Lower bounds on the constituent-quark mass differences

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Using a potential-model approximation to quantum chromodynamics and the masses of ground-state mesons and baryons as input, we obtain lower bounds on the constituent-quark mass differences. We find $m_d - m_u > 4.1 \pm 0.3$ MeV, $m_s - \overline{m} > 184 \pm 4$ MeV, $m_c - m_s > 1180 \pm 4$ MeV, and $m_b - m_c > 3343 \pm 4$ MeV, where $\overline{m}$ is the average mass of the u and d quarks.

In the absence of free quarks, quark masses cannot be directly measured but must be inferred from the properties of hadrons. At least three kinds of quark masses have been discussed in the literature: current-quark masses, invariant quark masses, and constituent-quark masses. In this paper we concentrate our attention on the constituent-quark masses. Current-quark masses have been reviewed by Gasser and Leutwyler[1] and invariant quark masses by Narison.[2]

The constituent quark masses are most appropriate for calculating the static properties of hadrons in an approximation to QCD which considers only valence quarks as constituents. In this approach, the contributions of gluons and quarks of the sea are ignored. We cannot begin to quote the many papers which use constituent-quark masses as a basis for calculation, but simply call attention to a review[3] which discusses this subject among others.

We assume at the outset that the five known quarks are ordered from lightest to heaviest: u, d, s, c, and b. Then, using rather general considerations, we show in this paper that the constituent-quark mass differences in MeV satisfy the lower bounds

$$\begin{aligned} &m_d - m_u > 4.1 \pm 0.3, \quad m_s - \overline{m} > 184 \pm 4 , \\ &m_c - m_s > 1180 \pm 4, \quad m_b - m_c > 3343 \pm 4 , \end{aligned} \tag{1}$$

where $\overline{m} = (m_u + m_d)/2$.

We discuss the quark masses within the framework of a potential-model approach, as this approach has proved most successful in enabling people to calculate the energy spectrum of hadrons. Unfortunately, the constituent-quark masses are not very well determined in potential models. This is because one can change the mass of a quark by a substantial amount without spoiling the agreement with experiment if one makes compensating changes in the parameters of the potential. One can get around this difficulty by considering the mass differences of the quarks. Provided the potential is flavor independent, the quark mass differences are much more stable than the quark masses themselves. For this reason, we emphasize mass differences here.

Although we use the potential-model approach, we do not need to consider any particular functional form for the potential. However, we need to assume that the Hamiltonian for the quark-antiquark or three-quark system satisfies certain properties. In particular, we divide the Hamiltonian into a sum of three terms:

$$H = H_0 + H_1 + H_2 , \tag{2}$$

where H_0 includes the rest energy, the kinetic energy, and that part of the strong interaction which is independent of spin variables, H_1 includes the spin-dependent part of the strong interaction in the Fermi-Breit approximation, and H_2 includes the electromagnetic interactions of the quarks. We treat H_1 and H_2 as first-order perturbations.

Because of the Feynman-Hellmann theorem,[4] if H_0 is the Hamiltonian of the nonrelativistic Schrödinger equation with a flavor-independent potential, then, as the mass of any of the constituent quarks increases, the energy eigenvalues E_{0n} (excluding the rest energy) of H_0 decrease. This fact was shown in a review of Quigg and Rosner.[5] Subsequent applications to quarkonium were given by Bertlmann and Ono[6] and Bertlmann and Martin.[7]

We now point out that we can apply the Feynman-Hellmann theorem to a much larger class of Hamiltonians H_0, including many-body Hamiltonians and a number of relativistic Hamiltonians. For example, if we apply the Feynman-Hellmann theorem to the two- or three-particle spinless Salpeter equation,[8] we can show that if the potential is flavor independent, the eigenvalues E_{0n} will decrease as any quark mass increases. We omit the proof, which is straightforward.

Because of the wide applicability of the Feynman-Hellmann theorem, it is plausible to assume that if the potential is flavor independent, then H_0 has the property such that E_{0n} decreases as any quark mass increases. How realistic is the assumption that the potential in H_0 is independent of quark flavor? At small distances between two quarks (or a quark-antiquark pair), the spin-independent potential is Coulomb type, with a running coupling strength α_s which depends on the square of the four-momentum transfer Q^2. If α_s is considered a function of Q^2, the potential depends upon the scale, and therefore on flavor through the quark masses. However, we can circumvent this problem by considering the Fourier transform of α_s which is a function of the distance r between two quarks or a quark-antiquark pair. Then at small distances, according to QCD perturbation theory, the potential can be written $V(r) = c\alpha_s(r)/r$, where $c = -\frac{4}{3}$ for mesons and $-\frac{2}{3}$ for baryons. This expression is flavor independent, as emphasized in a recent review.[9] It is usually assumed, partly from lattice QCD and partly on phenomenological grounds, that the confining part of the potential is also independent of flavor.

In the Fermi-Breit approximation to the QCD interaction, there exist flavor-dependent terms which are independent of spin. There is some controversy over the nature of these terms, and, furthermore, there is little or no empirical evidence for their existence.[9] In QED, the spin-independent Fermi-Breit contribution vanishes in the ground state. We shall confine our considerations to ground-state hadrons, and hope that the Fermi-Breit spin-independent contribution arising in QCD also vanishes, or at least is very small. In any case, we shall neglect any flavor-dependence in the potential in H_0.

On the other hand, the interactions H_1 and H_2 explicitly depend on flavor through the quark masses, and we do not neglect this flavor dependence. Furthermore, we also take into account an implicit flavor dependence in the expectation values of H_1 and H_2, using the fact that the flavor-independent potential is concave downward.[10]

In the ground state, only the color hyperfine interaction contributes to the expectation value of H_1. In order to obtain a reliable estimate for this expectation value, we need to examine the properties of the QCD potential under Lorentz transformations. The one-gluon-exchange contribution, which is Coulomb-type, is known to transform like the time component of a vector. Also, for baryons, this part of the potential is a sum of two-body potentials. On the other hand, the confining part of the potential is believed[9] to transform like a Lorentz scalar. If so, it does not contribute to the hyperfine interaction.

In a previous paper,[11] we used the above framework (except that we did not have the proofs based on the Feynman-Hellmann theorem) to obtain inequalities among the masses of ground-state baryons and mesons. *In all cases in which these inequalities could be compared with experiment, they turned out to be in agreement with the data.* This fact gives us confidence that our assumptions are reasonable. In this paper, we turn the arguments around to obtain inequalities among the quark mass differences. These inequalities turn out to be lower bounds.

Within our framework, the mass of a ground-state hadron containing quarks with masses m_i is

$$M=\sum_i m_i+E_0+E_1+E_2 \ , \tag{3}$$

where $\sum_i m_i+E_0$ is the lowest eigenvalue of H_0, and E_1 and E_2 are the expectation values of H_1 and H_2, respectively.

The energies E_1 and E_2 are given by

$$E_1=\sum_{1<j}\langle\nabla^2 V(r_{ij})\rangle\sigma_i\cdot\sigma_j \ , \tag{4}$$

$$E_2=\sum_{i<j}Q_iQ_j[\langle 1/r_{ij}\rangle-2\pi\langle\delta(r_{ij})\rangle\sigma_i\cdot\sigma_j/(3m_im_j)] \ , \tag{5}$$

where σ_i and σ_j are the Pauli spin matrices for the ith and jth quarks, Q_i and Q_j are their charges, and r_{ij} is the distance between them. In Eq. (4), $V(r_{ij})$ is the part of the potential between quarks i and j which transforms like the time component of a four-vector.

We now introduce a more explicit notation. For a meson, we let the eigenvalue of H_0 be $m_i+m_j+E_{ij}$ to show that the energy depends on the flavors of the quark it contains. We also define

$$\begin{aligned} R_{ij}&=\langle\nabla^2V(r_{ij})\rangle/(6m_im_j) \ ,\\ C_{ij}&=\alpha\langle 1/r_{ij}\rangle/3 \ ,\\ A_{ij}&=2\pi\alpha\langle\delta(r_{ij})\rangle/(9m_im_j) \ . \end{aligned} \tag{6}$$

Because of the Feynman-Hellmann theorem, if the mass of quark i or j increases, E_{ij} decreases. The quantities C_{ij}, A_{ij}, and R_{ij} are all positive definite.[11] Furthermore, if the mass of quark i or j increases, C_{ij} will increase, but A_{ij} and R_{ij} will decrease.[11] Briefly, the contraction of the wave function with increasing quark mass causes the expectation value of $1/r_{ij}$ to increase, while the explicit mass dependence of A_{ij} and R_{ij} causes them to decrease (except possibly for very heavy quarks such as the t quark).

For a baryon, we let the eigenvalue of H_0 be $m_i+m_j+m_k+E_{ijk}$. We define the positive-definite quantities

$$\begin{aligned} R_{ij,k}&=\langle\nabla^2V(r_{ij})\rangle/(6m_im_j) \ ,\\ C_{ij,k}&=\alpha\langle 1/r_{ij}\rangle/3 \ ,\\ A_{ij,k}&=2\pi\alpha\langle\delta(r_{ij})\rangle/(9m_im_j) \ . \end{aligned} \tag{7}$$

By writing the subscript k on $R_{ij,k}$, $C_{ij,k}$, and $A_{ij,k}$, we have noted explicitly that the matrix elements of a two-quark operator depends on the third or spectator quark through its dependence on the baryon wave function.[12]

We can now write the expression for the mass of any meson or baryon in terms of the quantities we have defined above. For the K mesons we obtain (in a notation that the symbol for a hadron denotes its mass)

$$\begin{aligned} K^+&=m_u+m_s+E_{us}-3R_{us}+2C_{qs}/3+2A_{qs} \ ,\\ K^0&=m_d+m_s+E_{ds}-3R_{ds}-C_{qs}/3-A_{qs} \ ,\\ K^{*+}&=m_u+m_s+E_{us}+R_{us}+2C_{qs}/3-2A_{qs}/3 \ ,\\ K^{*0}&=m_d+m_s+E_{ds}+R_{ds}-C_{qs}/3+A_{qs}/3 \ , \end{aligned} \tag{8}$$

where the symbol q stands for either a u or d quark, and we have neglected the mass difference between the u and d in C_{ij} and A_{ij}, since these are electromagnetic quantities and are already very small.

We can obtain the following expression from Eq. (8):

$$m_d-m_u=(3K^{*0}-3K^{*+}+K^0-K^+)/4+E_{us}-E_{ds}+C_{qs} . \tag{9}$$

Because $m_d>m_u$, we have $E_{ds}<E_{us}$. Also, $C_{qs}>0$. It then follows from Eq. (9) that

$$m_d-m_u>(3K^{*0}-3K^{*+}+K^0-K^+)/4 \ . \tag{10}$$

Using the experimental masses from the Particle Data Group[13] in the inequality (10), we obtain a lower bound for the d-u mass difference:

$$m_d - m_u > 4.1 \pm 0.3 \text{ MeV} , \tag{11}$$

where the error is experimental.

In obtaining the remaining bounds on quark mass differences, we shall use the symbol q for either the u or d quark, $\overline{m}$ for the average mass of these two quarks, and neglect all effects which violate isospin conservation. We estimate that this approximation will introduce an error of at most 4 MeV into our results. We obtain the following useful expressions for meson masses:

$$\begin{aligned} &\pi = 2\overline{m} + E_{qq} - 3R_{qq}, \quad \rho = 2\overline{m} + E_{qq} + R_{qq} , \\ &D = \overline{m} + m_c + E_{qc} - 3R_{qc}, \quad D^* = \overline{m} + m_c + E_{qc} + R_{qc} , \\ &B = \overline{m} + m_b + E_{qb} - 3R_{qb}, \quad B^* = \overline{m} + m_b + E_{qb} + R_{qb} . \end{aligned} \tag{12}$$

Using Eqs. (8) and (12), we obtain the following expressions for quark mass differences:

$$\begin{aligned} &m_s - \overline{m} = (3K^* + K - 3\rho - \pi)/4 + E_{qq} - E_{qs} , \\ &m_c - m_s = (3D^* + D - 3K^* - K)/4 + E_{qc} - E_{qs} , \\ &m_b - m_c = (3B^* + B - 3D^* - D)/4 + E_{qb} - E_{qc} . \end{aligned} \tag{13}$$

But from the Feynman-Hellmann theorem, we have

$$E_{qq} > E_{qs} > E_{qc} > E_{qb} . \tag{14}$$

Using the inequalities (14) in Eq. (13) and the meson masses from the Particle Data Group,[13] we obtain the following lower bounds on the quark mass differences in MeV:

$$\begin{aligned} &m_s - \overline{m} > 183 \pm 4 , \\ &m_c - m_s > 1180 \pm 4 , \\ &m_b - m_c > 3343 \pm 4 , \end{aligned} \tag{15}$$

where the errors of 4 MeV come from our neglect of isospin violations.

We next turn to the baryons. Using the same approximations as those employed in obtaining Eq. (12), we can obtain expressions for baryon masses. Several useful formulas are

$$\begin{aligned} &N = 3\overline{m} + E_{qqq} - 3R_{qq,q} , \\ &\Lambda = 2\overline{m} + m_s + E_{qqs} - 3R_{qq,s} , \\ &\Sigma = 2\overline{m} + m_s + E_{qqs} + R_{qq,s} - 4R_{qs,q} , \\ &\Delta = 3\overline{m} + E_{qqq} + 3R_{qq,q} , \\ &\Sigma^* = 2\overline{m} + m_s + E_{qqs} + R_{qq,s} + 2R_{qs,q} . \end{aligned} \tag{16}$$

Using Eq. (16), we obtain

$$m_s - \overline{m} = (2\Sigma^* + \Sigma + \Lambda - 2N - 2\Delta)/4 + E_{qqq} - E_{qqs} . \tag{17}$$

Then, using $E_{qqq} > E_{qqs}$ and the baryon masses from the Particle Data Group, we obtain the lower bound

$$m_s - \overline{m} > 184 \pm 4 \text{ MeV} , \tag{18}$$

in good agreement with the bound in (15).

We have not been able to find other useful bounds from the baryons, owing either to the absence of experimental knowledge of certain baryon masses or the fact the lower bounds are lower than the bounds we were able to obtain from the mesons. For example, we have been able to obtain the lower bound

$$m_c - m_s > \Lambda_c - \Lambda = 1169 \pm 4 \text{ MeV}, \tag{19}$$

but this is not as good as the bound given in (15).

Although most authors who obtain quark masses do not compare their results with those of other authors, we shall make a few selected comparisons. Some time ago, Lipkin[14] obtained the formula

$$m_s - m_u = \Lambda - p = 178 \text{ MeV}. \tag{20}$$

In our formalism, which takes into account effects neglected by Lipkin, we do not get Lipkin's result, but rather the inequality (18), which contradicts Eq. (20), although only by 8 ± 4 MeV. In obtaining this number, we have used (11) as well as (18).

Bertlmann and Ono[6] obtain the following quark masses in MeV:

$$\overline{m} = 310, \quad m_s = 620, \quad m_c = 1910, \quad m_b = 5270 ,$$

values which satisfy our lower bounds. Godfrey and Isgur,[15] in a detailed analysis of meson spectra, find the following values of the quark mass differences in MeV (with $\overline{m} = 220$):

$$\begin{aligned} &m_d - m_u = 8, \quad m_s - \overline{m} = 199, \\ &m_c - m_s = 1209, \quad m_b - m_c = 3349 , \end{aligned} \tag{21}$$

all in agreement with our bounds. Furthermore, Capstick and Isgur[16] are able to obtain a good description of the baryon spectrum using the same quark masses as in the meson case (neglecting isospin violations). On the other hand, Thakur[17] finds $m_s - \overline{m} = 175$ MeV, in agreement with Lipkin.

We do not know of any good reason why current- or invariant quark mass differences should be the same as constituent-quark differences. Nevertheless, we give a few comparisons. We use the notation that m_i^* is the running current-quark mass evaluated at a momentum transfer of 1 GeV and $\hat{m}_i$ is the invariant quark mass. The Particle Data Group[13] gives, as its best estimates of the current-quark masses (in MeV),

$$\begin{aligned} &m_u^* \simeq 5.6 \pm 1.1, \quad m_d^* \simeq 9.9 \pm 1.1, \quad m_s^* \simeq 199 \pm 33 , \\ &m_c^* \simeq 1350 \pm 50, \quad m_b^* \simeq 5000 , \end{aligned} \tag{22}$$

based on work by Dominguez and de Rafael[18] and Gasser and Leutwyler.[1] These masses lead to mass differences which agree with our bounds in (1) within the errors. Also Narison[19] finds

$$\begin{aligned} &m_u^* = 5.2 \pm 0.5, \quad m_d^* = 9.2 \pm 0.5, \quad m_s^* = 159.5 \pm 8.8 , \\ &m_c^* = 1400 \pm 60, \quad m_b^* = 5870 \pm 60 , \end{aligned} \tag{23}$$

$$\hat{m}_u = 8.7 \pm 0.8, \quad \hat{m}_d = 15.4 \pm 0.8, \quad \hat{m}_s = 266.7 \pm 14.7 \ ,$$
$$\hat{m}_c = 1920 \pm 180, \quad \hat{m}_b = 7890 \pm 90 \ . \tag{24}$$

We see that in Eqs. (23) and (24) only $m_s^* - \overline{m}^*$ fails to satisfy our lower bound.

In conclusion, using rather general considerations, we have been able to obtain lower bounds on the constituent quark mass differences. Although in some models these bounds are violated, we believe that one can obtain a consistent picture of the static properties of hadrons with quark masses which satisfy our bounds. As an example, Isgur and collaborators[15,16] have obtained quite good results for the properties of hadrons, using as input constituent quark masses which satisfy our lower bounds.

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