Integration of operator differential equations

Carl M. Bender
Physics Department, Washington University, St. Louis, Missouri 63130

Gerald V. Dunne
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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In a previous paper we introduced a method for obtaining exact solutions to the operator differential equations of quantum mechanics. In that paper we showed how to solve some simple quantum-mechanical models and we suggested that the method could be used to obtain exact solutions to the operator differential equations of more complicated models, such as the anharmonic oscillator whose Hamiltonian is $H=\frac{1}{2}p^2+\frac{1}{4}q^4$. In this paper we further sharpen the formalism and introduce the concept of a minimal solution. We then obtain the exact minimal solution to the operator differential equations arising from two different anharmonic-oscillator models whose Hamiltonians are $H=\frac{1}{2}p^2+\frac{1}{4}q^4$ and $H=\frac{1}{4}p^4+\frac{1}{4}q^4$.

I. INTRODUCTION

In a recent paper[1] we suggested that it may be possible to obtain exact closed-form solutions to an extremely wide class of operator differential equations. We considered the Heisenberg operator differential equations of motion for quantum Hamiltonians $H=H(p,q)$ describing quantum-mechanical systems having one degree of freedom. These operator differential equations take the form

$$\dot{q}=\frac{1}{i}[q,H]\ , \tag{1.1a}$$

$$\dot{p}=\frac{1}{i}[p,H]\ . \tag{1.1b}$$

In the past, equations of this kind have been regarded as intractable except for the one special case of the harmonic oscillator, $H=\frac{1}{2}p^2+\frac{1}{2}q^2$, where Eqs. (1.1) are linear and therefore the operator properties of p and q do not pose any difficulties. Indeed, the harmonic-oscillator operator differential equations

$$\dot{q}=p\ , \tag{1.2a}$$

$$\dot{p}=-q \tag{1.2b}$$

have the *explicit* solution

$$q(t)=q(0)\cos t+p(0)\sin t\ , \tag{1.3a}$$

$$p(t)=p(0)\cos t-q(0)\sin t\ . \tag{1.3b}$$

Observe from (1.3) that the explicit solution to the Heisenberg differential equations (1.1) consists of giving the operators $q(t)$ and $p(t)$ in terms of the operators $q(0)$ and $p(0)$ and time t.

Note that $q(0)$ and $p(0)$ satisfy the commutation relation

$$[q(0),p(0)]=i \tag{1.4a}$$

and that the exact solutions $q(t)$ and $p(t)$ satisfy the equal-time commutation relation

$$[q(t),p(t)]=i\ . \tag{1.4b}$$

Our idea for solving the Heisenberg operator differential equations (1.1) is to find a quantum analogue of the classical action-angle variable. Consider briefly the case in which the Heisenberg equations of motion are *classical*:

$$\dot{q}=\frac{\partial H}{\partial p}\ , \tag{1.5a}$$

$$\dot{p}=-\frac{\partial H}{\partial q}\ . \tag{1.5b}$$

We use the fact that the energy

$$E=\tfrac{1}{2}p^2(t)+V(q(t)) \tag{1.6}$$

is a constant of the motion. Solving (1.6) for $\dot{q}=p$ gives

$$\dot{q}(t)=\sqrt{2[E-V(q)]}\ . \tag{1.7}$$

Thus,

$$t=f(q(t))-f(q(0))\ , \tag{1.8a}$$

where

$$f(q)=\int_0^q\frac{dz}{\sqrt{2[E-V(z)]}}\ . \tag{1.8b}$$

Solving the algebraic equations (1.6) and (1.8) simultaneously gives the solution to the classical Heisenberg differential equations (1.5). In some cases this *algebraic* procedure poses no difficulty and an *explicit* solution for $q(t)$ and $p(t)$ is obtained. In other cases it is not possible to carry out the algebraic solution of these equations and in such cases the solution to the Heisenberg differential equations can only be given in *implicit* form. However, we emphasize that whether or not the solution can be presented *explicitly* or *implicitly,* once the Heisenberg differential equations have been integrated and replaced with a pair of simultaneous algebraic equations, we regard the equations of motion as *solved.*

The procedure for solving the *quantum* Heisenberg differential equations (1.1) is a simple generalization of the procedure used to solve the classical equations of motion (1.5). Our objective is to find a pair of simultaneous algebraic equations involving the operators $q(t)$, $p(t)$, $q(0)$, and $p(0)$. The first such equation expresses the fact that the Hamiltonian is time independent, whether or not the system is classical or quantum:

$$H(p(t),q(t))=H(p(0),q(0)) \ . \tag{1.9}$$

The second equation is the analogue of (1.8a). We will construct a function F which depends on both operators q and p and satisfies an equation like (1.8a):

$$F(p(t),q(t))-F(p(0),q(0))=t \ . \tag{1.10}$$

Constructing such an F amounts to integrating the operator differential equations. We regard the algebraic equations (1.9) and (1.10) as the solution to the operator equations of motion. In Ref. 1 we examined some elementary examples where these two equations can be solved *explicitly* for the operators $q(0)$ and $p(t)$. As in classical mechanics, there are models whose explicit solution may be difficult or even impossible to obtain. We have thus reduced the problem of integrating the quantum Heisenberg equations of motion to finding the function $F(q,p)$ in (1.10). Such a function F must satisfy the commutation relation

$$\frac{1}{i}[F,H]=1 \ . \tag{1.11}$$

This is clearly a quantum analogue of the familiar action-angle variables approach to classical mechanics[2]—H plays the role of the action variable and F the role of the angle variable. The rest of this paper is devoted to the solution of this commutation relation (1.11) for F. However, before proceeding to the solution of this equation it is important to point out that the solution is clearly not unique; obviously, any two functions F_1 and F_2 satisfying (1.11) must differ by a function of the Hamiltonian H: $F_1=F_2+\phi(H)$. In our quest for the function F we will find it convenient to introduce the concept of a *minimal solution F.* A precise statement of what we mean by a minimal solution will be given in Sec. III.

This paper is organized as follows. It is necessary to define an operator basis in terms of which the solution for the operator F will be expressed. These operator basis elements form an algebra whose detailed mathematical structure is described in Sec. II. In Sec. III we show how to use this operator basis to solve (1.11) for the harmonic-oscillator Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{2}q^2$. The operator differential equations for the Hamiltonians $H=\frac{1}{2}p^2+\frac{1}{4}q^4$ and $H=\frac{1}{4}p^4+\frac{1}{4}q^4$ are solved in Secs. IV and V, respectively.

II. ALGEBRA OF THE OPERATOR BASIS ELEMENTS

We will solve for the function $F(p,q)$ defined by the commutation relation (1.11) by expressing it as a sum over operator basis elements $T_{m,n}$:

$$F(p,q)=\sum_{m,n}\alpha_{m,n}T_{m,n} \ , \tag{2.1}$$

where $\alpha_{m,n}$ are c-number expansion coefficients which must be determined. In this section we define the mathematical properties of the operator basis elements $T_{m,n}$.

We define $T_{m,n}$ $(m\geq 0,\ n\geq 0)$ as the Weyl-ordered form of the classical function $p^m q^n$:

$$T_{m,n}=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}q^k p^m q^{n-k} \ . \tag{2.2}$$

The Weyl-ordered product can be rewritten using the commutation relation $[q,p]=i$, as

$$T_{m,n}=\frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}p^j q^n p^{m-j} \ . \tag{2.3}$$

It is interesting to note that $T_{m,n}$ may also be expressed as the totally symmetrized form containing m factors of p and n factors of q, normalized by dividing by the number of terms in the symmetrized expression. For example,

$$T_{1,1}=\tfrac{1}{2}(pq+qp),\quad T_{1,2}=\tfrac{1}{3}(pq^2+qpq+q^2p) \ ,$$

$$T_{2,2}=\tfrac{1}{6}(p^2q^2+q^2p^2+pqpq+qpqp+pq^2p+qp^2q) \ ,$$

$$T_{0,4}=q^4 \ .$$

The Weyl-ordered form of $T_{m,n}$ has the advantage that it allows us to define in a natural way the basis elements $T_{m,n}$ where $m<0$, $n\geq 0$ or $m\geq 0$, $n<0$: In the first case we use (2.2) and in the second case we use (2.3). In fact, it is possible to define $T_{m,n}$ where *both* indices m and n are negative integers, or even complex numbers (see Ref. 1). However, in the quantum models considered in this paper, the *minimal* solution for the function F can be expressed entirely in terms of the basis elements $T_{m,n}$, where m and n integers such that $n\geq 0$ and $m\leq 0$ (Ref. 3). Some examples of these basis elements represented in Weyl-ordered from are

$$T_{-1,1}=\frac{1}{2}\left[\frac{1}{p}q+q\frac{1}{p}\right] ,$$

$$T_{-2,1}=\frac{1}{2}\left[\frac{1}{p^2}q+q\frac{1}{p^2}\right] ,$$

$$T_{-1,3}=\frac{1}{8}\left[\frac{1}{p}q^3+3q\frac{1}{p}q^2+3q^2\frac{1}{p}q+q^3\frac{1}{p}\right] ,$$

$$T_{-4,0}=\frac{1}{p^4} \ ,$$

$$T_{-2,4}=\frac{1}{16}\left[\frac{1}{p^2}q^4+4q\frac{1}{p^2}q^3+6q^2\frac{1}{p^2}q^2+4q^3\frac{1}{p^2}q+q^4\frac{1}{p^2}\right] .$$

The basis elements $T_{m,n}$ form an algebra closed under multiplication. All the properties of this algebra stem from a single product formula

$$T_{m,n}T_{r,s}=\sum_{j=0}^{\infty}\frac{\left(\frac{i}{2}\right)^j}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{n!}{(n-k)!}\frac{m!}{(m+k-j)!}\frac{r!}{(r-k)!}\frac{s!}{(s+k-j)!}T_{m+r-j,n+s-j},\quad m,n,r,s\in\mathbb{Z}^+ . \tag{2.4a}$$

When m,n,r,s are not positive integers this product formula generalizes to

$$T_{m,n}T_{r,s}=\sum_{j=0}^{\infty}\frac{\left(\frac{i}{2}\right)^j}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{\Gamma(n+1)\Gamma(m+1)\Gamma(r+1)\Gamma(s+1)}{\Gamma(n-k+1)\Gamma(m+k-j+1)\Gamma(r-k+1)\Gamma(s+k-j+1)}T_{m+r-j,n+s-j} . \tag{2.4b}$$

Using (2.4) it is easy to see that the basis elements $T_{m,n}$ satisfy the following commutation and anticommutation relations:

$$[T_{m,n},T_{r,s}]=2\sum_{j=0}^{\infty}\frac{\left(\frac{i}{2}\right)^{2j+1}}{(2j+1)!}\sum_{l=0}^{2j+1}(-1)^l\binom{2j+1}{l}\frac{\Gamma(m+1)\Gamma(n+1)\Gamma(r+1)\Gamma(s+1)}{\Gamma(m-l+1)\Gamma(n+l-2j)\Gamma(r+l-2j)\Gamma(s-l+1)}\times T_{m+r-2j-1,n+s-2j-1} , \tag{2.5a}$$

$$\{T_{m,n},T_{r,s}\}_+=2\sum_{j=0}^{\infty}\frac{\left(\frac{i}{2}\right)^{2j}}{(2j)!}\sum_{l=0}^{2j}(-1)^l\binom{2j}{l}\frac{\Gamma(m+1)\Gamma(n+1)\Gamma(r+1)\Gamma(s+1)}{\Gamma(m-l+1)\Gamma(n+l-2j+1)\Gamma(r+l-2j+1)\Gamma(s-l+1)}\times T_{m+r-2j,m+s-2j} . \tag{2.5b}$$

Some interesting special cases of these commutation and anticommutation relations are

$$[q,T_{m,n}]=imT_{m-1,n} ,$$

$$[p,T_{m,n}]=-inT_{m,n-1} ,$$

$$\{q,T_{m,n}\}_+=2T_{m,n+1} ,$$

$$\{p,T_{m,n}\}_+=2T_{m+1,n} .$$

Thus, commuting with q and p has the effect of a lowering operator, and anticommuting with q and p has the effect of a raising operator, in the appropriate index.

Further special cases of the commutation relation (2.5a) which will be used later in this paper to evaluate the commutation relation (1.11) are

$$[q^2,T_{m,n}]=2imT_{m-1,n+1} , \tag{2.6a}$$

$$[p^2,T_{m,n}]=-2inT_{m+1,n-1} , \tag{2.6b}$$

$$[q^4,T_{m,n}]=4imT_{m-1,n+3}-im(m-1)(m-2)T_{m-3,n+1} , \tag{2.6c}$$

$$[p^4,T_{m,n}]=-4inT_{m+3,n-1}+in(n-1)(n-2)T_{m+1,n-3} . \tag{2.6d}$$

A more elaborate example of a commutation relation yielding an infinite sum of basis elements is

$$\left[\frac{1}{q},T_{m,n}\right]=-i\sum_{j=0}^{\infty}\left(-\frac{1}{4}\right)^j\frac{m!}{(m-2j-1)!}\times T_{m-2j-1,n-2j-2} . \tag{2.7}$$

III. MINIMAL SOLUTION OF THE HARMONIC OSCILLATOR

To illustrate the technique for determining the function F we consider the harmonic oscillator with Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{2}q^2$. Having expressed $F(p,q)$ as an arbitrary sum over basis elements $T_{m,n}$ as in (2.1) we substitute (2.1) into the defining relation (1.11) and use the results (2.6a) and (2.6b) to find

$$1=\sum_{m,n}\alpha_{m,n}(nT_{m+1,n-1}-mT_{m-1,n+1}) . \tag{3.1}$$

Hence, assuming completeness, we determine that the coefficients $\alpha_{m,n}$ satisfy the linear partial difference equation

$$(n+1)\alpha_{m-1,n+1}-(m+1)\alpha_{m+1,n-1}=\delta_{m,0}\delta_{n,0} . \tag{3.2}$$

This partial difference equation relates pairs of coefficients $\alpha_{m,n}$. If we represent the coefficients $\alpha_{m,n}$ (for all m,n) as dots on a integer planar lattice then it is clear that next-to-nearest-neighboring points on a diagonal whose slope is -1 are related (see Fig. 1).

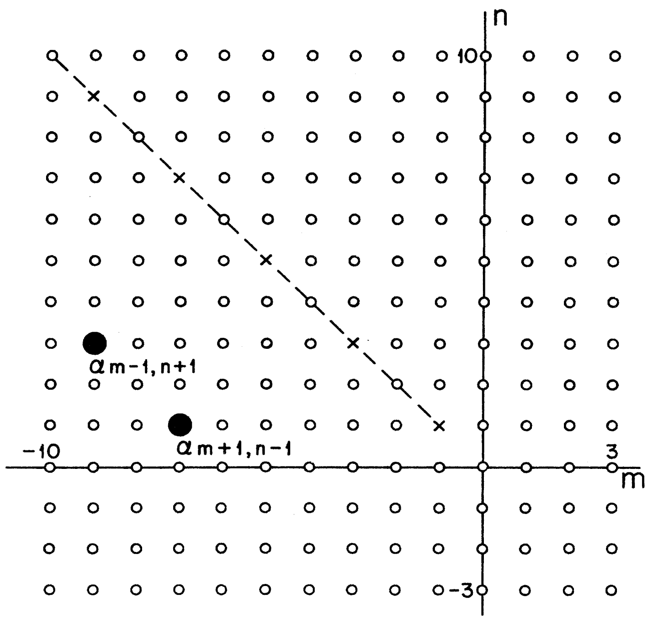

FIG. 1. The partial difference equation (3.2) for the harmonic oscillator relates the coefficients $\alpha_{m-1,n+1}$ and $\alpha_{m+1,n-1}$. The minimal solution consists of all $\alpha_{m,n}=0$ except $\alpha_{-2m-1,2m+1}$, $m=0,1,2,\ldots$, as indicated by crosses.

The simplest solution to (3.2) consists of taking as many $\alpha_{m,n}$'s as possible to vanish. Of course, it is not possible for *all* the coefficients $\alpha_{m,n}$ to vanish because of the presence of the inhomogeneous term on the right-hand side of (3.2). We construct what we call the *minimal solution* by starting with the partial difference equation with $m=n=0$, in which the inhomogeneous term is present, and deduce the smallest set of $\alpha_{m,n}$'s which are nonvanishing as a consequence of this partial difference equation. All other $\alpha_{m,n}$'s are set to zero. For the harmonic oscillator, with partial difference equation (3.2), the minimal set of nonzero $\alpha_{m,n}$'s lies on a diagonal line, passing through the origin, in the $m<0$, $n>0$ quadrant in Fig. 1. Specifically, the minimal solution to (3.2) is

$$\alpha_{-2m-1,2m+1}=\frac{(-1)^m}{2m+1}, \quad m=0,1,\ldots, \tag{3.3}$$

with all other $\alpha_{m,n}$'s vanishing. Thus a function $F(q,p)$ satisfying (1.11) is

$$F(p,q)=\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}T_{-2m-1,2m+1}\ . \tag{3.4}$$

It is interesting to note that this is in fact the Weyl-ordered form of the *classical* function

$$\theta=\arctan\left[\frac{q}{p}\right], \tag{3.5}$$

where θ is the angle of a point on the classical trajectory in phase space. This result clearly illustrates that the operator methods used in this paper generalize the notion of classical action-angle variables to the realm of quantum mechanics.

In the case of the harmonic oscillator, we can also use the exact solution (1.3) to find another function $\widetilde{F}(p,q)$ satisfying (1.11). As shown in Ref. 1, (1.3) implies that $q(t)[1/p(t)]$ is a function of $q(0)[1/p(0)]$ and we can further deduce that

$$\widetilde{F}(p,q)=\arctan\left[q\frac{1}{p}\right] \tag{3.6}$$

satisfies the commutation relation (1.11). Since both F and $\widetilde{F}$ satisfy the commutation relation (1.11), they must differ by a function of the Hamiltonian H. We have verified this by a lengthy calculation in which we have shown that $F-\widetilde{F}$ can be expressed as a series in inverse powers of the Hamiltonian H whose coefficients are Euler numbers:[4]

$$\begin{aligned} F(p,q)-\widetilde{F}(p,q)&=\frac{i}{2H}\sum_{n=0}^{\infty}E_{2n}\frac{1}{H^{2n}}\\ &=\frac{i}{2H}\int_0^{\infty}ds\frac{e^{-s}}{\cosh\left[\dfrac{s}{2H}\right]}\ . \end{aligned} \tag{3.7}$$

IV. MINIMAL SOLUTION FOR ANHARMONIC OSCILLATOR $H=\frac{1}{2}p^2+\frac{1}{4}q^4$

To find the function $F(p,q)$ satisfying (1.11) for the anharmonic-oscillator Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{4}q^4$ we substitute the general form for F in (2.1) into the commu-

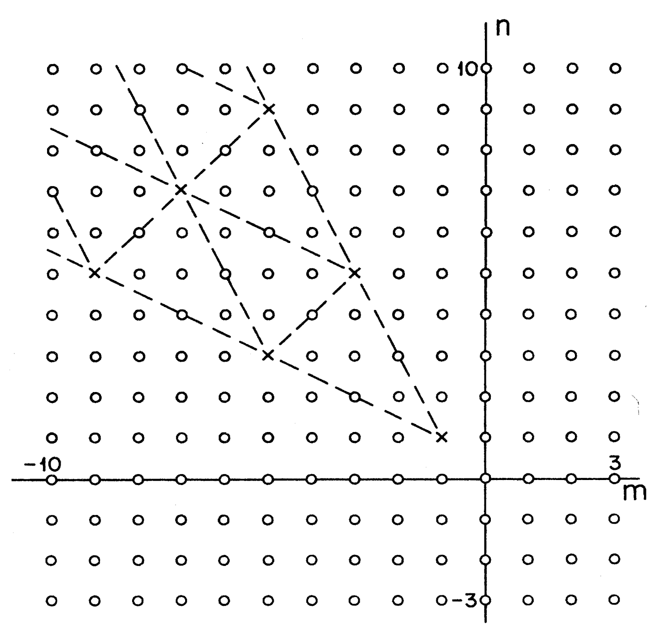

FIG. 2. Triplets of points related by the partial difference equation (4.2) for the anharmonic oscillator with $H=\frac{1}{2}p^2+\frac{1}{4}q^4$. The minimal solution consists of nonvanishing values of $\alpha_{m,n}$ indicated by crosses; all other $\alpha_{m,n}$'s vanish.

tation relation (1.11). Using (2.6b) and (2.6c) we find the analogue of (3.1),

$$1=\sum_{m,n}\alpha_{m,n}[nT_{m+1,n-1}-mT_{m-1,n+3}+\tfrac{1}{4}m(m-1)(m-2)T_{m-3,n+1}], \quad (4.1)$$

from which we deduce the partial difference equation satisfied by the coefficients $\alpha_{m,n}$:

$$(n+1)\alpha_{m-1,n+1}-(m+1)\alpha_{m+1,n-3}+\tfrac{1}{4}(m+1)(m+2)(m+3)\alpha_{m+3,n-1}=\delta_{m,0}\delta_{n,0}. \quad (4.2)$$

This partial difference equation relates triples of points on the integer lattice in Fig. 2 whose points correspond to coefficients $\alpha_{m,n}$. A careful analysis of (4.2) shows that a minimal solution exists consisting of certain nonzero values of $\alpha_{m,n}$. These nonzero values of $\alpha_{m,n}$ correspond to points in Fig. 2, lying in the quadrant $m<0$, $n>0$, and forming a triangular network as indicated in the figure.

A first step in solving (4.2) involves mapping the triangular network of nonvanishing values of $\alpha_{m,n}$ onto a triangular domain. On this domain the partial difference equation (4.2) works in much the same fashion as the difference equation that generates the binomial coefficients in Pascal's triangle. The relevant transformation of the independent variables is

$$M=-\tfrac{1}{6}(n+2m), \quad N=\tfrac{1}{6}(n-m). \quad (4.3)$$

We define a new dependent variable $A_{M,N}$ by

$$A_{M,N}=\alpha_{-2N-2M-1,4N-2M+1}=\alpha_{m-1,n+1}, \quad (4.4)$$

along with the constraint that $A_{M,N}=0$ for $M<0$, $N<0$, and for $M>N$.

The advantage of the M,N variables over the m,n variables is that the partial difference equation is first order:

$$(4N-2M+1)A_{M,N}+(2N+2M-1)A_{M,N-1}-\tfrac{1}{4}(2N+2M-1)(2N+2M-2)(2N+2M-3)A_{M-1,N-1}=\delta_{M,0}\delta_{N,0}. \quad (4.5)$$

A further transformation, this time of the dependent variable, reduces the partial difference equation to one whose coefficients are linear functions of M and N. We define

$$B_{M,N}=\frac{2^{-N}\Gamma(\frac{1}{2})}{\Gamma(M+N+\frac{1}{2})}A_{M,N} \quad (4.6)$$

and thus the partial difference equation becomes

$$(4N-2M+1)B_{M,N}+B_{M,N-1}-(N+M-1)B_{M-1,N-1}=\delta_{M,0}\delta_{N,0}. \quad (4.7)$$

In Table I we list the values of $B_{M,N}$ for the first few rows of the triangular network. It is not hard to derive simple closed-form expressions for special elements in this triangular array:

$$B_{0,N}=\frac{(-1)^N\Gamma(\frac{5}{4})}{4^N(N+\frac{5}{4})}, \quad (4.8)$$

$$B_{N,N}=\frac{1}{2N+1}, \quad N\geq 0, \quad (4.9)$$

$$B_{N-1,N}=\frac{-(2N+7)}{3(2N+1)(2N+3)}, \quad N\geq 1, \quad (4.10)$$

$$B_{N-2,N}=\frac{(N-1)(4N^2+34N+105)}{9(2N-1)(2N+1)(2N+3)(2N+5)}, \quad N\geq 2, \quad (4.11)$$

$$B_{N-3,N}=\frac{-(N-2)(8N^3+120N^2+808N+2517)}{27(2N-1)(2N+1)(2N+3)(2N+5)(2N+7)}, \quad N\geq 3, \quad (4.12)$$

$$B_{N-4,N}=\frac{(N-2)(N-3)(16N^4+368N^3+4052N^2+26\,224N+83\,289)}{6\times 81(2N-3)(2N-1)(2N+1)(2N+3)(2N+5)(2N+7)(2N+9)}, \quad N\geq 4. \quad (4.13)$$

However, the simplest way to express a general element $B_{M,N}$ of this array is in terms of a generating function; to wit, we define the generating function

$$g(x,y)=\sum_{N=0}^{\infty}\sum_{M=0}^{N}B_{M,N}x^My^N. \quad (4.14)$$

From (4.7) we can derive a first-order linear partial differential equation satisfied by $g(x,y)$:

$$-x(2+xy)g_x+y(4-xy)g_y+(1+y-xy)g=1. \quad (4.15)$$

Using the method of characteristics we can find the unique solution to this partial differential equation that satisfies the initial condition [from (4.8)]

$$g(0,y)=\sum_{N=0}^{\infty}\left[-\frac{y}{4}\right]^N\frac{\Gamma(\frac{5}{4})}{\Gamma(N+\frac{5}{4})}=e^{-y/4}\int_0^1 du\, e^{u^4 y/4} . \tag{4.16}$$

The solution is

$$g(x,y)=\int_0^1\frac{d\xi}{2\sqrt{\xi}(1-xy\xi)}\exp\left[\frac{1}{x^2y}\left[\frac{2}{3}-xy+\frac{(1-xy)^{3/2}(xy\xi-\frac{2}{3})}{(1-xy\xi)^{3/2}}\right]\right] . \tag{4.17}$$

In terms of the generating function g,

$$B_{M,N}=\frac{1}{N!M!}\left[\frac{\partial}{\partial x}\right]^M\left[\frac{\partial}{\partial y}\right]^N g(x,y)\bigg|_{x=y=0} . \tag{4.18}$$

Using this expression for the coefficients $B_{M,N}$ in the formula (2.1) for $F(p,q)$ gives a complete and exact minimal solution to the Heisenberg operator differential equations for the anharmonic oscillator with the Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{4}q^4$. The operator $F(p,q)$ satisfying (1.11) is

$$F(p,q)=\sum_{N=0}^{\infty}\sum_{M=0}^{N}2^N\frac{\Gamma(M+N+\frac{1}{2})}{\Gamma(\frac{1}{2})} \times B_{M,N}T_{-2N-2M-1,4N-2M+1} . \tag{4.19}$$

Before concluding this section, we point out that there is an interesting transformation that converts the array $B_{M,N}$ into a set of polynomials $P_J(N)$ all of which have positive integer coefficients. The relevant transformation is

$$B_{N-J,N}=\frac{\left[\frac{-4}{3}\right]^J}{J!}\frac{(N+J)!(2N-J)!}{(N-J)!(2N+2J+1)!}P_J(N) . \tag{4.20}$$

The first six such polynomials are

$$\begin{aligned}
&P_0(N)=1,\quad P_1(N)=2N+7 ,\\
&P_2(N)=4N^2+34N+105 ,\\
&P_3(N)=8N^3+120N^2+808N+2517 ,\\
&P_4(N)=16N^4+368N^3+4052N^2+26\,224N+83\,289 ,\\
&P_5(N)=32N^5+1040N^4+16\,600N^3+168\,460N^2\\
&\qquad+1\,089\,498N+3\,513\,915 .
\end{aligned} \tag{4.21}$$

TABLE I. Nonzero values of $B_{M,N}$.

	$M=0$	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$
$N=0$	1					
$N=1$	$-\frac{1}{5}$	$\frac{1}{3}$				
$N=2$	$\frac{1}{45}$	$-\frac{11}{105}$	$\frac{1}{5}$			
$N=3$	$-\frac{1}{585}$	$\frac{6}{385}$	$-\frac{13}{189}$	$\frac{1}{7}$		
$N=4$	$\frac{1}{9945}$	$-\frac{202}{135\,135}$	$\frac{305}{27\,027}$	$-\frac{5}{99}$	$\frac{1}{9}$	
$N=5$	$-\frac{1}{208\,845}$	$\frac{353}{3\,357\,585}$	$-\frac{23}{19\,305}$	$-\frac{100}{11\,583}$	$-\frac{17}{429}$	$\frac{1}{11}$

The polynomials $P_J(N)$ do not satisfy a simple recursion relation. However, they do satisfy a functional equation easily derivable from (4.7) and (4.20):

$$0=(2N-J)P_J(N)-3J(2N+2J-1)P_{J-1}(N-1) -2(N-J)P_J(N-1),\quad J<N . \tag{4.22}$$

V. MINIMAL SOLUTION OF THE OSCILLATOR $H=\frac{1}{4}p^4+\frac{1}{4}q^4$

In this section we consider the model Hamiltonian $H=\frac{1}{4}p^4+\frac{1}{4}q^4$. This Hamiltonian is the natural next step in our program of solving increasingly complex systems of quantum operator differential equations: $H=\frac{1}{2}p^2+\frac{1}{2}q^2$ gives rise to a two-term partial difference equation for the coefficients $\alpha_{m,n}$ in (2.1), $H=\frac{1}{2}p^2+\frac{1}{4}q^4$ gives rise to a three-term partial difference equation, and $H=\frac{1}{4}p^4+\frac{1}{4}q^4$ gives rise to a four-term partial difference equation.

To find the function $F(p,q)$ satisfying (1.11) for this anharmonic-oscillator Hamiltonian, we substitute the general form for F in (2.1) into (1.11) and use (2.6c) and (2.6d) to find the analogue of (3.1) and (4.1):

$$\begin{aligned}
1=\sum_{n,m}\alpha_{m,n}[\,&nT_{m+3,n-1}-mT_{m-1,n+3}\\
&-\tfrac{1}{4}n(n-1)(n-2)T_{m+1,n-3}\\
&+\tfrac{1}{4}m(m-1)(m-2)T_{m-3,n+1}] .
\end{aligned} \tag{5.1}$$

From this we deduce that the coefficients $\alpha_{m,n}$ must satisfy the partial difference equation

$$\begin{aligned}
&(n+1)\alpha_{m-3,n+1}-(m+1)\alpha_{m+1,n-3}\\
&\quad-\tfrac{1}{4}(n+1)(n+2)(n+3)\alpha_{m-1,n+3}\\
&\quad+\tfrac{1}{4}(m+1)(m+2)(m+3)\alpha_{m+3,n-1}=\delta_{m,0}\delta_{n,0} .
\end{aligned} \tag{5.2}$$

This partial difference equation relates quartets of points, which lie at vertices of rectangles, on the integer lattice in Fig. 3 whose points correspond to coefficients $\alpha_{m,n}$. Examining (5.2) we see that a minimal solution exists with nonzero values of $\alpha_{m,n}$ lying in the quadrant $m<0$, $n>0$, and forming a wedge-shaped network as indicated in Fig. 3.

To solve (5.2) we use the transformation

TABLE II. Nonzero values of $D_{M,N}$.

	$M=0$	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$
$N=0$	1								
$N=1$	$-\frac{3}{5}$	2	3						
$N=2$	$\frac{7}{15}$	$-\frac{174}{35}$	$\frac{201}{5}$	630	945				
$N=3$	$-\frac{77}{195}$	$\frac{668}{77}$	$-\frac{1438}{7}$	630	186 165	1 559 250	2 338 875		
$N=4$	$\frac{77}{221}$	$-\frac{13004}{1001}$	$\frac{6470}{11}$	$-\frac{170730}{11}$	$-716\,625$	73 284 750	2 125 101 825	14 898 633 750	22 347 950 625

$$N=\tfrac{1}{8}(n-m),\quad M=-\tfrac{1}{4}(n+m)\,, \tag{5.3}$$

which maps the points in the m,n plane for which the coefficient $\alpha_{m,n}$ is nonzero into a triangular lattice for which N runs from 0 to infinity and M runs from 0 to $2N$. We then define a new dependent variable $C_{M,N}$ by

$$C_{M,N}=\alpha_{-2M-4N-3,4N-2M+1}=\alpha_{m-3},\alpha_{n+1}\,, \tag{5.4}$$

along with the constraint $C_{M,N}=0$ for $M<0$, $N<0$, and $M>2N$. Observe that the partial difference equation for $C_{M,N}$ is first order:

$$2(2N-M+\tfrac{1}{2})C_{M,N}-(2N-M+\tfrac{1}{2})(4N-2M+2)(2N-M+\tfrac{3}{2})C_{M-1,N}+2(2N+M-\tfrac{1}{2})C_{M,N-1}$$
$$-(2N+M-\tfrac{1}{2})(2M+4N-2)(2N+M-\tfrac{3}{2})C_{M-1,N-1}=\delta_{M,0}\delta_{N,0}\,. \tag{5.5}$$

We can transform this partial difference equation into one with linear coefficients by defining

$$D_{M,N}=4^{-M}\frac{\Gamma\left[N-\dfrac{M}{2}+\dfrac{3}{4}\right]\Gamma\left[N-\dfrac{M}{2}+\dfrac{5}{4}\right]}{\Gamma\left[N+\dfrac{M}{2}+\dfrac{3}{4}\right]\Gamma\left[N+\dfrac{M}{2}+\dfrac{5}{4}\right]}C_{M,N}\,. \tag{5.6}$$

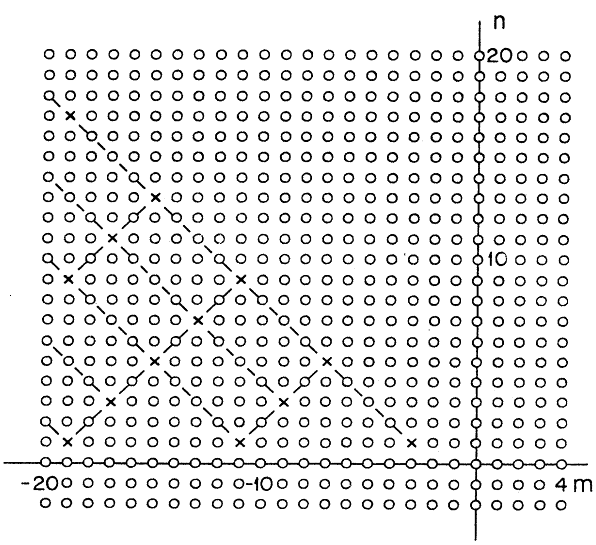

FIG. 3. Quartets of points related by the partial difference equation (5.2) for the anharmonic oscillator with $H=\frac{1}{4}p^4+\frac{1}{4}q^4$. The minimal solution consists of nonvanishing values of $\alpha_{m,n}$ indicated by crosses; all other $\alpha_{m,n}$'s vanish.

The partial difference equation becomes

$$\left[N+\frac{M}{2}+\frac{1}{4}\right]D_{M,N}-\left[N-\frac{M}{2}+\frac{1}{2}\right]D_{M-1,N}+\left[N-\frac{M}{2}-\frac{1}{4}\right]D_{M,N-1}-\left[N+\frac{M}{2}-\frac{1}{2}\right]D_{M-1,N-1}=\tfrac{1}{4}\delta_{M,0}\delta_{N,0}\,. \tag{5.7}$$

In Table II we list the values of $D_{M,N}$ for the first few rows of the triangular array. Some simple closed-form expressions for special elements in this array are

$$D_{0,N}=\frac{(-1)^N\Gamma(\frac{5}{4})\Gamma(N+\frac{3}{4})}{\Gamma(\frac{3}{4})\Gamma(N+\frac{5}{4})}\,, \tag{5.8}$$

$$D_{2N,N}=\frac{\Gamma(2N+\frac{1}{2})\Gamma(N+\frac{1}{2})^2\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})^3\Gamma(2N+\frac{3}{4})\Gamma(2N+\frac{5}{4})}\,. \tag{5.9}$$

A general element $D_{M,N}$ of this array can be expressed in terms of a generating function $h(x,y)$:

$$h(x,y)=\sum_{N=0}^{\infty}\sum_{M=0}^{2N}D_{M,N}x^My^N\,. \tag{5.10}$$

From (5.7) we can derive a first-order linear partial differential equation satisfied by $h(x,y)$:

$$\frac{x}{2}\left[\frac{1+x}{1-x}\right]h_x+y\left[\frac{1+y}{1-y}\right]h_y+\frac{1+3y-4xy}{4(1-x)(1-y)}h=\frac{1}{4(1-x)(1-y)}\,. \tag{5.11}$$

The unique solution to this partial differential equation that satisfies the initial condition

$$h(0,y)=\sum_{N=0}^{\infty}(-y)^N\frac{\Gamma(\frac{5}{4})\Gamma(N+\frac{3}{4})}{\Gamma(\frac{3}{4})\Gamma(N+\frac{5}{4})}$$
$$=F(\tfrac{3}{4},1;\tfrac{5}{4};-y)$$
$$=\int_0^1\frac{d\omega}{\sqrt{(1+y)^2-4y\omega^4}}\,, \tag{5.12}$$

is

$$h(x,y)=\int_0^1\frac{d\omega}{\sqrt{(1+y)^2-4y\omega^4(1+x-x\omega^2)^2}}\,. \tag{5.13}$$

In terms of this generating function,

$$D_{M,N}=\frac{1}{N!M!}\left[\frac{\partial}{\partial x}\right]^M\left[\frac{\partial}{\partial y}\right]^N h(x,y)|_{x=y=0}\,. \tag{5.14}$$

Using this expression for the coefficients $D_{M,N}$ in the formula (2.1) for $F(p,q)$ gives a complete and exact minimal solution to the Heisenberg operator differential equations for the anharmonic oscillator with Hamiltonian $H=\frac{1}{4}p^4+\frac{1}{4}q^4$. The operator $F(p,q)$ satisfying (1.11) is

$$F(p,q)=\sum_{N=0}^{\infty}\sum_{M=0}^{2N}4^M\frac{\Gamma\left[N+\frac{M}{2}+\frac{3}{4}\right]\Gamma\left[N+\frac{M}{2}+\frac{5}{4}\right]}{\Gamma\left[N-\frac{M}{2}+\frac{3}{4}\right]\Gamma\left[N-\frac{M}{2}+\frac{5}{4}\right]}\times D_{M,N}T_{-2M-4N-3,4N-2M+1}\,. \tag{5.15}$$

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[1]C. M. Bender and G. V. Dunne, Phys. Rev. D **40**, 2739 (1989).

[2]V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1980).

[3]Evidently, operators $T_{m,n}$ where $m<0$ and $n\geq 0$ must be defined as pseudodifferential operators. We reserve a detailed discussion of the action of such operators on vectors in Hilbert space for a future paper.

[4]The first few Euler numbers are $E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385, \ldots$.