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Path-integral representation of the wave function: The relativistic particle

Hitoshi Ikemori
Research Institute for Fundamental Physics, Kyoto University, Kyoto 606, Japan
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The path-integral representation for the wave function of a relativistic particle is studied as a model of parametrized theories. We construct the transition amplitude by means of a phase-space path integral and discuss some problems coming from the ambiguity in a gauge-fixing procedure. The explicit calculation in the canonical gauge reveals the discrepancy in the procedure of Batalin, Fradkin, and Vilkovisky as to the ambiguity of whether one has to adopt the Faddeev-Popov determinant or the absolute value of it.

I. INTRODUCTION

Recently it has been fashionable to attempt to apply quantum theory to the Universe as a whole.[1–4] Among such investigations of the quantum effect for the Universe, Coleman's argument or the mechanism that the wormhole configurations drive the cosmological constant to zero has been attracting vast attention.[5–8] At the basis of these arguments about quantum cosmology, the Hartle-Hawking proposal[1] for the wave function of the Universe plays an important role. Although the wave function of the Universe which satisfies the Wheeler-DeWitt equation[9–11]

$$\left[G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}}+\sqrt{h}\,{}^{(3)}R\right]\Psi[h_{ij}]=0 \tag{1.1}$$

is to be obtained by specifying an appropriate boundary condition, it is not easy to solve the equation in a direct way or to specify a physically authorized boundary condition. The Hartle-Hawking wave function is constructed by the path integral

$$\Psi[h_{ij}]=\int_{\mathcal{M}}\mathcal{D}g\,\exp(-S_E[g]) \tag{1.2}$$

over all Euclidean "no-boundary" four-manifolds $\mathcal{M}$ which have no boundary other than the only boundary specified by h_{ij}. However, it is not clear whether or not the no-boundary proposal can specify a boundary condition for the Wheeler-DeWitt equation uniquely. It is an essential but problematic point that they fuse the boundary (or initial) condition into the path-integral measure. Although the no-boundary proposal would be an ambitious idea in discussing the boundary (or initial) condition of the Universe, it might be advantageous to separate the two kinds of problems, namely, the path-integral measure and the boundary condition, at the present stage.

To discuss the problems of the boundary condition and the path-integral measure separately, it would be better to deal with the Lorentzian path integral rather than the Euclidean one.[12,13] In the approach with the Lorentzian path integral, it is preferable to construct the transition amplitude rather than the wave function. We do not have the wave function until we prepare a suitable initial condition and make a product of the transition amplitude and the initial-state function in an appropriate manner. We should declare what we mean by the transition amplitude and the product, however. Unlike naive expectations, it does not seem that the transition amplitude is an analogue of the propagator and the product is a simple superposition. It seems more likely that the transition amplitude is an analogue of the invariant delta function and the product is an analogue of the Klein-Gordon product in the terminology of the quantum theory of relativistic particles. The "propagator" is a Green's function of the Wheeler-DeWitt operator, which is not a solution of the Wheeler-DeWitt equation, and it leaves a delta functional when this operator is operated on it. So the superposition of it with any initial-state functional cannot make a solution for the Wheeler-DeWitt equation. Preferably what we call the invariant delta function seems to have more promising properties to be related to and to be made a product with an initial condition. Various authors have argued about the definition and the evaluation of the path integral, especially for minisuperspace models.[3,4,14,15] But it still would be obscure what we should obtain from the path integral and what we should employ as a definition to obtain the object.

In this paper we will study the path-integral representation for the wave function of a free relativistic particle as a model study for the wave function of the Universe.[14,16,17] It is well known that path-integral quantum mechanics of the relativistic particle resembles the quantum theory of gravitation in its nature as a parametrized theory or as a constrained system. As the Klein-Gordon equation is an analogue of the Wheeler-DeWitt equation from this point of view, to construct a solution for the former from the path-integral expression would be advisable.

We will argue that there are some difficulties in constructing a path-integral measure and in obtaining a definite answer when accomplishing path integration. The main part of this troublesome problem is caused by the gauge-fixing procedure. One usually employs the Faddeev-Popov procedure to evaluate path integration and this procedure requires setting the Faddeev-Popov determinant in order to compensate the symmetry which is fixed once by the gauge condition. The problem is whether one has to adopt the Faddeev-Popov determinant or the absolute value of it.

There is also the problem of an ambiguity in discretization to define the path integral, especially in connection with gauge fixing. Perhaps the situation may depend on the choice of the gauge condition; it seems that this problem could be avoided by being careful with the symmetry in the discretized action.

The purpose of this paper is to reveal the above-mentioned problems through the explicit construction of the path-integral representation of a transition amplitude for the free relativistic particle. Throughout this paper we employ the canonical gauge to fix the invariance of the system. The reason why we employ the canonical gauge is that this gauge makes it clear how the transition amplitude is related to the Klein-Gordon product and how the problem of the Faddeev-Popov determinant makes its appearance; moreover, the explicit calculation of the path integral with this gauge has never been performed to our knowledge.

In Sec. II we summarize some well-known properties of the solution for the Klein-Gordon equation for the sake of clarity. In Sec. III a brief remark on the Faddeev-Popov (FP) procedure is made and the discrepancy with the method of Batalin, Fradkin, and Vilkovisky (BFV) is pointed out. The path integral for the transition amplitude is constructed in Sec. IV. The explicit calculation is carried out in Sec. V and some problems concerning the gauge-fixing procedure are revealed in the canonical gauge. Section VI is left for summarizing and discussing our results.

II. WAVE FUNCTION AND KLEIN-GORDON PRODUCT

The wave function for the relativistic particle, that is to say, the solution of the relativistic wave equation, the so-called Klein-Gordon equation,

$$(-\Box+m^2)\Psi(x)=0 , \tag{2.1}$$

has the following properties.[18] The solution of the Klein-Gordon equation is generally given by

$$\begin{aligned}\Psi(x)&=\frac{1}{i(2\pi)^3}\int d^4p\,\delta(p^2+m^2)e^{ip\cdot x}\phi(p)\\&=\int d^3\mathbf{x}'\Delta(x-x';m^2)\overleftrightarrow{\partial}_{0'}\Psi_0(x') ,\end{aligned} \tag{2.2}$$

where $\overleftrightarrow{\partial}_0$ is the operator defined by

$$F\overleftrightarrow{\partial}_0 G\equiv F(\partial_0 G)-(\partial_0 F)G , \tag{2.3}$$

and $\Delta(x-x';m^2)$ is what is called the invariant delta function defined by

$$\Delta(x;m^2)\equiv\frac{1}{i(2\pi)^3}\int d^4p\,\epsilon(p_0)\delta(p^2+m^2)e^{ip\cdot x}$$

$$\left[\epsilon(x)=\mathrm{sgn}(x)\equiv\frac{x}{|x|},\ \epsilon(0)=0\right] . \tag{2.4}$$

It is convenient to write down the solution in the above form in order to see the relation with the initial condition. Actually we see that (2.2) is the solution under the initial conditions

$$\Psi(x)|_{x^0\to x'^0}=\Psi_0(x),\quad \partial_0\Psi(x)|_{x^0\to x'^0}=\partial_0\Psi_0(x) . \tag{2.5}$$

This is due to the properties of the invariant delta function, which itself is a solution of the Klein-Gordon equation under the conditions

$$\begin{aligned}&\Delta(x-x';m^2)|_{x^0\to x'^0}=0 ,\\&\partial_0\Delta(x-x';m^2)|_{x^0\to x'^0}=\delta^3(\mathbf{x}-\mathbf{x}') .\end{aligned} \tag{2.6}$$

If we define the inner product among the solutions of the Klein-Gordon equation, Ψ_1 and Ψ_2, in terms of the "Klein-Gordon product" as

$$(\Psi_1,\Psi_2)\equiv\frac{1}{i}\int d^3\mathbf{x}\,\Psi_1^*\overleftrightarrow{\partial}_0\Psi_2 \tag{2.7}$$

or

$$=\frac{1}{i}\int_\Sigma d\Sigma^\mu\Psi_1^*\overleftrightarrow{\partial}_\mu\Psi_2 \tag{2.8}$$

in covariant way, (Ψ_1,Ψ_2) is conserved and independent of the choice of the spacelike hypersurface Σ, but it does not lead to any positive-definite norm.

It should be remarked that the invariant delta function $\Delta(x-x';m^2)$ obeys a composition law of the form

$$\Delta\cdot\Delta=\Delta \tag{2.9}$$

with respect to the Klein-Gordon product, which is defined by

$$\int d^3\mathbf{x}'\Delta(x-x';m^2)\overleftrightarrow{\partial}_{0'}\Delta(x'-x'';m^2)=\Delta(x-x'';m^2) . \tag{2.10}$$

There is another kind of function,

$$\Delta^{(1)}(x;m^2)\equiv\frac{1}{i(2\pi)^3}\int d^4p\,\delta(p^2+m^2)e^{ip\cdot x} , \tag{2.11}$$

in the category of invariant functions; it obeys

$$\Delta^{(1)}\cdot\Delta^{(1)}=\Delta,\quad \Delta^{(1)}\cdot\Delta=\Delta^{(1)} , \tag{2.12}$$

which means that the composition does not close.

III. BFV PATH INTEGRAL AND FP DETERMINANT

It is well understood that the mechanics of the relativistic particle is characterized solely by the constraint

$$\mathcal{H}\equiv p^2+m^2=0 , \tag{3.1}$$

so it is described by the canonical action

$$S_{\mathrm{inv}}=\int_0^T d\tau(\dot{x}\cdot p-N\mathcal{H}) , \tag{3.2}$$

where τ is an arbitrary parameter which is introduced to treat all the components of the four-coordinate x^μ's on an equal footing. The above action is invariant under the transformation induced by $\mathcal{H}$ as a generator. The transformation law for the variables is explicitly

$$\delta x^\mu=2\epsilon p^\mu,\quad \delta p_\mu=0,\quad \delta N=\dot{\epsilon} , \tag{3.3}$$

for the generator $\mathcal{H}$, where ϵ is an infinitesimal transformation parameter which is an arbitrary function of τ.

The invariance of the above action can be seen easily as

$$\delta S_{\rm inv}=\int_0^T d\tau\frac{d}{d\tau}[\epsilon(p^2-m^2)]$$
$$=[\epsilon(p^2-m^2)]|_{\tau=0}^{\tau=T}$$
$$=0\ , \qquad (3.4)$$

provided $\epsilon(0)=\epsilon(T)=0$ at the end point. This transformation is another guise of the reparametrization of τ and the invariance of the action under this transformation is a reflection of the arbitrariness of τ.

Now what we want to evaluate is the transition amplitude represented by the path integral[14,16,17]

$$A(x'|x)=\int_{\begin{bmatrix}x(T)=x'\\ x(0)=x\end{bmatrix}}\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\exp(iS_{\rm inv})\ . \qquad (3.5)$$

In order to avoid a divergence caused by the gauge volume and to get a definite answer we shall make use of the gauge-fixing procedure of Batalin, Fradkin, and Vilkovisky (BFV).[19,20] The BFV path integral is defined by the functional integration containing the ghost variables $c,\bar{c}$ and the auxiliary variable b:

$$A(x'|x)=\int_{\begin{bmatrix}x(T)=x'\\ x(0)=x\end{bmatrix}}\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}\bar{c}\exp(iS_{\rm BRS})\ , \qquad (3.6)$$

over the exponential integrand with the Becchi-Rouet-Stora- (BRS-)invariant action $S_{\rm BRS}$,

$$S_{\rm BRS}=S_{\rm inv}+S_{\rm GF}+S_{\rm FP}\ , \qquad (3.7)$$

where $S_{\rm inv}$ is the previous gauge-invariant action and the remainders are introduced for the gauge-fixing procedure. $S_{\rm GF}$ is the gauge-fixing action to fix the invariance of $S_{\rm inv}$ and most simply of the form

$$S_{\rm GF}=-\int d\tau\, b\chi\ , \qquad (3.8)$$

where b is an auxiliary variable and the variation with respect to b leads to the gauge condition $\chi=0$. We may use the canonical gauge, such as

$$\chi\equiv x^0-f(\tau)=0 \qquad (3.9)$$

which we will employ in order to carry out the explicit calculation in the following sections. $S_{\rm FP}$ is the Faddeev-Popov ghost action to compensate the fixed gauge invariance as a guise of the BRS invariance and is of the form

$$S_{\rm FP}=\int d\tau\, c\{\mathcal{H},\chi\}\bar{c}\ , \qquad (3.10)$$

with the above gauge condition χ.

It is expected that the integration over N and b will lead to the delta functionals $\delta[\mathcal{H}]$ and $\delta[\chi]$, respectively. And it is also expected that the integration over $c,\bar{c}$ gives $\det\{\mathcal{H},\chi\}$. So it would be allowed to step into the next form:

$$A(x'|x)=\int\mathcal{D}x\,\mathcal{D}p\,\delta[\mathcal{H}]\delta[\chi]\det\{\mathcal{H},\chi\}\times\exp\left[i\int d\tau\,\dot{x}\cdot p\right]\ , \qquad (3.11)$$

where we should remark what the $c,\bar{c}$ integration leads to. It is the FP (Faddeev-Popov) determinant $\det\{\mathcal{H},\chi\}$ and is never the absolute value there of $|\det\{\mathcal{H},\chi\}|$, as long as we employ the naive Grassmann integral.

This fact seems to be in discrepancy with the original meaning of the gauge-fixing procedure with the path integral. To make things clear let us review the procedure. The transition amplitude defined first,

$$A(x'|x)=\int_{\begin{bmatrix}x(T)=x'\\ x(0)=x\end{bmatrix}}\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\exp(iS_{\rm inv})\ , \qquad (3.12)$$

suffers from a divergence caused by the gauge volume, that is, the volume of the gauge-equivalent classes. In order to avoid this divergence, we should choose a representative element for each class by means of a gauge-fixing condition. Suppose the gauge condition is $\chi=0$, the procedure we employ to obtain a gauge-independent result is the following. To begin with, let us make the normalization concerning the integration over the delta functional $\delta[\chi_\epsilon]$ with respect to the gauge transformation parameter ϵ as

$$\Delta_{\rm FP}[\chi]\int\mathcal{D}\epsilon\,\delta[\chi_\epsilon]=1\ , \qquad (3.13)$$

where χ_ϵ is derived from χ by the gauge transformation with the parameter ϵ, and $\Delta_{\rm FP}[\chi]$ is a factor required for the normalization. Inserting this into the above path integral for the transition amplitude makes no essential change, and it becomes

$$A(x|x')=\int\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\,\Delta_{\rm FP}[\chi]\int\mathcal{D}\epsilon\,\delta[\chi_\epsilon]\exp(iS_{\rm inv})\ . \qquad (3.14)$$

Because of the gauge invariance of $S_{\rm inv}$ we can separate the ϵ integration as

$$A(x|x')=\int\mathcal{D}\epsilon\int\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\,\Delta_{\rm FP}[\chi]\delta[\chi]\exp(iS_{\rm inv})\ . \qquad (3.15)$$

Now we are ready to omit the divergent integral $\int\mathcal{D}\epsilon$, which is just the gauge volume we should exclude to obtain the redefined transition amplitude:

$$A(x|x')\Longleftarrow A(x|x')/(\text{gauge volume})$$
$$=\int\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\,\Delta_{\rm FP}[\chi]\delta[\chi]\exp(iS_{\rm inv})\ . \qquad (3.16)$$

Taking into account the fact that χ_ϵ is actually of the form

$$\chi_\epsilon=\chi+\delta_\epsilon\chi=\chi+\epsilon\{\mathcal{H},\chi\}\ , \qquad (3.17)$$

it is required that

$$\Delta_{\rm FP}[\chi]=|\det\{\mathcal{H},\chi\}| \qquad (3.18)$$

and the transition amplitude becomes

$$A(x|x')=\int \mathcal{D}x\,\mathcal{D}p\,\delta[\mathcal{H}]|\det\{\mathcal{H},\chi\}|\delta[\chi] \times\exp\left[i\int d\tau\,\dot{x}\cdot p\right] . \tag{3.19}$$

Here we see the absolute value of the FP determinant $|\det\{\mathcal{H},\chi\}|$ in the above prescription.

As we have seen the ghost integration would give just the FP determinant itself and not the absolute value, so these results seem to lead to a discrepancy. This discrepancy concerning the FP determinant would cause an ambiguity depending on the choice of the gauge condition. Although this kind of problem may arise in field theory, it does not seem to cause any trouble as far as the perturbative features. It would be the problem of a nonperturbative feature and might be related to Gribov's problem of gauge fixing in the case of the field theory. We will not pursue this problem any more in this section until we define the path-integral measure in the next section.

IV. DEFINITION OF PATH INTEGRAL BY DISCRETIZATION

Now we are at the stage where we can define the path-integral measure in order to carry out the explicit calculation of the transition amplitude introduced in the previous section. At the first step to construct the path integral, let us divide the path into n pieces which have the end points $x^\mu(0), x^\mu(1), \ldots, x^\mu(n-1), x^\mu(n)$, where the first and last points are fixed to x^μ and x'^μ, respectively. The momenta $p_\mu(1), p_\mu(2), \ldots, p_\mu(n)$ are variables on the pieces or links in our case, and the associated Lagrange multiplier N is also discretized into $N_1, N_2, \ldots, N_n$. We mean that the τ parameter is discretized as

$$\begin{array}{ccccccc} 0=\tau_0 & \tau_1 & \tau_2 & \cdots & \tau_{n-1} & \tau_n=T \\ & \vee & \vee & & & \vee \\ & \Delta\tau_1 & \Delta\tau_2 & \cdots & & \Delta\tau_n \end{array} \tag{4.1}$$

at the ground of these assignments. With these preparations we assume that the invariant action is of the form

$$S_{\text{inv}}=\sum_{i=1}^{n}\Delta\tau_i\left[\frac{x(i)-x(i-1)}{\Delta\tau_i}\cdot p(i)-N_i[p(i)^2+m^2]\right] \tag{4.2}$$

according to the discretization.

Although we never know how we should define the path-integral measure in the case of the relativistic particle, we shall employ a naive measure,

$$\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\equiv\mathcal{N}\prod_{i=1}^{n-1}d^4x(i)\prod_{i=1}^{n}d^4p(i)\prod_{i=1}^{n}dN_i\ , \tag{4.3}$$

following the Liouville measure for the case of the nonrelativistic particle. We shall see the result derived by this choice later, when we will discuss whether or not it is adequate.

In order to carry out the calculation we should make use of the gauge-fixing procedure expressed in the previous section. Let us employ the canonical gauge

$$\chi\equiv x^0-f(\tau)=0\ , \tag{4.4}$$

where $f(\tau)$ is an arbitrary monotonic function of τ which satisfies $f(0)=x^0(0)$ and $f(T)=x^0(T)$. As the FP determinant is

$$\det\{\mathcal{H},\chi\}=\det(2p_0) \tag{4.5}$$

according to this gauge condition, the transition amplitude has the form of

$$A(x'|x)=\int_{\left[\begin{smallmatrix}x(0)=x\\ x(T)=x'\end{smallmatrix}\right]}\mathcal{D}x\,\mathcal{D}p\,\delta[p^2+m^2]\delta[x^0-f]\det(2p_0)\exp\left[i\int d\tau\,\dot{x}\cdot p\right] , \tag{4.6}$$

where we will not use the absolute value but the FP determinant itself for the present. Although we might expect that the above expression would be rendered into the form

$$A(x(n)|x(0))=\mathcal{N}\int\prod_{i=1}^{n-1}d^4x(i)\prod_{i=1}^{n}d^4p(i)\prod_{i=1}^{n}\delta(p(i)^2+m^2)\prod_{i=1}^{n-1}\delta(x^0(i)-f(i)) \times\prod_{i=1}^{n-1}[2p_0(i)]\exp\left[i\sum_{i=1}^{n}\Delta\tau_i\frac{x(i)-x(i-1)}{\Delta\tau_i}\cdot p(i)\right] , \tag{4.7}$$

in case of definition of the path integral by discretization, it would not work. We will see that it is not $\prod_{i=1}^{n-1}[2p_0(i)]$ but $\prod_{i=1}^{n-1}[p_0(i+1)+p_0(i)]$ which is preferred as the FP determinant in order to obtain a result independent of the gauge fixing: $f(\tau)$. The reason why we should choose the latter is as follows.

Although we have seen the gauge invariance of S_{inv} in the previous section we have not yet checked the invariance of the discretized version. Hereupon let us examine it as follows. The transformation which leaves the discretized action (4.2) invariant is of the form

$$\delta x^\mu(i)=\epsilon_i(p^\mu(i+1)+p^\mu(i)) \,,$$
$$\delta p_\mu(i)=0 \,, \qquad (4.8)$$
$$\delta N_i=\frac{\epsilon_i-\epsilon_{i-1}}{\Delta\tau_i} \,,$$

so we see that $S_{\rm inv}$ transforms as

$$\delta S_{\rm inv}=\epsilon_n(p(n)^2+m^2)-\epsilon_0(p(1)^2+m^2) \qquad (4.9)$$

and is invariant provided $\epsilon_0=\epsilon_n=0$ as before. We should remark upon the fact that $x^\mu(i)$ does not transform as $\delta x^\mu(i)=\epsilon_i(2p^\mu(i))$ but as $\delta x^\mu(i)=\epsilon_i(p^\mu(i+1)+p^\mu(i))$. On account of the fact that there is a gauge invariance of its own in case of the discretized action we should consider the FP determinant related to this transformation and employ

$$\det\{\mathcal{H},\chi\}=\det(2p_0) \Longrightarrow \prod_{i=1}^{n-1}[p_0(i+1)+p_0(i)] \qquad (4.10)$$

as stated above.

V. CALCULATION OF PATH INTEGRAL

Case I. FP determinant without the absolute value

Let us carry out the explicit calculation of the path-integral representation of the transition amplitude for the case where the FP determinant is

$$\Delta_{\rm FP}[\chi]=\det\{\mathcal{H},\chi\}=\det(2p_0) \qquad (5.1)$$

without the absolute value, which is written in the form

$$A(x(n)|x(0))=\mathcal{N}\int\prod_{i=1}^{n-1}d^4x(i)\prod_{i=1}^{n}d^4p(i)\prod_{i=1}^{n}\delta(p(i)^2+m^2)\prod_{i=1}^{n-1}\delta(x^0(i)-f(i)) \times\prod_{i=1}^{n-1}[p_0(i+1)+p_0(i)] \times\exp\left[i\sum_{i=1}^{n}\Delta\tau_i\frac{x(i)-x(i-1)}{\Delta\tau_i}\cdot p(i)\right] . \qquad (5.2)$$

First, we shall rewrite the argument in exp( ) as

$$i\sum_{i=1}^{n}[x(i)-x(i-1)]\cdot p(i)=-i\sum_{i=1}^{n-1}[p(i+1)-p(i)]\cdot x(i)+i[x(n)\cdot p(n)-x(0)\cdot p(1)] \qquad (5.3)$$

by means of resummation; then we have

$$A(x(n)|x(0))=\mathcal{N}\int\prod_{i=1}^{n}d^4p(i)\prod_{i=1}^{n-1}d^4x(i)\prod_{i=1}^{n}\delta(p(i)^2+m^2)\prod_{i=1}^{n-1}[p_0(i+1)+p_0(i)] \times\prod_{i=1}^{n-1}\delta(x^0(i)-f(i)) \times\prod_{i=1}^{n-1}\exp\{-i[p(i+1)-p(i)]\cdot x(i)\} \times\exp\{i[x(n)\cdot p(n)-x(0)\cdot p(1)]\} . \qquad (5.4)$$

As it is easy to perform the $d^4x(i)$ integration and to see that $d^3\mathbf{x}(i)$ integration yields $\delta^3(\mathbf{p}(i+1)-\mathbf{p}(i))$ and $dx^0(i)$ integration substitutes $f(i)$ for $x^0(i)$ because of the $\delta(x^0(i)-f(i))$ we have

$$A(x(n)|x(0))=\mathcal{N}\int\prod_{i=1}^{n}d^4p(i)\prod_{i=1}^{n}\delta(p(i)^2+m^2)\prod_{i=1}^{n-1}[p_0(i+1)+p_0(i)] \times\prod_{i=1}^{n-1}\delta^3(\mathbf{p}(i+1)-\mathbf{p}(i)) \times\prod_{i=1}^{n-1}\exp\{-i[p_0(i+1)-p_0(i)]f(i)\} \times\exp\{i[x(n)\cdot p(n)-x(0)\cdot p(1)]\} . \qquad (5.5)$$

Subsequently performing $d^3\mathbf{p}(\mathbf{i})$ integrations from $i=1$ to $n-1$ we have

$$A(x(n)|x(0))=\mathcal{N}\int d^4p(n)\prod_{i=1}^{n-1} dp_0(i)\prod_{i=1}^{n}\delta(-p_0(i)^2+\mathbf{p}(n)^2+m^2)$$
$$\times\prod_{i=1}^{n-1}[p_0(i+1)+p_0(i)]\prod_{i=1}^{n-1}\exp\{-i[p_0(i+1)-p_0(i)]f(i)\}$$
$$\times\exp\{i[\mathbf{x}(n)-\mathbf{x}(0)]\cdot\mathbf{p}(n)\}$$
$$\times\exp\{i[x^0(n)p_0(n)-x^0(0)p_0(1)]\}\ . \tag{5.6}$$

In order to take a step to the next stage we make use of the relations

$$\prod_{i=1}^{n}\delta(-p_0(i)^2+\mathbf{p}(n)^2+m^2)=\delta(p(n)^2+m^2)\prod_{i=1}^{n-1}\delta(p_0(i+1)^2-p_0(i)^2) \tag{5.7}$$

and

$$\delta(p_0(i+1)^2-p_0(i)^2)[p_0(i+1)+p_0(i)]=\frac{1}{2|p_0(i)|}[\delta(p_0(i+1)-p_0(i))+\delta(p_0(i+1)+p_0(i))][p_0(i+1)+p_0(i)]$$
$$=\epsilon(p_0(i))\delta(p_0(i+1)-p_0(i))\ . \tag{5.8}$$

Considering these relations we can accomplish the integration as follows:

$$A(x(n)|x(0))\ =\mathcal{N}\int d^4p(n)\prod_{i=1}^{n-1} dp_0(i)\prod_{i=1}^{n-1}\epsilon(p_0(i))\delta(p_0(i+1)-p_0(i))\delta(p(n)^2+m^2)$$
$$\times\prod_{i=1}^{n-1}\exp\{-i[p_0(i+1)-p_0(i)]f(i)\}$$
$$\times\exp\{i[\mathbf{x}(n)-\mathbf{x}(0)]\cdot\mathbf{p}(n)\}\exp\{i[x^0(n)p_0(n)-x^0(0)p_0(1)]\}$$
$$=\mathcal{N}\int d^4p(n)[\epsilon(p_0(n))]^{n-1}\delta(p(n)^2+m^2)\exp\{i[x(n)-x(0)]\cdot p(n)\}\ . \tag{5.9}$$

This leads to the result

$$A(x'|x)=\begin{cases}\Delta(x-x';m^2)\ \ (\text{for even } n)\ ,\\ \Delta^{(1)}(x-x';m^2)\ \ (\text{for odd } n)\ ,\end{cases} \tag{5.10}$$

which depends on the choice of n: namely, the number of pieces of discretized path. This is of course an unreasonable result. Let us investigate what causes this indefiniteness.

In order to clarify this, we examine the role of the FP determinant term within the path integral in the case of our gauge. The term consists of the product of $[p_0(i+1)+p_0(i)]$'s and we can regard each of them as a resultant of an operation of the differential operator $\overleftrightarrow{\partial}_0$ on either side. It means that the transition amplitude consists of units A_i's in terms of the Klein-Gordon product as

$$A=A_n\cdot A_{n-1}\cdots A_2\cdot A_1\ . \tag{5.11}$$

Each unit is of the form

$$A_i(x(i)|x(i-1))\propto\int d^4p(i)\delta(p(i)^2+m^2)\exp\{ip(i)\cdot[x(i)-x(i-1)]\}$$
$$\propto\Delta^{(1)}(x(i)-x(i-1);m^2)\ , \tag{5.12}$$

which is not the invariant delta function Δ but just $\Delta^{(1)}$ and this causes the trouble. As explained in Sec. II, Δ obeys the composition law $\Delta\cdot\Delta=\Delta$, in terms of the Klein-Gordon product, the $\Delta^{(1)}$ obeys the laws $\Delta^{(1)}\cdot\Delta^{(1)}=\Delta$ and $\Delta^{(1)}\cdot\Delta=\Delta^{(1)}$, instead, which do not close. Thus the product of the units depends on the number of discretization, as we have seen.

Case II. FP determinant with the absolute value

In this section we examine another possibility in the place of the FP determinant term, which is the absolute value of it. When we employ the absolute value of the FP determinant

$$\Delta_{\rm FP}[\chi]=|\det\{\mathcal{H},\chi\}|=|\det(2p_0)|\ , \tag{5.13}$$

the procedure of the calculation is changed as follows. As the transition amplitude we should evaluate

$$A(x(n)|x(0))=\mathcal{N}\int \prod_{i=1}^{n-1} d^4x(i)\prod_{i=1}^{n} d^4p(i)\prod_{i=1}^{n} \delta(p(i)^2+m^2) \times \prod_{i=1}^{n-1} \delta(x^0(i)-f(i)) \times \prod_{i=1}^{n-1} |p_0(i+1)+p_0(i)| \exp\left[i\sum_{i=1}^{n} \Delta\tau_i \frac{x(i)-x(i-1)}{\Delta\tau_i}\cdot p(i)\right] , \tag{14}$$

the essential change in the calculation to follow is that the relation (5.8) should be transferred into

$$\delta(p_0(i+1)^2-p_0(i)^2)|p_0(i+1)+p_0(i)|=\frac{1}{2|p_0(i)|}[\delta(p_0(i+1)-p_0(i))+\delta(p_0(i+1)+p_0(i))]|p_0(i+1)+p_0(i)| =\delta(p_0(i+1)-p_0(i)) . \tag{5.15}$$

According to this change the last stage of the integration (5.9) should be altered as

$$A(x(n)|x(0))=\mathcal{N}\int d^4p(n)\prod_{i=1}^{n-1} dp_0(i)\prod_{i=1}^{n-1} \delta(p_0(i+1)-p_0(i))\delta(p(n)^2+m^2) \times \prod_{i=1}^{n-1} \exp\{-i[p_0(i+1)-p_0(i)]f(i)\} \times \exp\{i[\mathbf{x}(n)-\mathbf{x}(0)]\cdot\mathbf{p}(n)\}\exp\{i[x^0(n)p_0(n)-x^0(0)p_0(1)]\}$$

$$=\mathcal{N}\int d^4p(n)\delta(p(n)^2+m^2)\exp\{i[x(n)-x(0)]\cdot p(n)\} . \tag{5.16}$$

This leads to the result

$$A(x'|x)=\Delta^{(1)}(x-x';m^2) , \tag{5.17}$$

which is independent of n.

Although we obtain a less ambiguous result than the case of the previous section where we deal with the FP determinant without the absolute value, we dare not declare it to be what we want. Because $\Delta^{(1)}$ does not obey the composition law and cannot be incorporated into the initial-state function, it is not suitable for clarifying the relation between the transition amplitude and the wave function or the initial condition.

VI. DISCUSSIONS

We have constructed a path-integral representation for the transition amplitude of the relativistic particle and have pointed out that there are some problems concerning the gauge-fixing procedure. The explicit calculation in the canonical gauge shows that one of them is an ambiguity in the discretization of the FP determinant term, which seems to be avoided if care is taken as to the gauge symmetry in the discretized action. The problem of greater significance is whether or not the FP determinant should be considered with the absolute value. The naive path-integral measure in the manner of Liouville and the canonical gauge lead us to the unacceptable result. That is, if we choose the FP determinant itself without the absolute value, the transition amplitude $A(x'|x)$ turns out to be $\Delta(x-x';m^2)$ or $\Delta^{(1)}(x-x';m^2)$ depending on whether the number of the discretization is even or odd, and if we choose the absolute value of the FP determinant instead, it turns out to be $\Delta^{(1)}(x-x';m^2)$ definitely.

Although the above fact seems to support the absolute value of the FP determinant, we should remark on the possibility of changing the definition of the path-integral measure. For example, if we use the odd-invariant measure

$$\mathcal{D}x\,\mathcal{D}p\equiv\mathcal{N}\prod_{i=1}^{n-1} d^4x(i)\prod_{i=1}^{n} \epsilon(p_0(i))d^4p(i) , \tag{6.1}$$

which is invariant under proper Lorentz transformation, we shall obtain the result

$$A(x'|x)=\Delta(x-x';m^2) \tag{6.2}$$

definitely even in the case without the absolute value. One may prefer to employ this modified measure in order to obtain the advantageous result, because Δ is more suitable as a transition amplitude than $\Delta^{(1)}$.

As described it seems that the problem of the FP determinant is not irrelevant to the path-integral measure in our case. But we do not know what the situation is in case of the configuration-space path integral, because our study has been restricted to the phase-space path integral. It is an interesting problem to investigate how the ambiguity about the FP determinant is to be settled in the former case. It is also a subject of great significance to investigate how a similar problem occurs in the case of field theory.[21] We would like to discuss these issues in the near future.

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[1]J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[2]A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).

[3]J. B. Hartle, in *High Energy Physics 1985*, proceedings of the Yale Summer School, New Haven, Connecticut, 1985, edited by M. J. Bowick and F. Gürsey (World Scientific, Singapore, 1986), Vol. 2, and references therein.

[4]*Quantum Cosmology*, edited by L. Z. Fang and R. Ruffini (World Scientific, Singapore, 1987).

[5]S. Coleman, Nucl. Phys. **B310**, 643 (1988).

[6]S. H. Giddings and A. Strominger, Nucl. Phys. **B321**, 481 (1989).

[7]A. Hosoya, Prog. Theor. Phys. **81**, 1248 (1989).

[8]S. Weinberg, Rev. Mod. Phys. **61**, 1 (1988), and references therein.

[9]J. A. Wheeler, in *Battle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[10]B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[11]M. McGuigan, Phys. Rev. D **38**, 3031 (1988).

[12]E. Farhi, Phys. Lett. B **219**, 403 (1989).

[13]J. M. Cline, Phys. Lett. B **224**, 53 (1989).

[14]J. J. Halliwell, Phys. Rev. D **38**, 2468 (1988).

[15]J. Louko, Class. Quantum Gravit. **4**, 581 (1987).

[16]J. B. Hartle and K. V. Kuchař, Phys. Rev. D **34**, 2323 (1986); J. Math. Phys. **25**, 57 (1984).

[17]C. Teitelboim, Phys. Rev. D **25**, 3159 (1982).

[18]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[19]E. S. Fradkin and G. Vilkovisky, Phys. Lett. **55B**, 224 (1975); I. A. Batalin and G. Vilkovisky, *ibid.* **69B**, 309 (1977); I. A. Batalin and E. S. Fradkin, *ibid.* **122B**, 157 (1983).

[20]M. Henneaux, Phys. Rep. **126**, 1 (1985).

[21]J. Polchinski, Phys. Lett. B **219**, 251 (1989).