Schilcher for numerous and stimulating discussions. This work was supported by the PROCOPE program of the Deutscher Akademischer Austauschdienst (DAAD).

APPENDIX A: THE CONFORMAL MAPPING

First we want to consider in detail the conformal mapping which transforms the integration contour in the physical cut plane to the unit circle. This conformal mapping, denoted with K^{-1}, is built of five well-known conformal mappings.

Shrinking of the radius R to 1:

$$t \to t/R .$$

Enlarging the cut to the origin [Fig. 11(a)]:

$$t \to \frac{t_{\text{th}}/R - t}{t t_{\text{th}}/R - 1} .$$

The cut disk to the upper half disk [Fig. 11(b)]:

$$t \to \sqrt{t} .$$

The half disk to the upper half plane [Fig. 11(c)]:

$$t \to \left[\frac{t+1}{t-1}\right]^2 .$$

The upper half plane to the unit disk [Fig. 11(d)]:

$$t \to \frac{t-i}{it-1} .$$

In practice we need the inverse mapping K in a very condensed form to save computer time. The best way is to treat the mapping K_1 from the right of the unit circle Γ_1 to the circle in the physical cut plane and the mapping K_2 from the left of the unit circle Γ_2 to the cut separately. Then we can represent both mappings as real functions in the following form.

$K_1:[-\pi/2,\pi/2]\to[0,2\pi]$,

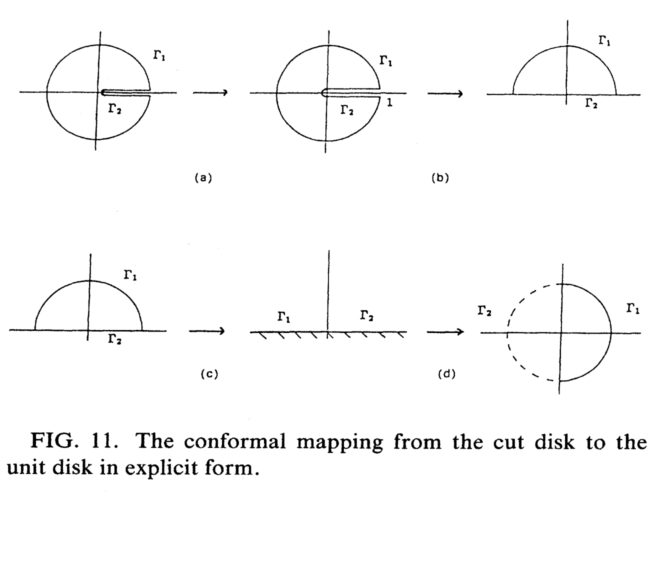

FIG. 11. The conformal mapping from the cut disk to the unit disk in explicit form.

$$\theta \rightarrow \arctan \frac{4[1-\cos\theta/(1+\sin\theta)]\sqrt{\cos\theta/(1+\sin\theta)}(t_{\text{th}}^2-1)}{[1-\cos\theta/(1+\sin\theta)]^2(t_{\text{th}}+1)^2-4\cos\theta/(1+\sin\theta)(t_{\text{th}}-1)^2} + \begin{cases} 0 & \text{for} -\pi/2 \le \theta \le -\theta_0 , \\ \pi & \text{for } -\theta_0 \le \theta \le \pi/2-\theta_0 , \\ 2\pi & \text{for } \pi/2-\theta_0 \le \theta \le \pi/2 , \end{cases} \tag{A1}$$

where θ_0 is the zero of the denominator.

K_2:$[\pi/2, 3\pi/2] \rightarrow [t_{\text{th}}, R]$,

$$\theta \rightarrow r \frac{t_{\text{th}}(1+v)^2+R(1-v)^2}{t_{\text{th}}(1-v)^2+R(1+v)^2}$$

$$\text{with } v = \left[\frac{-\cos\theta}{1+\sin\theta} \right]^{1/2} . \tag{A2}$$

So we have

$$K(e^{i\theta}) = \begin{cases} Re^{iK_1(\theta)} & \text{for } -\pi/2 \le \theta \le \pi/2 , \\ K_2(\theta) & \text{for } \pi/2 \le \theta \le 3\pi/2 . \end{cases} \tag{A3}$$

APPENDIX B: CONSTRUCTION OF THE WEIGHT FUNCTION C

In Sec. III we have introduced the weight function C, the so-called outer function. It is possible to construct C from the following three conditions: (1) C has no zeros on the whole unit disk (since it should be possible to extrapolate to every point on the unit disk); (2) C is analytic; (3) $|C(w)|=1/e(w)$ on Γ_1, $C(w)=\epsilon/\mu m(w)$ on Γ_2. From (i) and (ii) we know that also $\ln C$ is analytic in the unit disk. Now we consider the function $C'=\ln C$. We know from (iii) the real part of C' ($\text{Re}C'=\ln C$) on the boundary of the unit disk. Then we use the Schwarz-Villat formula, which produces the analytic function C' on the whole unit disk from its real part $\text{Re}C'$ on the unit circle and we get

$$C' \equiv \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\alpha \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{1}{e(e^{i\alpha})} + \frac{1}{2} \int_{\pi/2}^{3\pi/2} d\alpha \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{\epsilon}{\mu m(e^{i\alpha})} , \tag{B1}$$

the desired result.

APPENDIX C: TECHNIQUES OF NUMERICAL INTEGRATION

During the calculation of the final integral (12) in Sec. III there arise difficulties due to the discontinuity of the weight function C (Ref. 25), so we cannot use standard numerical integration techniques here.

Writing the Cauchy-Riemann equations for $\ln C(w)$ in natural coordinates on the unit disk

$$\frac{\partial\, \text{Im}(\ln C)}{\partial s} = -\frac{\partial \ln|C|}{\partial n_i} \quad (ds = d\theta,\ dn_i = -dr) \tag{C1}$$

tells us that in the vicinity of the discontinuity points $\theta = \pm\pi/2$ of the modulus of C on Γ, the phase of C varies extremely quickly. Therefore, the conventional numerical integration techniques cannot be used here. To solve the problem we consider the explicit form of the function C:

$$C(w) = \exp\left[\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{1}{e(e^{i\alpha})} d\alpha + \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{1}{m(e^{i\alpha})} d\alpha + \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \frac{e^{i\alpha}+w}{e^{i\alpha}-w} \ln \frac{\epsilon}{\mu} d\alpha \right] . \tag{C2}$$

We see the discontinuity comes only from the last term ($\sim \ln\epsilon/\mu$), since the functions e and m are chosen to be continuous (see Sec. III).

Therefore we treat the rapidly varying part separately,

$$C_{\text{rv}}(e^{i\alpha}) = \exp\left[\frac{1}{2} \ln \frac{\epsilon}{\mu} \int_{\pi/2}^{3\pi/2} \frac{e^{i\alpha}+e^{i\theta}}{e^{i\alpha}-e^{i\theta}} d\alpha \right] = \exp\left[\frac{i}{\pi} \ln \frac{\epsilon}{\mu} \ln \left[\tan \frac{\theta+\pi/2}{2} \right] \right] , \tag{C3}$$

and expand around $x = \pi/2 + \theta \approx 0$:

$$C_{\text{rv}}(e^{i\alpha}) = \exp\left[\frac{i}{\pi} \ln \frac{\epsilon}{\mu} (\tan x/2) \right] \simeq \exp\left[\frac{i}{\pi} \ln \frac{\epsilon}{\mu} \frac{x}{2} \right] = \left[\frac{x}{2} \right]^{(i/\pi)\ln\epsilon/\mu} . \tag{C4}$$

Now we use the advantage that the integrals of $x^{i[(1/\pi)\ln\epsilon/\mu]}$ multiplied by different powers of x, have simple, closed expressions $[r=(1/\pi)\ln\epsilon/\mu]$:

$$\int_{x_j}^{x_{j+1}} dx\, x^{ir} x^k = (x_{j+1}^{k+1+ir} - x_j^{k+1+ir})/(k+1+ir) . \tag{C5}$$

Hence, we shall split off the slowly varying part of C (denoted by C_{sv}), multiply it with the asymptotic expansion Π_{asy} and the Poisson kernel and approximate this product, in each interval $[x_j, x_{j+1}]$, by second-degree curves:

$$P(e^{i\theta}, W_\alpha)\Pi_{\text{asy}}(K(e^{i\theta}))C_{\text{sv}}(e^{i\theta}) \simeq \sum_{k=0}^{2} b_k x^k , \quad x_j \le x \le x_{j+1} . \tag{C6}$$

Combining (C5) and (C6), the integrals in every interval $[x_j, x_{j+1}]$ are simple expressions of the form $\sum_{k=0}^{2} a_k b_k$. In practice, we use these procedures for $x=\pi/2+\theta\leq 0.1$ and divide the interval [0,0.1] in 20–30 subintervals.

At another critical point $\theta=+\pi/2$ the problem is treated in the same manner, since

$$\ln\tan\left[x+\frac{\pi}{2}\right]=\ln\cot x$$
$$=\ln\frac{1}{\tan x}=-\ln\tan x\ .$$

[1]M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[2]N. A. Papadopoulos and H. Vogel, Phys. Lett. B **199**, 113 (1987); N. A. Papadopoulos, in *Proceedings of the International Europhysics Conference on High Energy Physics,* Uppsala, Sweden, 1987, edited by O. Botner (European Physical Society, Geneva, Switzerland, 1987), p. 326.

[3]N. F. Nasrallah, N. A. Papadopoulos, and K. Schilcher, Phys. Lett. **113B**, 61 (1982); Z. Phys. C **16**, 323 (1983); N. A. Papadopoulos, Hadr. Suppl. **2**, 646 (1986).

[4]S. Ciulli and T. D. Spearman, Nuovo Cimento **83A**, 352 (1984); S. Ciulli, C. Pomponiu, and I. Sabba-Stefanescu, Phys. Rep. **17**, 133 (1975); S. Ciulli, Inverse Problems and Non-Linear Evaluation, Proceedings of the RCP Workshop, Montpellier, 1979 (unpublished); G. Menessier, International Symposium on Hadron Interaction—Theory and Phenomenology, Bechyne, Czechoslovakia, 1988 (unpublished).

[5]S. Narison, in *Nonperturbative Aspects of QCD: Monte Carlo and Optimization,* proceedings of the 18th Summer School in Particle Physics, Gif-sur-Yvette, France, 1986, edited by A. Brezin *et al.* (Institut National de Physique Nucleaire et de Physique des Particules, Paris, 1987), p. 179; L. J. Reinders, A. Rubinstein, and S. Yazaki, Phys. Rep. **127**, 1 (1985).

[6]N. A. Papadopoulos, Report No. MZ-TH/88-11 (unpublished); J. E. Bowcock, S. Ciulli, and F. Geniet, Montpellier report (unpublished); S. Ciulli, F. Geniet, G. Menessier, and T. D. Spearman, Report No. PM-87/48 (unpublished); Phys. Rev. D **36**, 3494 (1987).

[7]S. Ciulli, C. Pomponiu, and I. Sabba-Stefanescu, Phys. Rep. **17**, 133 (1975).

[8]M. Kremer, N. A. Papadopoulos, and K. Schilcher, Phys. Lett. **143B**, 476 (1984).

[9]L. D. Landau and E. M. Lifshitz, *Relativistic Quantum Theory* (Pergamon, Oxford, England, 1971).

[10]N. F. Nasrallah, N. A. Papadopoulos, and K. Schilcher, Phys. Lett. **126B**, 379 (1973).

[11]F. Scheck, in *Proceedings of the International Europhysics Conference on High Energy Physics,* Bari, Italy, 1985, edited by L. Nitti and G. Preparata (Laterza, Bari, 1985).

[12]J. Bailey *et al.*, Nucl. Phys. **B150**, 1 (1979); A. Bramòn, E. Etim, and M. Greco, Phys. Lett. **29B**, 514 (1972); J. Calmet, S. Narison, M. Perrottet, and E. de Rafael, Rev. Mod. Phys. **49**, 21 (1977).

[13]E. Witten, Nucl. Phys. **B156**, 169 (1979).

[14]G. Veneziano, Nucl. Phys. **B159**, 213 (1979).

[15]V. A. Novikov, M. A. Shifman, A. I. Vainstein, and V. I. Zakharov, Phys. Lett. **86B**, 347 (1979); Nucl. Phys. **B174**, 378 (1980).

[16]V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B191**, 301 (1981).

[17]S. Narison, Z. Phys. C **26**, 209 (1984).

[18]K. L. Au, D. Morgan, and M. R. Pennington, Report No. RAL-86-076 (unpublished); G. J. Gounaris, R. Kögerler, and J. E. Paschalis, Phys. Lett. B **186**, 107 (1987).

[19]P. Di Vecchia, K. Fabricius, G. C. Rossi, and G. Veneziano, Nucl. Phys. **B192**, 392 (1981); Phys. Lett. **108B**, 323 (1982).

[20]A. S. Kronfeld, M. L. Laursen, G. Schierholz, and U.-J. Wiese, Nucl. Phys. **B292**, 330 (1987); **B292**, 349 (1987).

[21]Y. Arian and P. Woit, Phys. Lett. B **183**, 341 (1987).

[22]J. Hoek, M. Teper, and J. Waterhouse, Phys. Lett. B **180**, 112 (1986); Nucl. Phys. **B288**, 589 (1987).

[23]J. Smit and J. Vink, Nucl. Phys. **B284**, 234 (1987); Phys. Lett. B **194**, 433 (1987).

[24]M. Campostrini, A. Di Giacomo, and H. Panagopoulos, Phys. Lett. B **212**, 206 (1988); A. S. Kronfeld, in *Field Theory on the Lattice,* proceedings of the International Symposium, Fermilab, Batavia, Illinois, 1988, edited by A. Kronfeld *et al.* [Nucl. Phys. B (Proc. Suppl.) (in press)]; S. Itoh, Y. Iwasaki, and T. Yoshie, Phys. Rev. D **36**, 527 (1987).

[25]M. Ciulli and S. Ciulli, Comput. Phys. Commun. **18**, 215 (1979); I. Caprini, M. Sararu, C. Pomponiu, M. Ciulli, S. Ciulli, and I. S. Stefanescu, *ibid.* **18**, 305 (1979).