Multiphoton Monte Carlo event generator for Bhabha scattering at small angles

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We describe in this paper the application of the theory of Yennie, Frautschi, and Suura (YFS) to construct a Monte Carlo (MC) event generator with multiple-photon production for Bhabha scattering at low angles. The respective generator provides the four-momenta of the electron and positron and of all soft and hard photons with a proper treatment of the phase space and conservation of the total four-momentum. The final-state electron and positron are assumed to be visible above some minimum angle with respect to the beams (double tag). The QED matrix element in the algorithm is taken according to the YFS exponentiation scheme. The Monte Carlo program will be helpful in luminosity determination at experiments at the SLAC Linear Collider and the CERN collider LEP; it takes into account QED $O(\alpha)$ and the leading higher-order corrections. The important difference with the existing MC procedures is that the minimum energy above which photons are generated may be set arbitrarily low. Sample Monte Carlo data are illustrated in our discussion.

I. INTRODUCTION

The Bhabha-scattering process

$$e^+(p_1) + e^-(q_1) \rightarrow e^+(p_2) + e^-(q_2)$$

at the scattering angle range 1–40 mrad will be commonly used in all experiments at the CERN collider LEP and the SLAC Linear Collider (SLC) as a source of data for luminosity determination. The reason is that the cross section at such low angles is large and it comes almost exclusively from a pure QED source: namely, from t-channel photon exchange. The s-channel Z^0 contribution is below 3%, even on the top of the resonance.

Since the total cross section for all other processes will be calculated using a luminosity deduced from Bhabha event rates, any uncertainty in this measurement will propagate to all of them. Absolute normalization of the cross section is of particular importance for determination of the Z^0 line shape and for calculating the number of neutrinos using the total cross section at the top of resonance. It will be highly desirable to know the luminosity with 1% uncertainty.[1] This means that the theoretical cross section for the Bhabha process must be known with even better precision. This goal requires very good control over radiative effects. Most probably inclusion of $O(\alpha)$ radiative corrections is not enough and one will have to include at least the leading $O(\alpha^2)$ correction.

In the presence of the radiative effects it is practically impossible to calculate the integrated cross section without the help of the Monte Carlo method in the form of an event generator. It is especially true for low-angle Bhabha scattering due to the strong dependence on the minimum electron-positron angle with respect to the beam. Because of the sharp angular dependence of the differential cross section even the smallest smearing of the $e^{\pm}$ angles due to soft-photon emissions should be taken into account.

In this paper we shall describe the application of our recently developed Yennie-Frautschi-Suura[2] (YFS) Monte Carlo approach to $SU_{2L} \times U_1$ radiative corrections to Bhabha scattering with the consequence of solving all of the above problems. The present version of the program provides an arbitrary number of soft and hard photons. The implemented QED matrix element is not strictly valid, however, for two hard photons (above a few GeV). The probability of such two-hard-photon events is sufficiently small that the program in the present form can actually be used both for studies of topology of events and for calculating the total cross section with $\sim 1\%$ precision at low angles.

In the final version we plan to implement the QED matrix element based on the YFS scheme which will be valid for arbitrary number of soft photons and up to two hard photons. It should be stressed that the YFS scheme is rather well founded from the theoretical point of view and it allows us to sum up in a consistent way the real and virtual soft contributions in all orders of perturbative QED. It can be exploited as a very efficient tool in summing up the leading higher-order contributions without introducing theoretical uncertainties on what is actually done.

We would like to mention that similar work for μ-pair and τ-pair production is in progress.[3,4] The first version of the multiphoton initial-state radiation is incorporated

already in Ref. 5. For some details on the YFS approach we refer the reader to Refs. 2–4. The work presented in what follows is also related to another project for deep-inelastic *ep* scattering (see Ref. 6).

Our work is presented as follows. In the next section we give the respective representation of the differential cross section and a discussion of the relevant aspects of our Monte Carlo procedure for Bhabha scattering at low angles (1–140 mrad). In Sec. III we present some sample Monte Carlo data in comparison with the conventional single-bremsstrahlung Monte Carlo event generator similar to that of Berends and Kleiss in Ref. 7. Section IV contains our summary remarks.

II. DIFFERENTIAL DISTRIBUTIONS AND MONTE CARLO METHOD

Since the main emphasis in this paper is on the Monte Carlo procedure we start with a formula for the total cross section which includes phase-space integrals in a form ready for the Monte Carlo (MC) integration. The differential cross section for the process

$$e^+(p_1)+e^-(q_1) \to e^+(p_2)+e^-(q_2)+\gamma(k_1)+ \cdots$$
$$+\gamma(k_n) \tag{1}$$

is summed over n and integrated over the phase space in the following way:

$$\sigma = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^3q_2}{q_2^0} \frac{d^3p_2}{p_2^0} \prod_{i=1}^{n} \left[\frac{d^3k_i}{k_i^0} \widetilde{S}(k_i)\theta_\epsilon^{c.m.}(k_i) \right] \delta^4 \left[p_1+q_1-p_2-q_2-\sum_{i=1}^{n} k_i \right]$$

$$\times \exp \left[2\alpha \operatorname{Re}B + \int \frac{d^3k}{k^0} \widetilde{S}(k)[1-\theta_\epsilon^{c.m.}(k)] \right]$$

$$\times \left[\bar{\beta}_0(\mathscr{R}p_2,\mathscr{R}q_2) + \sum_{l=1}^{n} \bar{\beta}_1(\mathscr{R}p_2,\mathscr{R}q_2,k_l)\widetilde{S}(k_l)^{-1} \right] . \tag{2}$$

Let us explain the main ingredients in the above expression.

(a) The infrared singularity in the factors

$$\widetilde{S}(k) = -\frac{\alpha}{4\pi^2} \left[\frac{p_1}{p_1k} - \frac{p_2}{p_2k} - \frac{q_1}{q_1k} + \frac{q_2}{q_2k} \right]^2 \tag{3}$$

is excluded from the integration domain by means of the conventional energy cutoff in the center-of-mass system. This is done with help of

$$\theta_\epsilon^{c.m.}(k) = \theta(2k^0/\sqrt{s}-\epsilon) \tag{4}$$

which is equal zero for $k^0 < \epsilon\sqrt{s}/2$.

(b) The integral includes hard photons all over the complete phase space.

(c) The explicit dependence of the integrated cross section in Eq. (1) on the infrared cutoff ϵ (coming from the integration limits) is in fact completely counterbalanced by the Yennie-Frautschi-Suura[2] form factor

$$F_{YFS}(p_i,q_i,\epsilon) = \exp \left[2\alpha \operatorname{Re}B + \int \frac{d^3k}{k^0} \widetilde{S}(k)[1-\theta_\epsilon^{c.m.}(k)] \right]$$

$$= \exp[R(p_1,p_2)+R(q_1,q_2)+R(p_1,q_1)+R(p_2,q_2)-R(p_1,q_2)-R(p_2,q_1)] ,$$

$$R(p,q) = \frac{\alpha}{\pi} \left[\left[\ln\frac{2pq}{m_e^2}-1 \right] \ln\frac{s\epsilon^2}{4p^0q^0} + \frac{1}{2}\ln\frac{2pq}{m_e^2} \right] . \tag{5}$$

Neither the total cross section nor any other measurable quantity depends on ϵ. It plays only the role of a dummy parameter introduced to limit the multiplicity of very soft photons for the purpose of the numerical MC simulation. (It should be noted that the above expression for F_{YFS} is written in the leading-log approximation. The precise form is calculable and involves some additional dilogarithms.)

(d) The functions $\bar{\beta}_{0,1}$ are infrared finite and are easily calculable up to an $O(\alpha)$. For the purpose of the basic Monte Carlo program we take the lowest-order version of $\bar{\beta}_0$ only. It is up to a normalization constant equal to a Born differential cross section. The meaning of $\mathscr{R}p_2$ and $\mathscr{R}q_2$ is the following. Strictly speaking $\bar{\beta}_{0,1}$ are defined within the corresponding two- or three-body phase space and if in formula (2) there are some additional photons then in the arguments of $\bar{\beta}_{0,1}$ parameters the adjustment $p_2 \to \mathscr{R}p_2$ and $q_2 \to \mathscr{R}q_2$ has to be performed. It is done in such a way that one requires $p_1+q_1=\mathscr{R}p_2+\mathscr{R}q_2$ for $\bar{\beta}_0$ and $p_1+q_1=\mathscr{R}p_2+\mathscr{R}q_2+k_l$ for $\bar{\beta}_1(k_l)$. This procedure is related to the fact that in the YFS scheme infrared singular factors $\widetilde{S}$ are subtracted and the residua,

equal to $\bar{\beta}_i$, are taken at the singularity position ($k^0=0$). The $\mathscr{R}$ procedure concerns only $\bar{\beta}_i$ arguments and does not disturb the phase-space integral or four-momentum conservation. In our case we take

$$\bar{\beta}_0 = \frac{2s\alpha^2}{t_{\mathscr{R}}^2} \frac{1}{2} \left[1 + \left[\frac{u_{\mathscr{R}}}{s} \right]^4 + \left[\frac{t_{\mathscr{R}}}{s} \right]^4 \right] , \quad (6)$$

where $t_{\mathscr{R}}=(\mathscr{R}p_2-p_1)^2$, $u_{\mathscr{R}}=-s-t_{\mathscr{R}}$; the $\mathscr{R}$ procedure is specified in Ref. 5.

For completeness, let us recapitulate the discussion in Ref. 5 concerning this $\mathscr{R}$ operation. Specifically, we transform q_2 and p_2 to the rest frame $\mathbf{p}_2+\mathbf{q}_2=\mathbf{0}$. In this frame, $\mathbf{q}_2$ and $\mathbf{p}_2$ are scaled by a factor which corresponds to exclusion of the additional photons ($n_\gamma \geq 1$ for $\bar{\beta}_0$ and $n_\gamma \geq 2$ for $\bar{\beta}_1$, where n_γ is the total number of photons in the event candidate); the momenta are then boosted back to the c.m. system with a boost parameter which takes the respective exclusion of the additional photons into account. The resulting momenta then satisfy $\mathscr{R}q_2+\mathscr{R}p_2=p_1+q_1$ for $\bar{\beta}_0$ and $\mathscr{R}q_2+\mathscr{R}p_2+k=q_1+p_1$ for $\bar{\beta}_1(k)$. One may wonder how sensitive our numerical work is to the precise realization of this $\mathscr{R}$ operation, which was already anticipated in the original paper of Yennie, Frautschi, and Suura. The situation is the following. If one sums to all n on $\bar{\beta}_n$, one can show that the results of different $\mathscr{R}$-operation realizations are identical. If one stops to a finite number of $\bar{\beta}_n$, as we do, then one can use the properties of the change in the $\bar{\beta}_j$ under a change of $\mathscr{R}$-operation realization to show that the net change on our results will be of order $(\alpha/\pi)^{n'+1}$ if we stop our $\bar{\beta}_n$ sum at $n=n'$. For, if the additional photon energies all vanish, the two $\mathscr{R}$-operation realizations are identical. Hence, the change in our results associated with a change in the realization of $\mathscr{R}$ is infrared finite; it therefore will generate a change in our numerical results here of the order of $(\alpha/\pi)^2$, since there are no additional large infrared logarithms in $\bar{\beta}_0$ and $\bar{\beta}_1$ independent of the $\mathscr{R}$-operation realization for low-angle double-tag Bhabha

scattering. We have found that such a change is generally below the level of the pure weak radiative corrections to Bhabha scattering at low angles.

We shall now introduce the reader to the method of generating Monte Carlo events. The procedure of constructing Monte Carlo algorithms is generally the following: we shall gradually simplify the integrand and the phase-space limits such that at the end we obtain a simple distribution which can be easily generated with the help of the uniform random numbers. All these modifications have to be corrected for by appropriate reweighting and rejecting the events generated according to the simplified distribution. At the end of this section we shall summarize all weights which were introduced in the course of the simplifications. The exact integrated cross section is calculable numerically using the average weights from the MC generation and may be obtained with the arbitrary precision, a precision which increases with the number of generated events.

Let us first reject from the integrand all terms due to interference of the real bremsstrahlung among the positron and electron lines. It means that in the infrared factors

$$\tilde{S}(k_i)=\tilde{S}_p(k_i)+\tilde{S}_q(k_i)+\tilde{S}_{int}(k_i) ,$$

where

$$\tilde{S}_p(k) = \frac{-\alpha}{4\pi^2} \left[\frac{p_1}{p_1 k} - \frac{p_2}{p_2 k} \right]^2 ,$$

$$\tilde{S}_q(k) = \frac{-\alpha}{4\pi^2} \left[\frac{q_1}{q_1 k} - \frac{q_2}{q_2 k} \right]^2 , \quad (7)$$

$$\tilde{S}_{int}(k) = 2 \left[\frac{\alpha}{4\pi^2} \right] \left[\frac{p_1}{p_1 k} - \frac{p_2}{p_2 k} \right] \left[\frac{q_1}{q_1 k} - \frac{q_2}{q_2 k} \right] ,$$

we neglect $\tilde{S}_{int}$. After this change our modified master formula may be rewritten in the following way (we show only $\bar{\beta}_0$ for the purpose of illustration):

$$\sigma = \int \frac{d^3 q_2}{q_2^0} \frac{d^3 p_2}{p_2^0} F_{YFS}(p_i,q_i)\bar{\beta}_0(\mathscr{R}p_2,\mathscr{R}q_2) \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!n'!} \delta^4 \left[p_1+q_1-p_2-q_2-\sum_{i=1}^{n} k_i - \sum_{j=1}^{n'} k_j' \right]$$

$$\times \prod_{i=1}^{n} \frac{d^3 k_i}{k_i^0} \tilde{S}_p(k_i)\theta_\epsilon^{c.m.}(k_i) \prod_{j=1}^{n'} \frac{d^3 k_j'}{k_j'^0} \tilde{S}_q(k_j')\theta_\epsilon^{c.m.}(k_j') . \quad (8)$$

Now comes the most crucial step before the Monte Carlo algorithm can be set up. We are going to parametrize the phase-space integral with a set of special variables which will be related in a straightforward way to inform random numbers. The aim of this operation is the following: In the new variables we must entirely control the leading singularity in our integral which is roughly $X=(1/t^2)\Pi\tilde{S}(k_i)$. It means that with new variables we will be able to generate phase-space points according to X or, even better, this factor will be canceled by the Jacobian of the transformation from ordinary phase-space variables (four-momenta) to the new ones. In technical terms we are going to make importance sampling for the X factor. Needless to say, the total four-momentum-conserving delta function is the main obstacle in this task.

There are two major steps in the process of reorganizing the phase-space integral. First, we split the integral into a two-body phase space and two multibody subintegrals related to four-momenta $\Pi_p=p_2+\sum_{i=1}^{n} k_i$ and $\Pi_q=q_2+\sum_{j=1}^{n'} k_j'$ according to the kinematical tree depicted in Fig. 1:

$$\sigma = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!n'!} \int_{-\infty}^{t_{min}} \frac{dt}{s} \int_0^{2\pi} d\phi \int_{m_e^2}^{\infty} dM_p^2 \int_{m_e^2}^{\infty} dM_q^2 \int \delta^4 \left[\Pi_p - p_2 - \sum_{i=1}^{n} k_i \right] \frac{d^3 p_2}{p_2^0}$$

$$\times \prod_{i=1}^{n} d\omega(k_i) \theta_\epsilon^{c.m.}(k_i) \int \delta^4 \left[\Pi_q - q_2 - \sum_{j=1}^{n'} k_j' \right] \frac{d^3 q_2}{q_2^0} \prod_{j=1}^{n'} d\omega(k_j') \theta_\epsilon^{c.m.}(k_j') F_{YFS}(p_{i'}, q_{i'}) \bar{\beta}_0(\mathscr{R} p_2, \mathscr{R} q_2) \Theta(s, t, M_p^2, M_q^2) ,$$

(9)

where

$$d\omega(k) = \frac{d^3 k}{k^0} \tilde{S}(k), \quad M_\eta^2 = \Pi_\eta^2, \quad \eta = q, p, \quad t = (p_1 - \Pi_p)^2 .$$

The factor Θ keeps track of the upper limits on t, M_p^2, and M_q^2, i.e., normally $\Theta = 1$ but if t, M_p^2, M_q^2 are outside phase-space limits $\Theta = 0$. In the MC program the variables t, M_p^2, M_q^2 will be generated up to infinite limits and Θ will be implemented by rejection.

In the second step we parametrize the two multibody subintegrals in (9) in terms of the light-cone-style variables. In the case of the first subintegral, related to Π_p, we do it in the reference frame where $p_1^0 - p_2^0 = 0$ and $\mathbf{p}_1 + \mathbf{p}_2 = 0$. We call it the QRS_p frame. The integral can be rewritten as

$$\int dW_p^{(n)} = \int dM_p^2 \int \delta^4 \left[\Pi_p - p_2 - \sum_{i=1}^{n} k_i \right] \frac{d^3 p_2}{p_2^0} \prod_{i=1}^{n} d\omega(k_i) \theta_\epsilon^{c.m.}(k_i)$$

$$= \int \left[\frac{t_p}{t} \right] \prod_{i=1}^{n} d\omega(y_i, z_i, \phi_i) \theta_\epsilon^{c.m.}(k_i) ,$$

(10)

where $t_p = -2 p_1 p_2$ and

$$d\omega_i = d\omega(y_i, z_i, \phi_i)$$

$$= \frac{\alpha}{\pi} dy_i dz_i \frac{d\phi_i}{2\pi} \left[\frac{1}{y_i z_i} - \frac{m_e^2}{|t_p|} \left[\frac{1}{y_i^2} + \frac{1}{z_i^2} \right] \right] \theta \left[y_i - \frac{m_e^2}{|t_p|} z_i \right] \theta \left[z_i - \frac{m_e^2}{|t_p|} y_i \right] .$$

(11)

The new variables y_i, z_i, ϕ_i are related to k_i^μ in the QRS_p frame as follows: Let

$$k_i^0 = (\alpha_i + \beta_i) E_p, \quad k_i^1 = |\mathbf{k}_T| \cos\phi_i ,$$

$$k_i^3 = (-\alpha_i + \beta_i) E_p, \quad k_i^2 = |\mathbf{k}_T| \sin\phi_i ,$$

$$|\mathbf{k}_T| = 2 E_p \sqrt{\alpha_i \beta_i} ,$$

(12)

where E_p is a solution of a simple equation

$$t = Q_p^2 + 2 Q_p \sum_{i=1}^{n} k_i + \left[\sum_{i=1}^{n} k_i \right]^2 ,$$

$$Q_p = p_2 - p_1 = (Q_p^0, \mathbf{Q}_p) \equiv (0, 0, 0, 2 E_p) .$$

(13)

FIG. 1. Kinematical tree for multiphoton Bhabha scattering.

The Sudakov variables α_i and β_i are related to our variables y_i, z_i as

$$\alpha_i = \tilde{\alpha}_i \Big/ \left[1 - \sum_{k=1}^{n} \tilde{\beta}_k \right], \quad \tilde{\alpha}_i = y_i - \frac{m_e^2}{|t_p|} z_i ,$$

$$\beta_i = \tilde{\beta}_i \Big/ \left[1 - \sum_{k=1}^{n} \tilde{\beta}_k \right], \quad \tilde{\beta}_i = z_i - \frac{m_e^2}{|t_p|} y_i .$$

(14)

Summarizing we have introduced a three-level change of the variables

$$(k_i^\mu) \overset{(1)}{\leftrightarrow} (\alpha_i, \beta_i, \phi_i) \overset{(2)}{\leftrightarrow} (\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i) \overset{(3)}{\leftrightarrow} (y_i, z_i, \phi_i) ,$$

(15)

where the third transformation is rather cosmetic, for the purpose of the MC program. It is quite remarkable that for such a complicated change of the variables the Jacobian factor is so simple (t_p/t); nonetheless, this is what happens in (15). Finally, the phase-space limits in the new variables are also very simple:

$$0 < \tilde{\alpha}_i < 1, \quad 0 < \tilde{\beta}_i < 1, \quad 0 < \phi_i < 2\pi, \quad \sum_{i=1}^{n} \tilde{\beta}_i < 1 .$$

(16)

From now on we incorporate the condition $\sum_i \tilde{\beta}_i < 1$ into Θ.

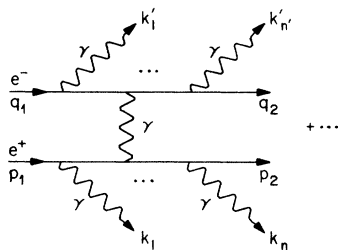The completely analogous change of the variables

$(k_j^{\prime\mu}) \leftrightarrow (y_j', z_j', \phi_j')$ is done in the second integral related to Π_q. In that case we work at the reference frame QRS_q, where $Q_q = q_2 - q_1 \equiv (0,0,0,2E_q)$, $\mathbf{q}_1 + \mathbf{q}_2 = 0$, and $t_q = Q_q^2 = -4E_q^2$.

With these transformations our master integral simplifies significantly:

$$\sigma = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!n'!} \int \frac{dt}{s} d\phi \left[\frac{t_p}{t}\right] \prod_{i=1}^{n} d\omega_i \theta_\epsilon^{\mathrm{c.m.}}(k_i) \left[\frac{t_q}{t}\right]$$

$$\times \prod_{j=1}^{n'} d\omega_j' \theta_\epsilon^{\mathrm{c.m.}}(k_j') F_{\mathrm{YFS}} \bar{\beta}_0 \Theta \qquad (17)$$

and the way to the Monte Carlo integration is open. The singularity $1/t^2$ is under control because the integration over t is explicit and the infrared factors $\tilde{S}(k_i)$ are also under control because the distribution $d\omega_i \simeq d \ln y_i d \ln z_i$ will be trivial to generate.

It should be stressed that, apart from the one modification, i.e., omission of $\tilde{S}_{\mathrm{int}}$, the formula (17) is precisely equivalent to our starting expression (2). The discussion of kinematics was far from complete, a lot of important details were omitted, and we will present them elsewhere.[8]

Although formula (17) brings us closer to the MC program, there are still important obstacles on the way. Let us list three of the most important ones. (1) The variables t_p and t_q are not independent but they are very complicated functions of all $y_i, z_i, \phi_i, y_j', z_j', \phi_j'$ and t. They enter into $d\omega_i$ and $d\omega_j'$ and, because of that, these distributions cannot be generated in their present form. (2) The same t_p and t_q and other dot products of fermion momenta enter into F_{YFS}. (3) The generation of our new variables $y_i, z_i, \ldots$ (in terms of which the densities are so simple) is also inhibited because the infrared singularity is eliminated out of the phase space with help of

$$\theta^{\mathrm{c.m.}}(\epsilon) \equiv \prod_{i=1}^{n} \theta_\epsilon^{\mathrm{c.m.}}(k_i) \prod_{j=1}^{n'} \theta_\epsilon^{\mathrm{c.m.}}(k_j') \qquad (18)$$

which is a very complicated object in terms of them. The reason is that the Lorentz transformations from QRS_p and QRS_q to the laboratory are involved.

We shall eliminate the above obstacles and the other ones and get a workable Monte Carlo algorithm by a series of simplifications of the integrand in (17) and/or enlargements of the integration domain. The resulting distribution (integrand) is easy to generate and all changes are properly countered later on by appropriate reweighting and/or rejecting MC events.

(S1) The first change is $d\omega_i \rightarrow d\tilde{\omega}_i$ in which we replace $|t_p|$ with its maximum value s:

$$d\tilde{\omega}_i = \left[\frac{\alpha}{\pi}\right] \frac{d\phi_i}{2\pi} \frac{d\tilde{y}_i d\tilde{z}_i}{\tilde{y}_i \tilde{z}_i} \theta \left[\tilde{y}_i - \frac{m_e^2}{s} \tilde{z}_i\right] \theta \left[\tilde{z}_i - \frac{m_e^2}{s} \tilde{y}_i\right],$$
$$\qquad (19a)$$

$$\tilde{y}_i = \tilde{\alpha}_i + \frac{m_e^2}{s} \tilde{\beta}_i, \quad \tilde{z}_i = \tilde{\beta}_i + \frac{m_e^2}{s} \tilde{\alpha}_i,$$

and, in the same step, we also put

$$\frac{t_p}{t} \frac{t_q}{t} \rightarrow 1. \qquad (19b)$$

(S2) The same is done with $d\omega_j'$.

(S3) Next we enlarge the phase space limits with the replacement $\Theta \rightarrow 1$. We recall that Θ takes care of upper limits on M_p^2, M_q^2 and of the conditions $\sum_i \tilde{\beta}_i < 1$ and $\sum_j \tilde{\beta}_j' < 1$.

(S4) The new distributions $d\tilde{\omega}_i$ and $d\tilde{\omega}_i'$ can be generated easily provided the lower boundary on $\tilde{y}_i, \tilde{z}_i$ is simple. We replace, therefore,

$$\theta^{\mathrm{c.m.}}(\epsilon) \rightarrow \theta^{qms}(\delta) = \prod_{i=1}^{n} \theta_i(\delta) \prod_{j=1}^{n'} \theta_j'(\delta), \qquad (20)$$

where

$$\theta_i(\delta) = \theta(\max(\tilde{y}_i, \tilde{z}_i) - \delta) \simeq \theta(\max(y_i, z_i) - \delta),$$
$$\theta_j'(\delta) = \theta(\max(\tilde{y}_j', \tilde{z}_j') - \delta) \simeq \theta(\max(y_j', z_j') - \delta). \qquad (21)$$

This modification is acceptable only if the new infrared domain defined by $\theta^{qms}(\delta) = 0$ falls entirely inside the old one defined by $\theta^{\mathrm{c.m.}}(\epsilon) = 0$. This may be always achieved by taking δ very small but the price may be high in terms of excessive rejection. We find that for $|t_{\min}|/s \simeq 3 \times 10^{-5}$ one has to use $\delta = 10^{-2}\epsilon$ and the rejection rate is around 90%.

(S5) The Yennie-Frautschi-Suura form factor we replace simply with a constant

$$F_{\mathrm{YFS}} \rightarrow \bar{F}_{\mathrm{YFS}} = \exp\left[-\frac{4\alpha}{\pi} \ln \frac{s}{m_e^2} \ln \frac{1}{\delta}\right]. \qquad (22)$$

(S6) For the sake of completeness we include in this list the modification

$$\tilde{S}_{\mathrm{int}} \rightarrow 0 \qquad (23)$$

which we have done earlier.

(S7) The next replacement, as the two previous ones, concerns rather the scattering amplitude than the phase space

$$\left[\bar{\beta}_0(\mathcal{R}p_2, \mathcal{R}q_2) + \sum_{l=1}^{n} \bar{\beta}_1(\mathcal{R}p_2, \mathcal{R}q_2, k_l)\tilde{S}(k_l)^{-1} + \cdots\right]$$

$$\rightarrow b_0 = \frac{2\alpha^2 s}{t^2}. \qquad (24)$$

This is usually a good approximation, especially if some (even mild) cutoff on the total photon energy is in place.

Altogether the approximate new distribution reads

$$\sigma_0 = \int_{|t_{\min}|}^{\infty} \frac{dt}{s} \int_0^{2\pi} d\phi \, \bar{F}_{\mathrm{YFS}} b_0$$

$$\times \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!n'!} \prod_{i=1}^{n} d\tilde{\omega}_i \theta_i(\delta)$$

$$\times \prod_{j=1}^{n'} d\tilde{\omega}_j' \theta_j'(\delta). \qquad (25)$$

Using the identity

$$\int d\widetilde{\omega}_i \theta_i(\delta) = \frac{2\alpha}{\pi} \ln \frac{s}{m_e^2} \ln \frac{1}{\delta} \qquad (26)$$

we find immediately

$$\sigma_0 = \int_{-t_{\min}}^{\infty} dt \frac{4\alpha^2 \pi}{t^2} = \frac{4\alpha^2 \pi}{|t_{\min}|} \qquad (27)$$

which is simply the lowest-order Bhabha cross section for $|t| > |t_{\min}|$.

The Monte Carlo result for the integrated cross section is given by

$$\sigma = \sigma_0 \langle w \rangle , \qquad (28)$$

where the average is taken over all events generated according to the approximate integrand in (25). The weight w consists of seven contributions corresponding to all simplifications made on the way from Eq. (2) to Eq. (25):

$$w = \prod_{k=S1}^{S7} w(k) . \qquad (29)$$

Let us trace back all these modifications giving explicit expressions for the $w(k)$ factors. The first weight is related to $d\omega_i \rightarrow d\widetilde{\omega}_i$ and is given by

$$w(S1) = \prod_{i=1}^{n} \frac{d\omega_i}{d\widetilde{\omega}_i}$$

$$= \left[\prod_{i=1}^{n} \frac{\widetilde{y}_i \widetilde{z}_i}{y_i z_i} \left[\frac{1}{y_i z_i} - \frac{m_e^2}{|t_p|} \left[\frac{1}{y_i^2} + \frac{1}{z_i^2} \right] \right] \right] \left[\frac{t_p}{t} \right] \qquad (30)$$

and the second weight $w(S2)$ is defined in a completely analogous way for $d\omega_i' \rightarrow d\widetilde{\omega}_i'$. The next three weights are listed below

$$w(S3) = \Theta, w(S4) = \theta^{\text{c.m.}}(\epsilon), w(S5) = F_{\text{YFS}}/\overline{F}_{\text{YFS}} . \qquad (31)$$

The next weight, attendant to our omitting $\widetilde{S}_{\text{int}}$ at the very beginning of our simplification procedure, is equivalent to

$$w(S6) = \prod_{i=1}^{n+n'} \left[1 + \frac{\widetilde{S}_{\text{int}}(k_i)}{\widetilde{S}_p(k_i) + \widetilde{S}_q(k_i)} \right] . \qquad (32)$$

Finally the last one is the model weight

$$w(S7) = \frac{1}{b_0} \left[\overline{\beta}_0(\mathcal{R}p_2, \mathcal{R}q_2) \right.$$

$$\left. + \sum_{l=1}^{n} \overline{\beta}_1(\mathcal{R}p_2, \mathcal{R}q_2, k_l) \widetilde{S}(k_l)^{-1} + \cdots \right] . \qquad (33)$$

It is in this way that we realize the YFS theory, on an event-by-event basis, for Bhabha scattering.

The description of our Monte Carlo procedure will be complete if we specify the model weight in (33). We have used the recipe in Ref. 3 together with the results in Ref. 7. Hence, we have, for our normalization in (6),

$$\frac{1}{2}\overline{\beta}_0(p_2, q_2) = \frac{d\sigma(1 \text{ loop})}{d\Omega} - 2\,\text{Re}(\alpha B)\frac{d\sigma_0}{d\Omega} , \qquad (34)$$

$$\frac{1}{2}\overline{\beta}_1(p_2, q_2, k) = \frac{d\sigma^{B1}}{d\Omega\, d\Omega_\gamma k\, dk} - \widetilde{S}(k)\frac{d\sigma_0}{d\Omega} , \qquad (35)$$

where $d\sigma(1 \text{ loop})/d\Omega$ is the one-loop differential cross section in Eq. (2.8) of Ref. 7 with the soft photon part omitted, $d\sigma^{B1}/d\Omega\, d\Omega_\gamma k\, dk$ is given in Eq. (3.1) of Ref. 7, and $d\sigma_0/d\Omega$ is the respective Born cross section in Eq. (2.1) of Ref. 7. [Note that the right-hand side (RHS) of (34) can be obtained directly from Eq. (2.8) of Ref. 7 by substracting $(\ln F_{\text{YFS}})d\sigma_0/d\Omega$ from the RHS of the latter equation.] The virtual infrared function B is taken from Refs. 2 and 3. Hence we see that (2)–(35) represent a complete description of our YFS Monte Carlo approach to higher-order radiative corrections to Bhabha scattering at low angles near the Z^0 resonance.

In particular, the fact that $d\sigma_0/d\Omega$ in Eq. (2.1) of Ref. 7 does not include the Z^0 exchange means that the resulting $\overline{\beta}_{0,1}$ are only appropriate for small c.m. scattering angles $\lesssim 140$ mrad near $\sqrt{s} = M_{Z^0}$. It is, of course, primarily for this reason that we refer to this initial Bhabha program (BHLUMI) as a luminosity-monitor program, to be used in the small scattering angle regime where the QED t-channel exchange is dominant. Of course, sufficiently far below the Z^0, the restriction to small angles may be ignored; for, at such values of $\sqrt{s}$, the Z^0 exchange is negligible for most practical purposes. The effect of inclusion of the Z^0 exchange in $d\sigma_0$ will be presented elsewhere. (In fact, the A-Z^0 interference effects are already available in the most recent version of BHLUMI FORTRAN; this version of BHLUMI is available from the authors upon request.)

In the next section, we illustrate the results of this application of our YFS Monte Carlo method with some explicit Monte Carlo data.

III. YFS MONTE CARLO RESULTS

In this section we wish to illustrate the type of results which we have generated with our YFS Monte Carlo approach to Bhabha scattering at low angles. We recall that the raison d'être of this particular application of our work is the use of such low angle scattering for the luminosity monitors at SLC and LEP near the Z^0 resonance. Accordingly, we immediately focus our attention on the luminosity-monitor scenario for the Mark II detector at the SLC; for the Mark II is currently in operation at the SLC.

Specifically, we consider the MINISAM small-angle monitor regime $15.2 \leq \theta_{\overline{e}} \leq 25$ mrad ($16.2 \leq \theta_e \leq 24.5$ mrad) for the $\overline{e}$ (e) center-of-momentum scattering angles $\theta_{\overline{e}}$ (θ_e). Further, we require $E_f' > 0.5\sqrt{s}/2$ and we require $E_e' + E_{\overline{e}}' > 0.6\sqrt{s}$, where E_f' is the final-state energy of f in the c.m. system, $f = e, \overline{e}$. We impose this scenario both on our BHLUMI YFS Monte Carlo program (the round dots in Fig. 2) and on the familiar Berends-Kleiss[7]-type one-photon Monte Carlo program (the crosses in Fig. 2) which the user can obtain from BHLUMI as an option in its input file. (We have verified that our

one-photon comparison Monte Carlo program agrees with that of Ref. 7 below the level of 0.1%.) We see that, while the two Monte Carlo programs are close throughout the Z^0 region, for high precision, the YFS-type simulation is desirable. The statistical errors on the points in Fig. 1 are of the size of the dots themselves.

One may wonder what would happen if we included the collinear photons' energy with that of the e^+ and e^- in the energy cuts on E_f' in Fig. 2. We have checked that this does affect the results in Fig. 2 at the level of 1%.

The question then is does one need the multiphotons at the level of 1% Z^0 physics, presuming the systematic errors will be at that level for the foreseeable future. What our detailed simulations have to say about this can be illustrated at the 92-GeV point in Fig. 2. For 6×10^5 1γ Monte Carlo events and 6×10^5 YFS events, the respective total cross sections are 246.8 ± 0.3 and 246.4 ± 0.7 nb. [Here, we have restored the nonleading constant terms to $R(p,q)$ in (5)]. These numbers are less than 1σ apart and, further, they are less than 0.2% apart. This is fortuitous, however, because the 1γ result depends on the famous k_0 parameter, the soft-photon cutoff in units of $\sqrt{s}/2$. The quoted result has $k_0 = 0.01$, so that this 1γ result is uncertain at the level of $k_0 = 1\%$, on general grounds. Hence, if one wants a k_0-independent prediction of the respective radiative corrections at the level of $\sim 1\%$, one must include multiple-photon effects. Our main message is that, prior to Fig. 2, there was no systematic calcula-

tion, for a realistic detector scenario, of the size of the multiple-photon effects in Bhabha scattering in the luminosity regime, although *ad hoc* naive exponentiation extensions of order-α^3 calculations existed which only allow for the produced final-state four-vectors to include at most one real-photon four-vector. Hence, while these extensions gave a good qualitative guesstimate of the effects of the higher-order radiative effects, their accuracy could not be trusted at the level of $\sim 1\%$. BHLUMI (YFS) has no naive *ad hoc* procedures in it so that it represents a truly systematic and rigorous calculation of the higher-order (multiphoton) radiative effects in Bhabha scattering in the SLC-LEP luminosity scenario. It is pleasing that, on the one hand, the total effects are small on an absolute scale but, on the other hand, they are in fact necessary for certainty in the highest precision work.

Finally, let us give the reader a view of the total weight w in (29) for the type of data in Fig. 2. Specifically, in Fig. 3, we show a histogram for w for the BHLUMI YFS
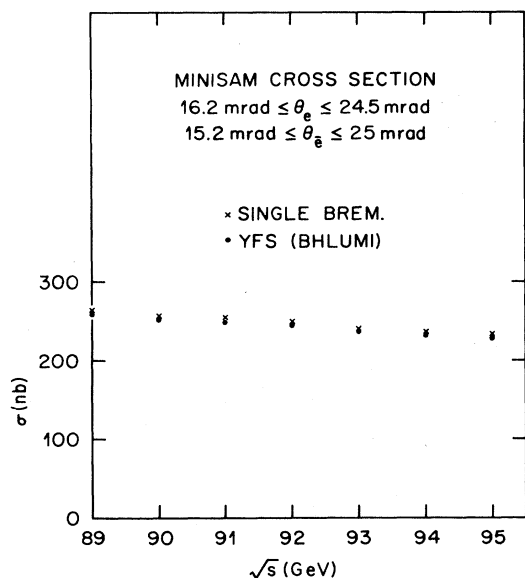

FIG. 2. Luminosity-monitor results for $\sqrt{s} \sim M_{Z^0}$: the dots represent our multiple-photon Monte Carlo result for $\bar{\beta}_0 + \bar{\beta}_1$; the crosses represent the one-photon Monte Carlo result of the type of that by Berends and Kleiss (Ref. 7). The statistical error is the size of the dot. The monitor configuration is that of the MINISAM at the Mark II at the SLC: $16.2 \leq \theta_e \leq 24.5$ mrad, $15.2 \leq \theta_{\bar{e}} \leq 25$ mrad, where θ_f is the c.m. scattering angle of f, $f = e^-, e^+$; $E_e' + E_{\bar{e}}' \geq 0.6\sqrt{s}$ and $E_f' > 0.5\sqrt{s}/2$ in the c.m. system, where E_f' is the final-state energy of f, $f = e^-, e^+$.
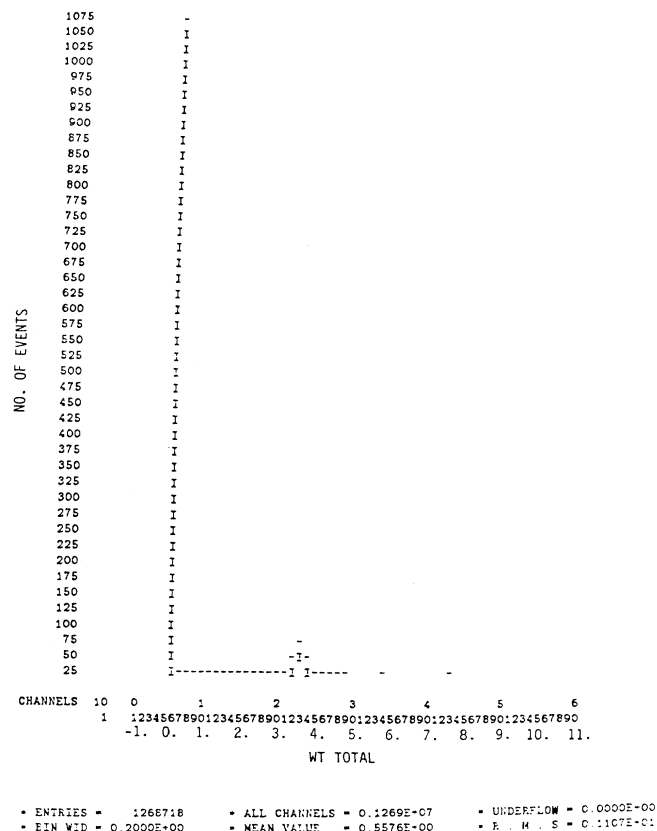

FIG. 3. Histogram of the total weight WT in our Monte Carlo program BHLUMI at $\sqrt{s} = 92$ GeV $= M_{Z^0}$ for the kinematical restrictions on E_f' in Fig. 2. The ordinate is the number of events, the abscissa is WT and is given in tens of bins, bins, and the respective values of WT. Some statistics of the histogram are also shown: the total number of events (1.2687×10^6), the bin size (0.2), the mean value in bin units (0.5576), etc. The histogram explains our efficiency of 1/12.7. It also shows that the weights are not fluctuating wildly.

point at $\sqrt{s} = 92$ GeV in Fig. 2. The statistics is 1.27×10^6 events. What we see is that w, referred as WT in the histogram, is reasonably well controlled, with the expected tail for rare configurations. The mean value of w is 0.56 with a root-mean-squared fluctuation of 1.1, so that, indeed, the fluctuations give an important tail to the distribution which is otherwise primarily centered near its average. We see, however, that the values of w do not have wild excursions so that, indeed, our numerical work does indeed converge at the level of $\sim 0.1\%$ and, hence, our approach allows an arbitrarily accurate simulation of our model (33) in principle.

We conclude that the way to precision measurement of the luminosities at SLC and LEP is open. In this way, the high-precision checks of the $SU_{2L} \times U_1$ model are made realistic after all. More detailed applications of BHLUMI will appear elsewhere.[8]

IV. SUMMARY, REMARKS, OUTLOOK

What we have shown in this paper is that the YFS theory allows one to realize, in a rigorous theoretical framework, an event-by-event description by Monte Carlo methods, of the low-angle Bhabha scattering at $\sqrt{s} \sim M_{Z^0}$ (we emphasize $\sqrt{s} \sim M_{Z^0}$ because of many reasons, the most technical of which is that BHLUMI is restricted to low angles when $\sqrt{s} \sim M_{Z^0}$), the type of Bhabha scattering which is crucial for the luminosity determinations at the SLC and LEP experimental scenarios. In this way, we feel that, in view of our previous work which realizes $e^+e^- \rightarrow f\bar{f} + n\gamma$, $f \neq e$, by similar methods, we now have software tools with a rigorous basis which permit a realistic determination of the interplay between detector cuts on the one hand and higher-order $SU_{2L} \times U_1$ radiative effects on the other at SLC and LEP.

We would like to extend our results to wide-angle Bhabha scattering near the Z^0 in the not-too-distant future. This work is in progress.

We would like to emphasize that, at energies of the SLAC and DESY storage rings PEP and PETRA, the use of our BHLUMI program would provide an independent check of the luminosities used in the respective physics scenarios. We understood that[9] such a check is in progress.

Concerning the results of related works, we should also note the work of Karlen in Ref. 10. He, however, has focused on an event generator for detected $\overset{(-)}{e}\gamma$, γ and $\overset{(-)}{e}$ final states in the process $e^+e^- \rightarrow e^+e^- + (2)\gamma$, so that at most two real-photon four-vectors are produced, with at least one of $\{e^+, e^-\}$ required to stay below a veto angle of ~ 15 mrad; it is therefore outside of the SLC-LEP luminosity monitor regime. Hence, his kinematical regime is orthogonal in this respect to the regime which we have analyzed in this paper (and for which BHLUMI is optimized numerically). In a later work we plan to explore also this single- or zero-tag regime in order to determine in a precise way the effects of the final states with more than two photons on single- and zero-tag physics.

In summary, the application of the YFS theory to the Monte Carlo simulation of double-tag Bhabha scattering at low angles for $\sqrt{s} \sim M_{Z^0}$ has been achieved. Taken together with our YFS (2) FORTRAN Monte Carlo program in Refs. 5 and 8, our BHLUMI FORTRAN program then completes our Monte Carlo realization of the YFS approach to higher-order radiative effects in the $SU_{2L} \times U_1$ theory in a way which is then applicable directly to the SLC and LEP scenarios.

ACKNOWLEDGMENTS

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