Quantum field theory in a colliding plane-wave background

Ulvi Yurtsever
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125
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Although linear quantum field theory on the background of a single gravitational plane wave is trivial, colliding plane-wave solutions are likely to feature interesting quantum effects; these might yield insight into similar effects in other more general inhomogeneous and dynamical spacetimes. This paper presents the initial results of an investigation into the behavior of quantum fields in colliding plane-wave backgrounds. Throughout the paper, we restrict our attention to the analysis of quantum field theory on the Khan-Penrose spacetime. Since spacetime is flat before the arrival of either plane wave, there exists a unique, well-defined set of in modes and a corresponding in vacuum state. We introduce a physically plausible prescription for constructing a unique canonical set of out-mode solutions, and we evaluate the Bogolubov transformation between the in and out modes explicitly for a massless scalar field propagating on the Khan-Penrose spacetime. We then use these results to approximately compute the spectrum of created particles in the out region. Next, we study the quantity $\langle T_{\mu\nu}\rangle \equiv \langle 0,\text{in}|T_{\mu\nu}|0,\text{in}\rangle$, the renormalized in-vacuum expectation value of the stress-energy tensor for a massless, conformally coupled ($\xi=\frac{1}{6}$) scalar field. In a colliding plane-wave spacetime, $\langle T_{\mu\nu}\rangle$ vanishes everywhere except in the interaction region. To compute $\langle T_{\mu\nu}\rangle$ in the interaction region, we make a number of assumptions about its general form; these assumptions are entirely reasonable for the specific geometry of the Khan-Penrose spacetime, but they may not hold for a general colliding plane-wave solution. Combined with the conservation property of $\langle T_{\mu\nu}\rangle$ and the choice of $\xi=1/6$, our assumptions reduce the determination of $\langle T_{\mu\nu}\rangle$ throughout the interaction region to the solution of a coupled system of first-order partial differential equations for two functions. These equations cannot be solved exactly; but they can be used to obtain crucial information on the behavior of $\langle T_{\mu\nu}\rangle$ near the singularity of the Khan-Penrose spacetime. Although our method of computing $\langle T_{\mu\nu}\rangle$ is unlikely to be adequate for other colliding plane-wave solutions we use the information obtained through our calculations to speculate about $\langle T_{\mu\nu}\rangle$ in more general colliding gravitational-wave spacetimes. We argue that these speculations have important consequences for cosmology, but they must be verified by further calculations.

INTRODUCTION

Quantum field theory in a curved background spacetime generally involves difficult and tedious calculations. Although the general formalism of the theory is by now well understood and fairly standard,[1] many of the more extensively studied applications focus on background spacetimes which possess either a very high degree of symmetry (such as de Sitter space[2] or Robertson-Walker spacetimes[3]) or a special geometric property (e.g., spacetimes that are flat[4] or conformally flat[5]). The most important exceptions to this are the black-hole solutions; the tremendous physical significance of quantum black holes has motivated many researchers to study quantum field theory on a black-hole background with great detail and rigor.[6] On the other hand, the work so far on the equally important subject of quantum field theory in a dynamical, anisotropic, and inhomogeneous spacetime has been mostly confined to perturbation-theoretical analyses around highly symmetric and/or conformally flat backgrounds.[7]

Very simple, exact examples of dynamical spacetimes are provided by the gravitational plane-wave solutions.[8,9] The theory of (linear) quantum fields propagating on an arbitrary gravitational plane-wave spacetime has been investigated by Gibbons,[10] and more recently by Klimcik.[11] For a "sandwich" plane wave,[9] i.e., for a plane-wave spacetime whose curvature is sandwiched between the two flat, null wave fronts of the wave, the spacetime is flat both before and after the wave's passage. Therefore, for any linear quantum field propagating on a sandwich plane-wave background, it is possible to define unique in- and out-mode solutions and corresponding in- and out-vacuum states unambiguously. The Bogolubov transformation[1] connecting these two sets of mode solutions (for a massive scalar field) is studied by Gibbons,[10] and he finds that no particles are created in the out region, and that the renormalized expectation value of the stress-energy tensor in the in vacuum (or equivalently in the out vacuum) vanishes throughout spacetime. Quantum field theory (linear) on the background of a single plane-wave solution is essentially trivial.

From a general point of view this result is not terribly surprising; plane-wave spacetimes have a high degree of symmetry (in general there is a five-parameter group of isometries), and perhaps more importantly they satisfy the Huygens's principle[12] (i.e., a linear scalar field experiences no backscattering while propagating through a plane wave). Moreover, results analogous to those of Gibbons[10] are well known to be true for semiclassical plane-wave solutions in quantum electrodynamics and in other gauge theories.[13] Clearly, for the study of nontrivial quantum effects in a dynamical spacetime, examples of background solutions more complicated than single plane waves must be considered.

Colliding plane-wave solutions furnish examples of dynamical vacuum spacetimes significantly more complicated than plane waves (their isometry groups are in general only two dimensional); and the structures of these solutions are correspondingly much richer. As a result of the combined contributions of various workers over the last two decades, the local and global structures of arbitrary colliding gravitational plane-wave spacetimes are by now fully understood; for references and a detailed exposition see Refs. 9, 14, 15, and 16, and the references cited therein. In particular, it is now known that generic gravitational plane-wave collisions create spacetime singularities with a rich, nontrivial asymptotic structure.[15,16] Although these plane-symmetric singularities are always homogeneous in the x, y directions (directions along which the orbits of the plane-symmetry-generating Killing vectors lie), their asymptotic structure is inhomogeneous-Kasner, with the asymptotic Kasner axes and exponents across the singularity in general depending on the z coordinate (the spacelike coordinate along which the colliding plane waves propagate and in which spacetime is inhomogeneous). Moreover, even when the colliding waves are not exactly plane symmetric, the focusing[17,9] of each gravitational wave by the other is strong enough to create spacetime singularities provided the colliding almost-plane waves[9] are sufficiently large in transverse size and sufficiently intense in amplitude.[18,16] It is likely that these non-plane-symmetric singularities are hidden behind event horizons, i.e., that collisions of almost-plane waves create black holes (see Ref. 18, especially Fig. 2 and the discussion that follows it in Sec. I). If this is the case, then the singularities inside these black holes have a highly inhomogeneous and anisotropic local structure. Therefore, when endowed with the opposite time orientation, a colliding gravitational-wave spacetime becomes a good local model for a highly inhomogeneous and anisotropic initial singularity (with "white holes"[19]), in which initial vacuum inhomogeneities decay through the emission of gravitational radiation (see Fig. 2 below). Similarly, when its time orientation is reversed, a colliding plane-wave solution can be regarded as an exact, idealized model of such an initial singularity, with all its spatial inhomogeneity being restricted to the z direction.

It is clear that unlike single plane-wave solutions, quantum field theory on a colliding plane-wave background will have nontrivial features, and the above discussion suggests that these features are well worth investigating. The purpose of this paper is to report on the initial results of such an investigation. The formidable difficulty of the calculations involved has forced us to restrict our attention to a single specific colliding plane-wave solution throughout the paper; and as the earliest and perhaps the best-known solution, we have chosen the Khan-Penrose[20] spacetime to be our fixed background. The main results from our study of quantum field theory on the Khan-Penrose background are summarized in the following two paragraphs.

Although like every colliding plane-wave spacetime the Khan-Penrose solution has a Minkowskian in region (i.e., the region that lies before the collision, prior to the passage of either plane wave), unlike a single sandwich plane wave, its out region (i.e., the interaction region that lies to the future of the collision) is curved. Since both of the two Killing vectors in the interaction region are spacelike, there is no unique way to specify out-mode solutions and an out-vacuum state. We introduce a physically plausible prescription for constructing a unique canonical set of out-mode solutions, and we evaluate the Bogolubov transformation between the in and out modes explicitly for a massless scalar field propagating on the Khan-Penrose spacetime. We then use these results to approximately compute the spectrum of created particles in the out region. Our prescription for constructing the out modes is applicable in a straightforward way to any arbitrary colliding plane-wave spacetime and to fields with arbitrary spin, thus, we expect that the results on particle creation in the general case will be similar to the results reported here.

Since it is exceedingly difficult to compute the in-mode solutions in the interaction region, we cannot evaluate any of the in-vacuum Green's functions[1] there, and consequently, we have no way of determining the response of a particle detector.[1,21] Therefore, in order to further understand the behavior of quantum fields in the interaction region, it becomes necessary to study other quantities that are not sensitive to our (rather arbitrary) choice of the out-vacuum state. One such quantity is $\langle 0,\text{ in}|\, T_{\mu\nu}\,|0,\text{ in}\rangle$; the renormalized in-vacuum expectation value of the stress-energy tensor for a massless, conformally coupled ($\xi=\frac{1}{6}$) scalar field. Moreover, we find that it is not too difficult to compute the in-vacuum Green's functions *near* the initial null boundaries of the interaction region, exactly *along* the boundaries and to first order in the displacement away from them. Using standard point-splitting renormalization,[1] this allows us to compute $\langle T_{\mu\nu}\rangle \equiv \langle 0,\text{ in}|\, T_{\mu\nu}\,|0,\text{ in}\rangle$ *exactly* along the boundaries. To compute $\langle T_{\mu\nu}\rangle$ elsewhere in the interaction region, we make a number of assumptions about its general form; these assumptions are entirely reasonable for the specific geometry of the Khan-Penrose spacetime, but they may not hold for a general colliding plane-wave solution. Combined with the conservation property of $\langle T_{\mu\nu}\rangle$ and the choice of $\xi=\frac{1}{6}$, our assumptions reduce the determination of $\langle T_{\mu\nu}\rangle$ throughout the interaction region to the solution of a coupled system of first-order partial differential equations for two functions. These equations still cannot be solved exactly; but they can be used to obtain crucial information on the behavior of $\langle T_{\mu\nu}\rangle$ near the singularity of the Khan-Penrose spacetime. Although our method of computing $\langle T_{\mu\nu}\rangle$ is unlikely to be adequate in other colliding plane-wave solutions, we use the information obtained through our calculations to speculate about $\langle T_{\mu\nu}\rangle$ in more general colliding gravitational-wave spacetimes. We argue that these speculations have important consequences for cosmology, but they must be verified by further calculations.

Our sign conventions are $(-,+,+)$ in the terminology of Misner, Thorne, and Wheeler;[22] our notation and all other conventions are the same as those of Ref. 1. Unless otherwise indicated, we use Planck units in which $\hbar = c = G = 1$.

THE KHAN-PENROSE SPACETIME

The two-dimensional geometry of the Khan-Penrose solution is depicted in Fig. 1. In the interaction region I (where $u > 0$, $v > 0$), the metric is given by

$$\begin{aligned} g_{\rm I} = & \frac{(1-\hat{u}^2-\hat{v}^2)^{3/2}}{(1-\hat{u}^2)^{1/2}(1-\hat{v}^2)^{1/2}[\hat{u}\hat{v}+(1-\hat{u}^2)^{1/2}(1-\hat{v}^2)^{1/2}]^2}\,du\,dv \\ & -(1-\hat{u}^2-\hat{v}^2)\left[\frac{(1-\hat{u}^2)^{1/2}+\hat{v}}{(1-\hat{u}^2)^{1/2}-\hat{v}}\right]\left[\frac{(1-\hat{v}^2)^{1/2}+\hat{u}}{(1-\hat{v}^2)^{1/2}-\hat{u}}\right]dx^2 \\ & -(1-\hat{u}^2-\hat{v}^2)\left[\frac{(1-\hat{u}^2)^{1/2}-\hat{v}}{(1-\hat{u}^2)^{1/2}+\hat{v}}\right]\left[\frac{(1-\hat{v}^2)^{1/2}-\hat{u}}{(1-\hat{v}^2)^{1/2}+\hat{u}}\right]dy^2 , \end{aligned} \tag{1}$$

with

$$\hat{u} \equiv \frac{u}{a}, \quad \hat{v} \equiv \frac{v}{b}, \tag{2}$$

where a and b are the focal lengths[9] of the u wave (the colliding plane wave whose initial wave front is $\{u=0\}$) and of the v wave (the colliding plane wave whose initial wave front is $\{v=0\}$), respectively (Fig. 1). In the regions labeled II (where $u>0$, $v<0$) and III (where $u<0$, $v>0$) in Fig. 1, the metric is

$$\begin{aligned} g_{\rm II} &= du\,dv - (1+\hat{u})^2dx^2 - (1-\hat{u})^2dy^2 \\ &= dU\,dV - dX^2 - dY^2 , \end{aligned} \tag{3a}$$

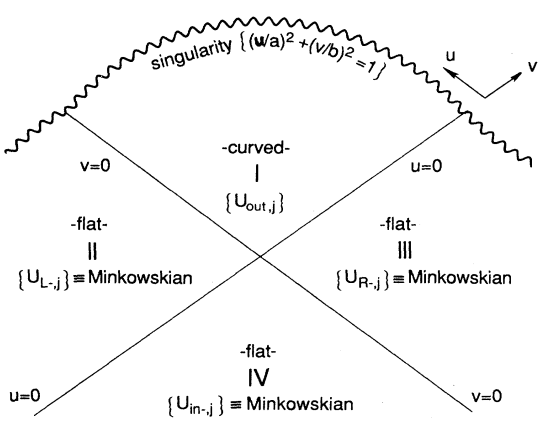

FIG. 1. The two-dimensional geometry of the Khan-Penrose spacetime. (The actual four-dimensional geometry is more complicated; see Refs. 23 and 9.) Both of the two colliding sandwich plane waves are impulsive; i.e., their initial (past) and final (future) wave fronts coincide. Consequently, the metric is continuous but not continuously differentiable across the null boundaries (wave fronts) $\{u=0\}$ and $\{v=0\}$; the spacetime curvature has a delta-function discontinuity along these boundaries. The curved interaction region I is bounded by a spacelike curvature singularity located at $\{\hat{u}^2+\hat{v}^2=1\}$. The modes $u_{{\rm in},i}$, $u_{L-,i}$, and $u_{R-,i}$ are constructed such that they are Minkowskian throughout the flat regions IV, II, and III, respectively.

$$\begin{aligned} g_{\rm III} &= du\,dv - (1+\hat{v})^2dx^2 - (1-\hat{v})^2dy^2 \\ &= dU\,dV - dX^2 - dY^2 , \end{aligned} \tag{3b}$$

while in region IV (where $u<0$, $v<0$) it is

$$g_{\rm IV} = du\,dv - dx^2 - dy^2 . \tag{4}$$

The Minkowskian coordinates (X,Y,U,V) in the flat regions II and III are related to the coordinates (x,y,u,v) by

$$\begin{aligned} &X = (1+\hat{u})x , \quad Y = (1-\hat{u})y , \\ &U = u , \quad V = v + (1+\hat{u})\frac{x^2}{a} - (1-\hat{u})\frac{y^2}{a} \quad \text{in II} , \end{aligned} \tag{5a}$$

$$\begin{aligned} &X = (1+\hat{v})x , \quad Y = (1-\hat{v})y , \\ &U = u + (1+\hat{v})\frac{x^2}{b} - (1-\hat{v})\frac{y^2}{b} , \quad V = v \quad \text{in III} . \end{aligned} \tag{5b}$$

Both of the two colliding sandwich plane waves in the Khan-Penrose spacetime are impulsive; i.e., their initial (past) and final (future) wave fronts coincide. Consequently, the metric is continuous but not continuously differentiable across the null boundaries (wave fronts) $\{u=0\}$ and $\{v=0\}$; the spacetime curvature has a delta-function discontinuity along these boundaries. The curved interaction region I is bounded by a spacelike curvature singularity located at $\{\hat{u}^2+\hat{v}^2=1\}$ (Fig. 1). For more details on the Khan-Penrose solution, see Refs. 20, 23, 15, and 9.

NORMALIZED MODES FOR A SCALAR FIELD AND THE IN-VACUUM STATE

The natural in region for a colliding plane-wave spacetime is the flat, Minkowskian region lying before the collision, prior to the passage of either colliding wave (region IV in Fig. 1). The canonical, normalized in modes for a real, massless scalar field are the modes that have the standard Minkowski form[1] throughout this flat in-region IV:

$$u_{\text{in},\mathbf{k}}(x,y,u,v) = \frac{1}{[2\omega_k(2\pi)^3]^{1/2}} \times \exp\{-i[\tfrac{1}{2}(\omega_k+k_z)u + \tfrac{1}{2}(\omega_k-k_z)v - k_x x - k_y y]\} \quad \text{in IV}, \tag{6}$$

where

$$\omega_k \equiv (k_x{}^2 + k_y{}^2 + k_z{}^2)^{1/2} . \tag{6a}$$

On a plane-symmetric vacuum spacetime with the metric

$$g = e^{-M(u,v)}du\,dv - F^2(u,v)\,dx^2 - G^2(u,v)\,dy^2 , \tag{7a}$$

a massless scalar field satisfies the wave equation ($R \equiv 0$)

$$\Box\phi = 2e^M\left[2\phi_{,uv} + \left[\frac{F_{,u}}{F} + \frac{G_{,u}}{G}\right]\phi_{,v} + \left[\frac{F_{,v}}{F} + \frac{G_{,v}}{G}\right]\phi_{,u}\right] - \frac{1}{F^2}\phi_{,xx} - \frac{1}{G^2}\phi_{,yy} = 0 . \tag{7b}$$

Applying Eqs. (7) to Eqs. (3) and using the continuity of the modes across the null boundaries $\{u=0\}$ and $\{v=0\}$, it is not very difficult to compute the expressions of the in modes (6) in regions II and III of the Khan-Penrose spacetime: In region II (where $u>0$, $v<0$),

$$u_{\text{in},\mathbf{k}}(u,v,x,y) = \frac{1}{[2\omega_k(2\pi)^3]^{1/2}} \exp\{i[k_x x + k_y y + \tfrac{1}{2}(k_z-\omega_k)v]\} \times \frac{1}{\sqrt{1-\hat{u}^2}} \exp\left\{i\left[\frac{1}{2(k_z-\omega_k)}\left[\frac{k_x{}^2}{1+\hat{u}} + \frac{k_y{}^2}{1-\hat{u}}\right]u\right]\right\} \quad \text{in II}, \tag{8a}$$

and in region III (where $u<0$, $v>0$)

$$u_{\text{in},\mathbf{k}}(u,v,x,y) = \frac{1}{[2\omega_k(2\pi)^3]^{1/2}} \exp\{i[k_x x + k_y y - \tfrac{1}{2}(k_z+\omega_k)u]\} \times \frac{1}{\sqrt{1-\hat{v}^2}} \exp\left\{-i\left[\frac{1}{2(k_z+\omega_k)}\left[\frac{k_x{}^2}{1+\hat{v}} + \frac{k_y{}^2}{1-\hat{v}}\right]v\right]\right\} \quad \text{in III}. \tag{8b}$$

The vacuum state $|0,\text{in}\rangle$ associated with the in modes (6) and (8) is called the in vacuum. It is not possible to compute the in modes explicitly in the interaction region I; we will return to this problem when we discuss the in-vacuum Green's functions.

We now describe how to construct two new sets of normalized mode solutions for a massless scalar field. Although we continue to work with the Khan-Penrose solution, our construction applies equally well to any arbitrary colliding plane-wave spacetime. The modes $u_{L-,j}$ are constructed so that they are Minkowskian throughout the (left) flat region II (i.e., for $u>0$, $v<0$):

$$u_{L-,\mathbf{k}}(X,Y,U,V) = \frac{1}{[2\omega_k(2\pi)^3]^{1/2}} \exp\{-i[\tfrac{1}{2}(\omega_k+k_z)U + \tfrac{1}{2}(\omega_k-k_z)V - k_x X - k_y Y]\} \quad \text{in II}, \tag{9}$$

where the coordinates (X,Y,U,V) are defined by Eq. (5a). The modes $u_{R-,j}$ are constructed so that they are Minkowskian throughout the (right) flat region III (i.e., for $u<0$, $v>0$):

$$u_{R-,\mathbf{k}}(X,Y,U,V) = \frac{1}{[2\omega_k(2\pi)^3]^{1/2}} \exp\{-i[\tfrac{1}{2}(\omega_k+k_z)U + \tfrac{1}{2}(\omega_k-k_z)V - k_x X - k_y Y]\} \quad \text{in III}, \tag{10}$$

where the coordinates (U,V,X,Y) are defined by Eq. (5b). (See Fig. 1.) By appealing to the uniqueness theorem for the solutions of the scalar wave equation (7b), it is not hard to see that the modes $u_{L-,j}$ and $u_{R-,j}$ are uniquely determined *throughout* the colliding plane-wave spacetime once their expressions (9) and (10) in the flat regions II and III are given. In fact, if the v wave were absent, i.e., if spacetime contained only the single impulsive plane wave $\{u=0\}$ (the u wave), then the modes $u_{L-,j}$ would be precisely the u-wave out-mode solutions. Similarly, if the u wave were absent, i.e., if spacetime contained only the single impulsive plane wave $\{v=0\}$ (the v wave), then the modes $u_{R-,j}$ would be the v-wave out-mode solutions (Fig. 1).

To compute the Bogolubov transformations between the modes $u_{\text{in},j}$, $u_{L-,j}$, and $u_{R-,j}$, we put

$$u_{L-,i} = \sum_j (\alpha^L{}_{ij}\,u_{\text{in},j} + \beta^L{}_{ij}\,u^*{}_{\text{in},j}) , \tag{11a}$$

$$u_{R-,i} = \sum_j (\alpha^R{}_{ij}\,u_{\text{in},j} + \beta^R{}_{ij}\,u^*{}_{\text{in},j}) , \tag{11b}$$

which give[1]

$$\alpha^L{}_{ij} = (u_{L-,i}, u_{\text{in},j}),\quad \beta^L{}_{ij} = -(u_{L-,i}, u^*{}_{\text{in},j}), \tag{12a}$$

$$\alpha^R{}_{ij} = (u_{R-,i}, u_{\text{in},j}),\quad \beta^R{}_{ij} = -(u_{R-,i}, u^*{}_{\text{in},j}). \tag{12b}$$

From the linearity of the scalar field equation and the continuity and uniqueness of the mode solutions, it follows that with exactly the same coefficients expansion (11a) is valid in the spacetime of the single u wave (with the v wave absent), and similarly that with exactly the same coefficients expansion (11b) is valid in the spacetime of the single v wave (with the u wave absent); i.e., the transformations (11a) and (11b) are precisely the Bogolubov transformations between the in and out modes of the u wave and of the v wave, respectively. We now turn to the explicit computation of these transformations.

Consider first the transformation (11a) and (12a) between the modes $u_{L-,j}$ and $u_{\text{in},j}$. To compute the coefficients $\alpha^L{}_{ij}$ and $\beta^L{}_{ij}$ in the spacetime of the single u wave, we recall that (as explained by Gibbons[10]) the null $\{u=U=\text{const}\}$ planes can be used as partial Cauchy surfaces on which the Klein-Gordon inner product[1] can be evaluated as

$$\begin{aligned}(\psi,\phi) &= -i\int_{U=\epsilon}(\psi \overleftrightarrow{\nabla}^\mu \phi^*)\,\Omega_\mu \\ &= -i\int_{U=\epsilon}(\psi\phi^*{}_{,V} - \phi^*\psi_{,V})\,dV\,dX\,dY,\end{aligned} \tag{13}$$

where Ω is the volume form $\frac{1}{2}dU\wedge dV\wedge dX\wedge dY$ in Minkowski space. Thus, Eqs. (12a) can be written as

$$\alpha^L{}_{\mathbf{kk'}} = -i\int_{U=0^+}\left[u_{L-,\mathbf{k}}\frac{\partial}{\partial V}u^*{}_{\text{in},\mathbf{k'}} - u^*{}_{\text{in},\mathbf{k'}}\frac{\partial}{\partial V}u_{L-,\mathbf{k}}\right]dV\,dX\,dY, \tag{14a}$$

$$\beta^L{}_{\mathbf{kk'}} = i\int_{U=0^+}\left[u_{L-,\mathbf{k}}\frac{\partial}{\partial V}u_{\text{in},\mathbf{k'}} - u_{\text{in},\mathbf{k'}}\frac{\partial}{\partial V}u_{L-,\mathbf{k}}\right]dV\,dX\,dY. \tag{14b}$$

After expressing the in modes $u_{\text{in},\mathbf{k}}$ [Eqs. (8a)] in terms of the (U,V,X,Y) coordinates [Eqs. (5a)] and evaluating the V integral, Eqs. (14) become

$$\begin{aligned}\alpha^L{}_{\mathbf{kk'}} = &-\tfrac{1}{2}(k'_z-\omega_{k'}+k_z-\omega_k)\,\delta[k_z-\omega_k-(k'_z-\omega_{k'})]\frac{1}{4\pi^2\sqrt{\omega_k\omega_{k'}}}\\ &\times\int\exp\left[i\left[(k_x-k'_x)+(k_y-k'_y)Y+\tfrac{1}{2}(k'_z-\omega_{k'})\frac{X^2-Y^2}{a}\right]\right]dX\,dY,\end{aligned} \tag{15a}$$

$$\begin{aligned}\beta^L{}_{\mathbf{kk'}} = &-\tfrac{1}{2}[k'_z-\omega_{k'}-(k_z-\omega_k)]\,\delta[(k_z-\omega_k)+(k'_z-\omega_{k'})]\frac{1}{4\pi^2\sqrt{\omega_k\omega_{k'}}}\\ &\times\int\exp\left[i\left[(k_x+k'_x)+(k_y+k'_y)Y+\tfrac{1}{2}(k'_z-\omega_{k'})\frac{Y^2-X^2}{a}\right]\right]dX\,dY.\end{aligned} \tag{15b}$$

The argument of the delta function in Eq. (15b) is negative definite since $\omega_k - k_z \geq 0$ [cf. Eq. (6a)]; this implies $\beta^L{}_{\mathbf{kk'}} \equiv 0$ (in accordance with Gibbons[10]). A further evaluation of Eq. (15a) now yields

$$\begin{aligned}\alpha^L{}_{\mathbf{kk'}} = &-\tfrac{1}{2}(k'_z-\omega_{k'}+k_z-\omega_k)\,\delta[k_z-\omega_k-(k'_z-\omega_{k'})]\frac{a}{2\pi(k'_z-\omega_{k'})\sqrt{\omega_k\omega_{k'}}}\\ &\times\exp\left[i\left[\frac{a}{2(k'_z-\omega_{k'})}[(k_y-k'_y)^2-(k_x-k'_x)^2]\right]\right],\end{aligned} \tag{16a}$$

$$\beta^L{}_{\mathbf{kk'}} \equiv 0. \tag{16b}$$

For the computation of the coefficients $\alpha^R{}_{ij}$ and $\beta^R{}_{ij}$, we proceed in exactly the same way as above. The final result is

$$\begin{aligned}\alpha^R{}_{\mathbf{kk'}} = &\tfrac{1}{2}(k'_z+\omega_{k'}+k_z+\omega_k)\,\delta[k_z+\omega_k-(k'_z+\omega_{k'})]\frac{b}{2\pi(k'_z+\omega_{k'})\sqrt{\omega_k\omega_{k'}}}\\ &\times\exp\left[-i\left[\frac{b}{2(k'_z+\omega_{k'})}[(k_y-k'_y)^2-(k_x-k'_x)^2]\right]\right],\end{aligned} \tag{17a}$$

$$\beta^R{}_{\mathbf{kk'}} \equiv 0. \tag{17b}$$

THE OUT VACUUM STATE AND PARTICLE CREATION

According to Eqs. (16), (17), and (11), we have

$$u_{L-,i} = \sum_j \alpha^L{}_{ij}\, u_{\text{in},j} \,, \tag{18a}$$

$$u_{R-,i} = \sum_j \alpha^R{}_{ij}\, u_{\text{in},j} \,; \tag{18b}$$

the vacuum states $|0, L-\rangle$ and $|0, R-\rangle$ associated with the modes $u_{L-,i}$ and $u_{R-,i}$ are both equivalent to the in vacuum $|0, \text{in}\rangle$. Now, consider an inertial observer O in the Minkowskian in region IV, moving into the flat region II on a timelike geodesic which initially in region IV has the form $v = cu - v_0$, with $c \gg 1$, $v_0 > 0$ (Fig. 1). Just before O crosses the null boundary $\{v=0\}$ between regions II and I and enters the interaction region I, his motion is entirely confined to the flat, Minkowskian region II, where the preferred vacuum state is the standard Minkowski vacuum $|0_M, \text{II}\rangle \equiv |0, L-\rangle$ associated with the modes $u_{L-,i}$ [cf. Eq. (9)]. As O crosses the boundary $\{v=0\}$, he first feels the v wave, followed by the gravitational field due to the nonlinear interaction and scattering of the two colliding waves. However, since O moves to the "right" in Fig. 1 with arbitrarily large speed ($c \gg 1$), i.e., in the same direction as the u wave and in the opposite direction to the v wave, he observes the v wave as blue-shifted by the factor c and the u wave as red-shifted by the factor $1/c$; as $c \to \infty$, all influence on O from the u wave will be obliterated by this red-shift. Therefore, in the limit $c = \infty$, when his inertial trajectory degenerates to the null geodesic $u = 0$ (Fig. 1), the observer O will not be able to distinguish his experiences from those of any other inertial observer moving on the spacetime of the *single* v wave, with the u wave absent. But the sole effect of the passage of the single v wave is to change the initial Minkowskian vacuum state (which for O is the state $|0, L-\rangle$ associated with the modes $u_{L-,i}$) in accordance with the Bogolubov transformation given by the right-hand side of Eq. (18b) (see also Fig. 1). Hence, defining a new set of mode solutions $u_{L+,i}$ by

$$u_{L+,i} \equiv \sum_j \alpha^R{}_{ij}\, u_{L-,j} \,, \tag{19a}$$

it is physically reasonable to restrict the canonical out modes $u_{\text{out},i}$ to satisfy

$$u_{\text{out},i}(u=0,v,x,y) = u_{L+,i}(u=0,v,x,y), \quad v \ge 0 \,. \tag{20a}$$

When applied to an observer who enters the interaction region from the flat region III, moving "leftward" on a timelike geodesic initially given in region IV by $u = cv - u_0$ (with $c \gg 1$ and $u_0 > 0$), the above argument yields

$$u_{\text{out},i}(u,v=0,x,y) = u_{R+,i}(u,v=0,x,y), \quad u \ge 0 \,, \tag{20b}$$

where the modes $u_{R+,i}$ are defined by

$$u_{R+,i} \equiv \sum_j \alpha^L{}_{ij}\, u_{R-,j} \,. \tag{19b}$$

The initial data (20) combined with the field equation $\Box u_{\text{out},i} = 0$ [cf. Eqs. (7) and (1)] constitute a well-defined initial-value problem for each mode $u_{\text{out},i}$. The solution of this initial-value problem determines each $u_{\text{out},i}$ uniquely throughout the interaction region $\text{I} \equiv \{u > 0,\ v > 0\}$, and consequently throughout the spacetime.

To compute the Bogolubov transformation

$$u_{\text{out},i} = \sum_j (\alpha_{ij}\, u_{\text{in},j} + \beta_{ij}\, u^*{}_{\text{in},j}) \tag{21}$$

between the in modes $u_{\text{in},i}$ and the out modes $u_{\text{out},i}$, it is not necessary to calculate the out modes explicitly. Combining Eqs. (19) and (18) with Eqs. (20), and introducing the quantities

$$\gamma_{il} \equiv \sum_j \alpha^L{}_{ij}\, \alpha^R{}_{jl} \,, \quad \eta_{il} \equiv \sum_j \alpha^R{}_{ij}\, \alpha^L{}_{jl} \,, \tag{22}$$

we find that

$$\alpha_{ij} = \tfrac{1}{2}(\gamma_{ij} + \eta_{ij}) \,, \tag{23}$$

and that β_{ij} are found by solving the integral equations

$$\tfrac{1}{2}\sum_j (\gamma_{ij} - \eta_{ij})\, u_{\text{in},j}(u,v=0,x,y) = \sum_j \beta_{ij}\, u^*{}_{\text{in},j}(u,v=0,x,y), \quad u \ge 0 \,, \tag{24a}$$

$$-\tfrac{1}{2}\sum_j (\gamma_{ij} - \eta_{ij})\, u_{\text{in},j}(u=0,v,x,y) = \sum_j \beta_{ij}\, u^*{}_{\text{in},j}(u=0,v,x,y), \quad v \ge 0 \,. \tag{24b}$$

In practice, the calculations involved in the computation of γ_{ij}, η_{ij}, and β_{ij} from Eqs. (22) and (24) are rather tedious and difficult. Even for the Khan-Penrose spacetime, the results cannot be obtained in exact analytic form without approximations. In the following, we give the final results of our computations for the Khan-Penrose background, but we do no discuss the intermediate steps. All results below are valid in the long-wavelength limit $(\omega_k \sqrt{ab}\,) \ll 1$.

Using Eqs. (16a) and (17a) and assuming $a = b$ for simplicity, Eqs. (22) give

$$\gamma_{\mathbf{k}\mathbf{k}'} = \eta_{\mathbf{k}'\mathbf{k}}$$
$$= -\frac{a^2}{4\pi\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} \exp\left[\frac{ia}{2}\left[\frac{k_x{}^2-k_y{}^2}{\omega_k-k_z}+\frac{k'_x{}^2-k'_y{}^2}{\omega_{k'}+k'_z}\right]\right]\cos\left[ma\left[\frac{k_y}{\omega_k-k_z}+\frac{k'_y}{\omega_{k'}+k'_z}\right]\right]$$
$$\times\cos\left[ma\left[\frac{k_x}{\omega_k-k_z}+\frac{k'_x}{\omega_{k'}+k'_z}\right]\right]\cos\left[\tfrac{1}{2}a\left(\omega_k+\omega_{k'}+k'_z-k_z\right)\right], \tag{25}$$

where

$$m \equiv [(\omega_{k'}+k'_z)(\omega_k-k_z)]^{1/2}. \tag{25a}$$

When inserted into Eqs. (24) together with Eqs. (8), Eqs. (25) yield

$$\beta_{\mathbf{k}\mathbf{k}'} = -\frac{a^3}{4\pi^2}\frac{1}{\sqrt{\omega_k\omega_{k'}}}\left[\frac{k_x{}^2-k_y{}^2}{k_x{}^2+k_y{}^2}\right]\frac{k_zk'_z}{|k'_z|} \times \ln\left[\frac{\omega_{k'}+|k'_z|}{\omega_{k'}-|k'_z|}\right]. \tag{26}$$

The number density of created particles[1] in the out region I,

$$n_{\mathbf{k}} \equiv \langle 0,\text{ in}\,|\,a^\dagger_{\text{out},\mathbf{k}}\,a_{\text{out},\mathbf{k}}\,|\,0,\text{ in}\rangle = \int d^3k'\,|\beta_{\mathbf{k}\mathbf{k}'}|^2, \tag{27}$$

is quadratically divergent (in k) with $\beta_{\mathbf{k}\mathbf{k}'}$ given by Eq. (26). Introducing a high-frequency cutoff k_F [in an "actual" physical collision we expect $k_F \sim (\lambda_1\lambda_2)^{-1/2}$, where λ_1 and λ_2 are the wavelengths of the colliding waves], we obtain

$$n_{\mathbf{k}}{}^{(a=b)} = \frac{a^6k_F{}^2}{24\pi}\frac{1}{\omega_k}\left[\frac{k_x{}^2-k_y{}^2}{k_x{}^2+k_y{}^2}\right]^2 k_z{}^2 \tag{28a}$$

In the general ($a\neq b$) case the result is

$$n_{\mathbf{k}} = \frac{a+b}{4ab}\frac{(ab)^3k_F{}^2}{24\pi}\left[\frac{k_x{}^2-k_y{}^2}{k_x{}^2+k_y{}^2}\right]^2 \times\frac{[(a-b)\omega_k+(a+b)k_z]^2}{(a+b)\omega_k+(a-b)k_z}. \tag{29a}$$

We can rewrite Eqs. (28a) and (29a) in the form

$$n_{\mathbf{k}}{}^{(a=b)} = a^3(ak_F)^2(ak_z)^2\left[\frac{k_x{}^2-k_y{}^2}{k_x{}^2+k_y{}^2}\right]^2\frac{k_BT}{\omega_k}, \tag{28b}$$

and

$$n_{\mathbf{k}} = (ab)^{3/2}(\sqrt{ab}\,k_F)^2\left[\tfrac{1}{2}(a-b)\omega_k+\tfrac{1}{2}(a+b)k_z\right]^2 \times\left[\frac{k_x{}^2-k_y{}^2}{k_x{}^2+k_y{}^2}\right]^2\frac{k_BT}{\omega_k+\dfrac{a-b}{a+b}k_z}, \tag{29b}$$

where the last factors are consistent with the long-wavelength $[(\omega_k\sqrt{ab}\,)\ll 1]$ limit of a (Bose-Einstein) thermal distribution with temperature

$$k_BT \equiv \frac{1}{24\pi\sqrt{ab}}, \tag{30}$$

or, in cgs units [and since the dimensionless part of the phase-space term $(ab)^{3/2}$ remains uncertain within our approximation],

$$T = \frac{\hbar c}{k_B}\frac{\text{geometric factor}}{\sqrt{ab}}. \tag{30a}$$

IN-VACUUM GREEN'S FUNCTIONS IN THE INTERACTION REGION

The most reliable way to obtain information about particle creation is to determine the response of a particle detector[21,1] moving through the out region. This requires [see Eqs. (3.54) and (3.55) of Ref. 1] the calculation in the interaction region of the in-vacuum Green's function

$$G^+(x,x') \equiv \langle 0,\text{ in}\,|\,\phi(x)\phi(x')\,|\,0,\text{ in}\rangle = \sum_i u_{\text{in},i}(x)\,u^*_{\text{in},i}(x') \tag{31}$$

and consequently of the in-mode solutions $u_{\text{in},i}(x)$.

After introducing the separation of variables

$$u_{\text{in},\mathbf{k}} \equiv \frac{1}{[(2\pi)^3 2\omega_k]^{1/2}} \times\exp[i(k_xx+k_yy)](1-\hat{u}^2-\hat{v}^2)^{-1/2}f_{\mathbf{k}}(u,v) \tag{32}$$

and using Eqs. (7), (1), and (8), the initial-value problem for the in modes in the interaction region reduces to

$$f_{\mathbf{k},uv} + \delta_{\mathbf{k}}(u,v)f_{\mathbf{k}} = 0, \tag{33a}$$

$$f_{\mathbf{k}}(u=0,v)=\exp\left[-\frac{i}{2}\frac{1}{k_z+\omega_k}\left[\frac{k_x{}^2}{1+\hat{v}}+\frac{k_y{}^2}{1-\hat{v}}\right]v\right], \quad v\geq 0, \tag{33b}$$

$$f_{\mathbf{k}}(u,v=0)=\exp\left[\frac{i}{2}\frac{1}{k_z-\omega_k}\left[\frac{k_x{}^2}{1+\hat{u}}+\frac{k_y{}^2}{1-\hat{u}}\right]u\right], \quad u\geq 0,$$

where

$$\delta_k(u,v) \equiv \frac{\hat{u}\hat{v}}{ab(1-\hat{u}^2-\hat{v}^2)^2} + \frac{k_x{}^2[(1-\hat{u}^2)^{1/2}-\hat{v}]^2[(1-\hat{v}^2)^{1/2}-\hat{u}]^2 + k_y{}^2[(1-\hat{u}^2)^{1/2}+\hat{v}]^2[(1-\hat{v}^2)^{1/2}+\hat{u}]^2}{4(1-\hat{u}^2-\hat{v}^2)^{3/2}(1-\hat{u}^2)^{1/2}(1-\hat{v}^2)^{1/2}[\hat{u}\hat{v}+(1-\hat{u}^2)^{1/2}(1-\hat{v}^2)^{1/2}]^2} . \tag{34}$$

Equations (33) and (34) cannot be solved exactly (no Riemann function[24] is available). Nevertheless, by a careful perturbation analysis using the method of strained coordinates, it is possible to obtain the following result which is exact *along* the boundaries $\{u=0,\ v\geq 0\}$ and $\{v=0,\ u\geq 0\}$, but which is accurate only to first order in the displacement away from them (i.e., along these boundaries the true $u_{\mathrm{in},k}$ have the same values and the same normal derivatives as the functions below; from here on we will refer to this kind of accuracy as "first order"):

$$u_{\mathrm{in},k}(u,v,x,y) \equiv \frac{1}{[(2\pi)^3 2\omega_k]^{1/2}} \exp[i(k_x x + k_y y)](1-\hat{u}^2-\hat{v}^2)^{-1/2} \times \exp\left[-\frac{i}{2}\frac{1}{\omega_k+k_z}\left[\frac{k_x{}^2}{1+\hat{v}}+\frac{k_y{}^2}{1-\hat{v}}\right]v - \frac{i}{2}\frac{1}{\omega_k-k_z}\left[\frac{k_x{}^2}{1+\hat{u}}+\frac{k_y{}^2}{1-\hat{u}}\right]u\right] . \tag{35}$$

Inserting Eq. (35) into Eq. (31), we obtain an expression for $G^+(x,x')$ with the same first-order accuracy as the above $u_{\mathrm{in},k}$. Obviously, this $G^+(x,x')$ is not accurate enough to calculate the response of a particle detector in the interaction region, so we will not need its explicit form. However, the Green's function

$$G^{(1)}(x,x') \equiv G^+(x,x') + G^+(x',x) \tag{36}$$

will be useful in the next section. Its first-order expression according to the above computations is

$$G^{(1)}(x,x') = -\frac{(1-\hat{u}^2-\hat{v}^2)^{-1/2}(1-\hat{u}'^2-\hat{v}'^2)^{-1/2}}{2\pi^2} \times \left[[(\Delta_u{}^2-D_u{}^2)(\Delta_v{}^2-D_v{}^2)]^{1/2} - \left[\frac{1-D_u/\Delta_u-D_v/\Delta_v}{1+D_u/\Delta_u+D_v/\Delta_v}\right]^{1/2}(y-y')^2 - \left[\frac{1+D_u/\Delta_u+D_v/\Delta_v}{1-D_u/\Delta_u-D_v/\Delta_v}\right]^{1/2}(x-x')^2\right]^{-1} , \tag{37}$$

where

$$\Delta_u \equiv \frac{a(1+\hat{u}\hat{u}')(\hat{u}-\hat{u}')}{(1-\hat{u}^2)(1-\hat{u}'^2)} , \qquad D_u \equiv \frac{a(\hat{u}+\hat{u}')(\hat{u}-\hat{u}')}{(1-\hat{u}^2)(1-\hat{u}'^2)} ,$$
$$\Delta_v \equiv \frac{b(1+\hat{v}\hat{v}')(\hat{v}-\hat{v}')}{(1-\hat{v}^2)(1-\hat{v}'^2)} , \qquad D_v \equiv \frac{b(\hat{v}+\hat{v}')(\hat{v}-\hat{v}')}{(1-\hat{v}^2)(1-\hat{v}'^2)} . \tag{37a}$$

THE RENORMALIZED IN-VACUUM STRESS-ENERGY TENSOR

For notational convenience, in this section we rename the coordinates $\hat{u}$ and $\hat{v}$ as u and v (i.e., $u\equiv\hat{u}=u/a,\ v\equiv\hat{v}=v/b$), and express the interaction-region Khan-Penrose metric (1) in the form

$$g_{\mathrm{I}} = e^{-M}du\,dv - e^{-U}(e^{V}dx^2 + e^{-V}dy^2) , \tag{38}$$

where

$$M = -\ln\left[ab\frac{(1-u^2-v^2)^{3/2}}{(1-u^2)^{1/2}(1-v^2)^{1/2}[uv+(1-u^2)^{1/2}(1-v^2)^{1/2}]^2}\right] , \tag{39a}$$

$$U = -\ln(1-u^2-v^2) , \tag{39b}$$

$$V = \ln\left[\frac{(1-u^2)^{1/2}+v}{(1-u^2)^{1/2}-v}\right] + \ln\left[\frac{(1-v^2)^{1/2}+u}{(1-v^2)^{1/2}-u}\right] . \tag{39c}$$

According to the Hadamard-Wald[25] point-splitting renormalization,[1] the final renormalized in-vacuum expectation value $\langle 0, \text{in} | T^{(f)}{}_{\mu\nu} | 0, \text{in} \rangle \equiv \langle T^{(f)}{}_{\mu\nu} \rangle$ of the stress-energy tensor

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \xi[(\phi\phi_{;\nu})_{;\mu} + (\phi\phi_{;\mu})_{;\nu}] + (2\xi - \tfrac{1}{2}) g_{\mu\nu} g^{\rho\sigma} \phi_{;\rho}\phi_{;\sigma} \tag{40}$$

for a massless, scalar field in a vacuum spacetime is obtained by the following procedure.

First, by acting on the renormalized Green's function $G^{(1)}(x,x') - G^{(1)}{}_{DS}(x,x')$ with the appropriate (symmetrized) differential operator [cf. Eq. (40)] and taking the limit $x \to x'$, the quantity $\langle 0, \text{in} | T^{(\text{ren})}{}_{\mu\nu} | 0, \text{in} \rangle \equiv \langle T^{(\text{ren})}{}_{\mu\nu} \rangle$ is constructed (see Chap. 6 of Ref. 1). Here $G^{(1)}$ is the *exact* in-vacuum Green's function and

$$G^{(1)}{}_{DS}(x,x') \equiv \frac{1}{4\pi^2}[-\det g(x)]^{-1/4} \times \left[-\frac{1}{\sigma} - a_1(x,x')(\gamma + \tfrac{1}{2}\ln|\sigma|) - \tfrac{1}{2} a_2(x')\sigma(\gamma - \tfrac{1}{2} + \tfrac{1}{2}\ln|\sigma|) \right], \tag{41}$$

where γ is Euler's constant, 2σ is the proper geodesic distance between x and x', and in a normal coordinate system y^α centered at x',

$$a_1(x,x') \equiv -\tfrac{1}{3} a_{\alpha\beta}(x') y^\alpha y^\beta, \qquad a_2(x') \equiv \tfrac{1}{3} a^\lambda{}_\lambda(x'), \tag{42}$$
$$a_{\alpha\beta} \equiv \tfrac{1}{60} R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta}.$$

Next, the final result $\langle T^{(f)}{}_{\mu\nu} \rangle$ is computed by the relation[25]

$$\langle T^{(f)}{}_{\mu\nu}(x) \rangle = \langle T^{(\text{ren})}{}_{\mu\nu}(x) \rangle - \frac{1}{64\pi^2} a_2(x) g_{\mu\nu}(x). \tag{43}$$

Since the exact in-vacuum Green's function $G^{(1)}(x,x')$ is not available in the interaction region, the above procedure cannot be straightforwardly applied to compute $\langle T^{(f)}{}_{\mu\nu} \rangle$. Nevertheless, the first-order result Eq. (37) allows us to compute $\langle T^{(f)}{}_{\mu\nu} \rangle$ *exactly* along the boundaries of the interaction region. Before discussing this result, however, we will make a number of physically plausible assumptions on the form of $\langle T^{(f)}{}_{\mu\nu} \rangle$ throughout spacetime, which will eventually allow us to compute it everywhere for the conformally coupled case $\xi = 1/6$.

First note that, as is proved in detail in Gibbons,[10] $\langle T^{(f)}{}_{\mu\nu} \rangle$ vanishes throughout spacetime for a single plane wave, consequently, in a colliding plane-wave spacetime it vanishes everywhere except in the interaction region. We assume that in the subset of spacetime lying *after* the passage of both waves,

$$\langle T^{(f)}{}_{uu} \rangle = \langle T^{(f)}{}_{vv} \rangle. \tag{44}$$

In the Khan-Penrose spacetime, this subset coincides with the interaction region, but in a more general colliding plane-wave solution it may be smaller. Note that in the coordinate system $t \equiv \tfrac{1}{2}(u+v)$, $z \equiv \tfrac{1}{2}(v-u)$, the condition (44) is equivalent to $\langle T^{(f)}{}_{tz} \rangle \equiv 0$; hence our assumption means that there is no momentum flux in the z direction after the passage of the colliding waves. (This clearly will not be the case if the colliding waves are not identical.) We also assume that the algebraic structure of $\langle T^{(f)}{}_{\mu\nu} \rangle$ respects the symmetries of spacetime; in the specific case of the Khan-Penrose solution, this means

$$\langle T^{(f)}{}_{ux} \rangle = \langle T^{(f)}{}_{uy} \rangle = \langle T^{(f)}{}_{vx} \rangle = \langle T^{(f)}{}_{vy} \rangle = \langle T^{(f)}{}_{xy} \rangle \equiv 0, \tag{45}$$

and

$$\langle T^{(f)}{}_{\hat{x}\hat{x}} \rangle = \langle T^{(f)}{}_{\hat{y}\hat{y}} \rangle, \tag{46}$$

where $\mathbf{e}_{\hat{x}}$ and $\mathbf{e}_{\hat{y}}$ are the orthonormal vector fields in the x and y directions. [Eq. (46) implies the same relation for $\langle T^{(\text{ren})}{}_{\mu\nu} \rangle$; cf. Eq. (43).]

We turn now to the analysis of $\langle T^{(f)}{}_{\mu\nu} \rangle$ on the boundaries of the interaction region. From this point on, we restrict our attention to the conformally coupled $\xi = \tfrac{1}{6}$ case, but observe that since spacetime is vacuum ($R \equiv 0$), ξ enters the analysis only through Eq. (40), i.e., only in the computation of $\langle T^{(\text{ren})}{}_{\mu\nu} \rangle$. Since in the conformally coupled (massless) case $\langle T^{(\text{ren})}{}_{\mu\nu} \rangle$ is traceless,[1,25] Eqs. (46) and (43) give

$$\langle T^{(\text{ren})}{}_{xx} \rangle = 2e^M e^{V-U} \langle T^{(\text{ren})}{}_{uv} \rangle,$$
$$\langle T^{(\text{ren})}{}_{yy} \rangle = 2e^M e^{-V-U} \langle T^{(\text{ren})}{}_{uv} \rangle, \tag{47}$$

and

$$\langle T^{(f)}{}_{xx} \rangle = 2e^{V-U} \left[e^M \langle T^{(f)}{}_{uv} \rangle + \frac{1}{64\pi^2} a_2 \right], \tag{48a}$$

$$\langle T^{(f)}{}_{yy} \rangle = 2e^{-V-U} \left[e^M \langle T^{(f)}{}_{uv} \rangle + \frac{1}{64\pi^2} a_2 \right] \tag{48b}$$

throughout the interaction region.

Now, before carrying out the actual computation, it is possible to guess the value of $\langle T^{(\text{ren})}{}_{\mu\nu} \rangle$ along the boundaries of the interaction region using the following argument: From Eq. (35), it is seen that the in modes and their first derivatives are continuous across the surfaces $\{u=0\}$ and $\{v=0\}$. [This is true only in the massless case; for a massive ($m \neq 0$) scalar field the in modes are C^0 but not C^1 along the boundaries.] In principle, this should imply that the Green's functions $G^{(1)}$ and $G^{(1)}{}_{DS}$ and their first derivatives are also continuous across these boundaries. [For the explicit expressions (37) and (41) this is *not* the case, presumably since the evaluation of the Green's functions involves infinite summations over the in modes. However, these discontinuities should not be manifest in physically observable quantities; we expect them to cancel in the renormalization process.] Since the actual renormalization through Eqs. (40)–(43) involves only first-order differentiations of $G^{(1)}-G^{(1)}{}_{DS}$, and since before the boundaries $\{u=0\}$ and $\{v=0\}$ $\langle T^{(\text{ren})}{}_{\mu\nu}\rangle = \langle T^{(f)}{}_{\mu\nu}\rangle \equiv 0$, we must have

$$\langle T^{(\text{ren})}{}_{\mu\nu}\rangle = 0 \quad \text{on } \{u=0\}\cup\{v=0\}\ . \tag{49}$$

The above result is verified by explicit computation using Eq. (37), Eqs. (40)–(43), and the standard prescription for point-splitting renormalization as described on p. 195 of Ref. 1. We will not display these calculations here, but instead go on with the analysis of $\langle T^{(f)}{}_{\mu\nu}\rangle$ in the interior of the interaction region.

In any spacetime with the metric (38)

$$R^{\mu\beta\gamma\delta}R_{\mu\beta\gamma\delta} = 16e^{2M}[2(M_{,uv})^2 + (M_{,u}V_{,u}+V_{,uu}-V_{,u}U_{,u})(M_{,v}V_{,v}+V_{,vv}-V_{,v}U_{,v})]\ . \tag{50}$$

When combined with Eqs. (39) and (42), for the Khan-Penrose spacetime Eq. (50) gives

$$a_2 \equiv \tfrac{1}{180}R^{\mu\beta\gamma\delta}R_{\mu\beta\gamma\delta} = \frac{64}{180a^2b^2}\frac{1}{\alpha^7}\frac{\theta^4}{[(1-u^2)(1-v^2)]^{1/2}} \times (2\theta^5 - 8\theta^3u^2v^2 + 6\theta u^4v^4 - \theta^2u^3v^3 + 3\theta^4uv - 2u^5v^5)\ , \tag{51}$$

where

$$\alpha \equiv 1-u^2-v^2\ , \qquad \theta \equiv uv + [(1-u^2)(1-v^2)]^{1/2}\ . \tag{52}$$

In particular, on the boundaries $\{u=0,\ v\geq 0\}\cup\{v=0,\ u\geq 0\}$ of the interaction region

$$a_2(u=0,v,x,y) = \frac{128}{180a^2b^2}\frac{1}{(1-v^2)^3}, \qquad v\geq 0\ , \tag{53a}$$

$$a_2(u,v=0,x,y) = \frac{128}{180a^2b^2}\frac{1}{(1-u^2)^3}, \qquad u\geq 0\ . \tag{53b}$$

Combined with Eqs. (49) and (53), Eq. (43) now implies that the final, renormalized stress-energy tensor $\langle T^{(f)}{}_{\mu\nu}\rangle$ on these boundaries, given by the expressions

$$\langle T^{(f)}{}_{uv}\rangle|_{\{u=0\}\cup\{v=0\}} = -\frac{1}{128\pi^2}e^{-M}a_2(x), \qquad u\geq 0,\ v\geq 0\ , \tag{54a}$$

$$\langle T^{(f)}{}_{xx}\rangle|_{\{u=0\}\cup\{v=0\}} = \frac{1}{64\pi^2}e^{V-U}a_2(x), \qquad u\geq 0,\ v\geq 0\ , \tag{54b}$$

$$\langle T^{(f)}{}_{yy}\rangle|_{\{u=0\}\cup\{v=0\}} = \frac{1}{64\pi^2}e^{-V-U}a_2(x), \qquad u\geq 0,\ v\geq 0\ , \tag{54c}$$

and

$$\langle T^{(f)}{}_{\mu\nu}\rangle|_{\{u=0\}\cup\{v=0\}} \equiv 0 \qquad \text{for } \mu\nu \neq uv,\ vu,\ xx, \text{ or } yy\ , \qquad u\geq 0,\ v\geq 0\ , \tag{54d}$$

has a Casimir-type form,[1,4] with a negative energy density $\langle T^{(f)}{}_{tt}\rangle$ (that diverges to $-\infty$ near the singularity), and positive (anisotropic) pressures $\langle T^{(f)}{}_{zz}\rangle = -\langle T^{(f)}{}_{tt}\rangle$, $\langle T^{(f)}{}_{xx}\rangle$, and $\langle T^{(f)}{}_{yy}\rangle$ (that diverge to $+\infty$ near the singularity).

To calculate $\langle T^{(f)}{}_{\mu\nu}\rangle$ in the interior of the interaction region, we use the conservation equations

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}T^{\alpha\mu} \equiv 0\ . \tag{55}$$

When applied to the tensor $\langle T^{(f)}{}_{\mu\nu}\rangle$ using Eqs. (38), (39), (44)–(46), and (48), Eqs. (55) reduce to the system

$$e^{M}Q_{,u} + e^{U}P_{,v} = \frac{1}{16\pi^2}e^{-U}U_{,v}a_2\ , \tag{56a}$$

$$e^{U}P_{,u} + e^{M}Q_{,v} = \frac{1}{16\pi^2}e^{-U}U_{,u}a_2\ , \tag{56b}$$

where the functions $P(u,v)$ and $Q(u,v)$ are defined by

$$\langle T^{(f)}{}_{uv}\rangle \equiv \tfrac{1}{4}e^{2U-M}P,$$
$$\langle T^{(f)}{}_{uu}\rangle = \langle T^{(f)}{}_{vv}\rangle \equiv \tfrac{1}{4}e^{U}Q\ . \tag{57}$$

In terms of P and Q, the only remaining nonzero components of $\langle T^{(f)}{}_{\mu\nu}\rangle$ are

$$\langle T^{(f)}{}_{xx}\rangle = \tfrac{1}{2}e^{V-U}\left[e^{2U}P + \frac{1}{16\pi^2}a_2\right], \tag{58a}$$

$$\langle T^{(f)}{}_{yy}\rangle = \tfrac{1}{2}e^{-V-U}\left[e^{2U}P + \frac{1}{16\pi^2}a_2\right]. \tag{58b}$$

Using Eqs. (57) and the boundary values given by Eqs. (54), we can rewrite Eqs. (56) in the form of a well-defined initial-value problem: The evolution equations for P and Q in the interaction region are

$$e^{M-U}Q_{,u} + P_{,v} = -\frac{1}{32\pi^2}(e^{-2U})_{,v}\,a_2\,, \tag{59a}$$

$$e^{M-U}Q_{,v} + P_{,u} = -\frac{1}{32\pi^2}(e^{-2U})_{,u}\,a_2\,, \tag{59b}$$

and the initial values are

$$P(u=0,v) = -\frac{1}{45\pi^2a^2b^2}\frac{1}{1-v^2}\,,\quad v \geq 0\,, \tag{60a}$$

$$P(u,v=0) = -\frac{1}{45\pi^2a^2b^2}\frac{1}{1-u^2}\,,\quad u \geq 0\,, \tag{60b}$$

$$Q(u=0,v) = Q(u,v=0) \equiv 0,\quad u \geq 0,\ v \geq 0\,. \tag{60c}$$

The initial-value problem (59) and (60) cannot be solved in closed form. However, it follows from Eqs. (59) that

$$(e^{M-U}Q_{,u})_{,u} - (e^{M-U}Q_{,v})_{,v} = -\frac{1}{32\pi^2}[(e^{-2U})_{,v}a_{2,u} - (e^{-2U})_{,u}a_{2,v}]\,. \tag{61}$$

Therefore, if the quantity a_2 [Eq. (51)] were a function of $\alpha = e^{-U}$ *only* [cf. Eq. (52)], then the right-hand side of Eq. (61) would vanish identically, and combined with the initial data (60c) this would imply $Q \equiv 0$ throughout the interaction region. But a_2 clearly *does* depend on θ as well as on α [Eqs. (51) and (52)], hence Q cannot be identically zero. Nevertheless, our analysis in Ref. 15 shows that as $\alpha \rightarrow 0$, i.e., near the singularity $\{u^2+v^2=1\}$, a_2 asymptotically *is* a function of α only, i.e., as $\alpha \rightarrow 0$ $R^{\mu\nu\lambda\delta}R_{\mu\nu\lambda\delta}$ is asymptotically independent of θ. [See Secs. III and IV of Ref. 15, especially Eqs. (2.19), (3.23), (3.33)–(3.35), (3.38), and (4.7). Also observe that $R^{\mu\nu\lambda\delta}R_{\mu\nu\lambda\delta} = 32\Psi_2{}^2 + 16\Psi_0\Psi_4$ in any parallel-polarized colliding plane-wave spacetime.] We conclude that Q is asymptotically negligible compared to P near the singularity $\{\alpha=0\}$.

To obtain some information on $\langle T^{(f)}{}_{\mu\nu}\rangle$ near the singularity, we compute P on the central timelike world line $u=v$, using the ordinary differential equation along this world line to which both Eq. (59a) and Eq. (59b) reduce when Q is neglected. The final result is

$$P(u,u) = \frac{1}{45\pi^2a^2b^2}\left[\frac{1}{1-2u^2}\{\tfrac{3}{10}[1-(1-2u^2)^5] + u^2 + u^4 - 4u^6 + 2u^8\} - 1\right]. \tag{62}$$

Although initially it is negative, $P(u,u)$ rapidly changes sign and diverges to $+\infty$ as $u \rightarrow 1/\sqrt{2}$, i.e., as the world line $u=v$ approaches the singularity $\{u^2+v^2=1\}$. This means that deep inside the interaction region, away from the initial null surfaces $\{u=0\}$ and $\{v=0\}$, $\langle T^{(f)}{}_{\mu\nu}\rangle$ no longer has the Casimir-type form (54), but instead has an anisotropic "inflationary" form where the energy density as well as the two pressures along the Killing x, y directions are positive, while the pressure along the inhomogeneous z direction is negative [Eqs. (57) and (58)]. In other words, $\langle T^{(f)}{}_{\mu\nu}\rangle$ near the singularity has a form that introduces a positive, effective cosmological constant "in the z direction"; it is reasonable to expect that this would tend to smooth out the z-dependent inhomogeneities of the Khan-Penrose spacetime when the back reaction of $\langle T^{(f)}{}_{\mu\nu}\rangle$ on the geometry is taken into account.

Although our method of computing $\langle T^{(f)}{}_{\mu\nu}\rangle$ for the Khan-Penrose spacetime is unlikely to be adequate in other colliding plane-wave solutions, we speculate that, at least in its qualitative aspects, the information obtained through our calculations in this paper is applicable to more general colliding gravitational-wave spacetimes. In particular, for a highly inhomogeneous and anisotropic initial singularity which can be modeled by a time-reversed colliding gravitational-wave spacetime (see Fig. 2 and the discussion in the introductory section of this paper), we speculate that $\langle\, 0,\text{ out}\,|\,T_{\mu\nu}\,|\,0,\text{ out}\,\rangle$ near the singularity will have an inflationary form, and will tend to dissipate the initial inhomogeneities by causing a local de Sitter-type expansion through its back reaction on the geometry. [Here $|0,\text{ out}\rangle$ is the vacuum state which is

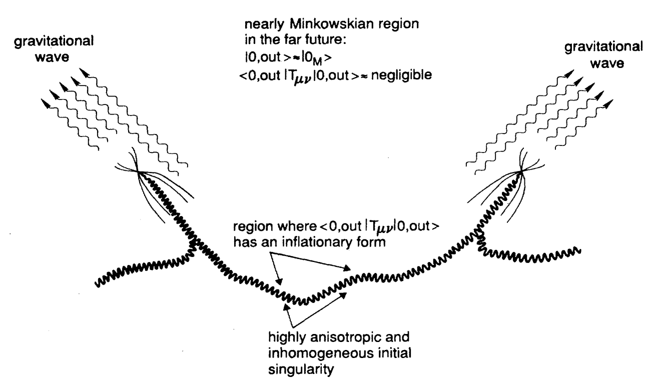

FIG. 2. Although our method of computing $\langle T^{(f)}{}_{\mu\nu}\rangle$ for the Khan-Penrose spacetime is unlikely to be adequate in other colliding plane-wave solutions, we speculate that in its qualitative aspects the information obtained through our calculations is applicable to more general colliding gravitational-wave spacetimes. In particular, for a highly inhomogeneous and anisotropic initial singularity which can be modeled by a time-reversed colliding gravitational-wave spacetime (see the discussion in the introductory section), we speculate that $\langle\, 0,\text{ out}\,|\,T_{\mu\nu}\,|\,0,\text{ out}\,\rangle$ near the singularity has an inflationary form, and tends to dissipate the initial inhomogeneities by causing a local de Sitter-type expansion through its back reaction on the geometry. Here $|0,\text{ out}\rangle$ is the vacuum state which is nearly Minkowskian throughout the nearly-flat, out-region in the far future. With the reversed time orientation, this state is analogous to the in vacuum which we use throughout the paper.

nearly Minkowskian throughout the nearly flat out region in the far future (Fig. 2); this state is the time-reversed analogue of the in vacuum which we have used throughout the paper.] If it is granted, either on the basis of the anthropic principle[26] or of quantum cosmology,[27] that quantum fields initially start out in a vacuum state close to $|0, \text{out}\rangle$, then the above result (if true) might have interesting consequences for cosmology. Our speculations are consistent with the results of previous particle-production computations for anisotropic cosmological models,[7] but they must be verified with further calculations along the lines of this paper.

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