Bogomol'nyi equations for non-Abelian gauge theories

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We find a bound for the two-dimensional classical action (four-dimensional energy per unit length) for an SU(N) gauge theory with spontaneous symmetry breaking. This bound is saturated when a particular relation between coupling constants holds; the corresponding vortex configuration satisfies a first-order system of differential equations—the Bogomol'nyi equations.

I. INTRODUCTION AND RESULTS

Vortices, monopoles, and instantons are interesting topological objects that arise as regular classical solutions in gauge theories and have relevant physical implications in quantum field theory.[1,2]

Originally, Abelian vortex and non-Abelian monopole solutions were discovered by studying the second-order Euler-Lagrange equations for gauge theories with spontaneous symmetry breaking.[3,4] It was soon realized[5] that at certain "critical" values of the coupling constants one can find these solutions by solving a system of first-order nonlinear coupled equations, called the Bogomol'nyi equations, rather than the more involved system of second-order equations. Moreover, the energy per unit length of the vortex or the energy of the monopole are bounded below by a topological invariant, the vortex number n or monopole charge n, respectively. It is remarkable that at the critical point the energy of an n-vortex (n-monopole) configuration is n times the energy of a single-vortex (-monopole) one. This additivity of the bound implies that vortices (monopoles) do not interact.

Concerning instanton solutions to Yang-Mills equations of motion, they were originally found by saturating the four-dimensional equivalent of this bound.[6] The equation that arises from its saturation is the famous first-order self-duality equation.

It is interesting to note that in these three cases, exact solutions to the first-order system have been found.[6–8]

Apart from the fact that a first-order system is much simpler to analyze than the original equations of motion (in particular in what concerns existence and uniqueness of the solutions[9]) Bogomol'nyi equations are very attractive in the context of the recently discovered topological field theories.[10,11] Indeed, new two- and three-dimensional topological field theories have been constructed by Becchi-Rouet-Stora-Tyutin (BRST) quantization of the Langevin equations associated with the Bogomol'nyi equations.[12–14] As is well known, vortices (monopoles) can be considered as instantons in a $d=2$ ($d=3$) Euclidean space-time. From this point of view, they have a vanishing $d=2$ ($d=3$) energy-momentum tensor as it is also the case for $d=4$ instantons. This is a very appealing property concerning topological field theories which exhibit as a basic property the vanishing of the (quantum) energy-momentum tensor.

For non-Abelian gauge theories with symmetry breaking ($d=3$) the critical point at which Bogomol'nyi equations hold corresponds to the the so-called Prasad-Sommerfield limit[7] in which there is no Higgs-boson coupling, $\lambda=0$. This is a very peculiar spontaneous symmetry-breaking case and, from the topological-field-theory viewpoint, a bit disappointing: one needs Bogomol'nyi equations in order to construct a topological theory but it seems impossible (in the non-Abelian case) to introduce a Higgs-boson coupling while keeping the topological nature of the theory.

In this paper we show that happily it is possible to construct Bogomol'nyi equations for a non-Abelian gauge theory in $d=2$ Euclidean space-time with no necessity of considering the $\lambda=0$ limit.

With this purpose, we first consider an SU(2) gauge theory with two Higgs fields ϕ and ψ in order to have complete symmetry breaking. As is well known, vortex solutions (static, axially symmetric solutions in $d=4$ or alternatively, instanton $d=2$ Euclidean solutions) exist in this non-Abelian case.[15,16] Inspired in the Ansatz leading to these solutions and proceeding in the manner of Bogomol'nyi[5] we are able to find a bound to the $d=2$ action ($d=4$ energy per unit length) which reads

$$\begin{aligned} S &\geq \tfrac{1}{2}\pi e^2\eta^4|(2k+n)\hat{\psi}^3_\infty| \quad \text{if } \lambda_1 \geq \tfrac{1}{8}\ , \\ S &\geq \sqrt{2\lambda_1}\,\pi e^2\eta^4|(2k+n)\hat{\psi}^3_\infty| \quad \text{if } \lambda_1 \leq \tfrac{1}{8}\ . \end{aligned} \tag{1.1}$$

Here η is the ϕ-field vacuum expectation value, ψ^3_∞ is the value of the third component of the ψ field at infinity, n is the topological charge [$n=0,1$ since in the SU(2) case the relevant homotopy group is Z_2], and $k\in Z$.

In contrast with the Abelian vortex, monopole, and instanton cases, this bound is not a topologically invariant quantity. The difference in the bounds associated with distinct members of the same class is related to the fact that the topological charge of SU(N) vortices is defined modulo N but physical quantities may depend on the actual value of the magnetic flux associated with this topological charge.[17] The presence of k in (1.1) shows the nontopological character of the bound.

The bound (1.1) is saturated when the following Bogomol'nyi equations hold:

$$\mathbf{F}_{\mu\nu} \mp \epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}(\boldsymbol{\phi}^2-1)=0 \, , \qquad D_\mu \boldsymbol{\phi} \pm \epsilon_{\mu\nu}\hat{\boldsymbol{\psi}} \wedge D_\nu \boldsymbol{\phi}=0, \quad D_\mu \boldsymbol{\psi}=0 \, , \tag{1.2}$$

where $\mathbf{A}_\mu$ is the gauge field satisfying the boundary condition

$$\lim_{\rho\to\infty} A_\mu = A_\mu^3 \sigma_3 \, . \tag{1.3}$$

Note that the original Abelian Bogomol'nyi equations[5] can be obtained from (1.2) by reduction of the SU(2) model to an SO(2) one taking

$$\psi^1=\psi^2=\phi^3=0, \quad A_\mu^1=A_\mu^2=0 \, . \tag{1.4}$$

We also find the general SU(N) Bogomol'nyi bound [see Eq. (3.10)] and we also present the corresponding Bogomol'nyi equations [Eqs. (3.12)–(3.14)].

As we stated above, one of the applications of Bogomol'nyi equations is related to the construction of topological field theories, not only because they have physical relevance but also because they provide a field-theoretical method of arriving at instanton moduli space. In Ref. 14 the first example of a topological field theory with explicit spontaneous symmetry breaking was presented and the Abelian instanton (vortex) moduli space analyzed. We hope that Bogomol'nyi equations for non-Abelian vortices will provide, following the approach developed in Refs. 13 and 14, a topological field theory with a richer structure. We will report on this subject elsewhere.

The paper is organized as follows. In Sec. II we briefly describe the SU(2) non-Abelian vortex solutions and then we explain how the bound to the action and the Bogomol'nyi equations can be found. In Sec. III we extend the discussion to the general SU(N) case. Finally, we leave for the Appendix details of the calculations.

II. NON-ABELIAN BOGOMOL'NYI EQUATIONS: THE SU(2) CASE

A. Non-Abelian vortices

As we stated in the Introduction, vortices can be considered either as static, axially symmetric solutions to the four-dimensional Euler-Lagrange equations or as instantons in a two-dimensional Euclidean space. We shall take this last point of view.

Vortex configurations exist whenever the gauge symmetry is spontaneously broken via Higgs fields, leaving invariant the vacuum under a certain subgroup H of the gauge group G. The magnetic flux is quantized and this is intimately related to the nontriviality of the relevant homotopy group.

For G=SU(N), vortex solutions have been found for maximum symmetry breaking.[15] This can be achieved taking N Higgs fields in the adjoint representation and choosing an adequate symmetry-breaking potential. In this case $H=Z_N$ and $\pi_1(G/H)=Z_N$. One then has, for G=SU(N), $N-1$ topologically inequivalent solutions [compare with the Abelian case, G=U(1), $H=I$ and one has $\pi_1(G/H)=Z$]. If we call θ the angle characterizing the direction at infinity, a mapping belonging to the n-homotopy class satisfies, when one makes a turn around a closed contour,

$$g_n(2\pi)=e^{i2\pi n/N}g_n(0), \quad g_n \in \mathrm{SU}(N) \, . \tag{2.1}$$

For simplicity we shall first consider the SU(2) case leaving for the next section the general SU(N) case. The action for the two-dimensional (Euclidean) SU(2) Higgs model is

$$S=\int d^2x \left[\tfrac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}_{\mu\nu}+\tfrac{1}{2}D_\mu\boldsymbol{\phi}\cdot D_\mu\boldsymbol{\phi}+\tfrac{1}{2}D_\mu\boldsymbol{\psi}\cdot D_\mu\boldsymbol{\psi} + V(\boldsymbol{\phi},\boldsymbol{\psi})\right] , \tag{2.2}$$

where $\mathbf{A}_\mu$ is the gauge field,

$$\mathbf{F}_{\mu\nu}=\partial_\mu \mathbf{A}_\nu-\partial_\nu \mathbf{A}_\mu+e\,\mathbf{A}_\mu \wedge \mathbf{A}_\nu \, , \tag{2.3}$$

and $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are the Higgs fields in the adjoint representation. The covariant derivative of these scalar fields is

$$D_\mu\boldsymbol{\phi}=\partial_\mu\boldsymbol{\phi}+e\,\mathbf{A}_\mu\wedge\boldsymbol{\phi} \, . \tag{2.4}$$

We choose, for the potential,

$$V(\boldsymbol{\phi},\boldsymbol{\psi})=\frac{g}{8}(\boldsymbol{\phi}^2-\eta^2)^2+\frac{g'}{8}(\boldsymbol{\psi}^2-\eta'^2)^2\frac{g'}{8}(\boldsymbol{\phi}\cdot\boldsymbol{\psi})^2 \tag{2.5}$$

so as to ensure complete symmetry breaking.

It is convenient to go over to dimensionless variables:

$$\boldsymbol{\phi}\to\eta\boldsymbol{\phi}, \quad \boldsymbol{\psi}\to\eta'\boldsymbol{\psi}, \quad \mathbf{A}_\mu\to\eta\,\mathbf{A}_\mu, \quad \rho\to\frac{1}{e\eta}\rho \, . \tag{2.6}$$

Then, the action reads

$$S=e^2\eta^4\int d^2x \left[\tfrac{1}{4}(\mathbf{F}_{\mu\nu})^2+\tfrac{1}{2}(D_\mu\boldsymbol{\phi})^2+\frac{1}{2}\frac{\eta'^2}{\eta^2}(D_\mu\boldsymbol{\psi})^2 +\lambda_1(\boldsymbol{\phi}^2-1)^2+\lambda_2\frac{\eta'^4}{\eta^4}(\boldsymbol{\psi}^2-1)^2 +\lambda_3\frac{\eta'^2}{\eta^2}(\boldsymbol{\phi}\cdot\boldsymbol{\psi})^2\right] , \tag{2.7}$$

where

$$\mathbf{F}_{\mu\nu}=\partial_\mu\mathbf{A}_\nu-\partial_\upsilon\mathbf{A}_\mu+\mathbf{A}_\mu\wedge\mathbf{A}_\nu \, , \qquad D_\mu=\partial_\mu+\mathbf{A}_\mu\wedge \, , \qquad \lambda_1=\frac{g^2}{8e^2}, \quad \lambda_2=\frac{g'^2}{8e^2}, \quad \lambda_3=\frac{g''^2}{8e^2} \, , \tag{2.8}$$

$\lambda_1,\lambda_2,\lambda_3$ are dimensionless parameters.

The corresponding equations of motion are

$$D_\mu D_\mu\boldsymbol{\phi}=\frac{\delta V}{\delta\boldsymbol{\phi}} \, , \tag{2.9a}$$

$$D_\mu D_\mu\boldsymbol{\psi}=\frac{\delta V}{\delta\boldsymbol{\psi}} \, , \tag{2.9b}$$

$$D_\mu \mathbf{F}_{\mu\nu} = \frac{\eta'^2}{\eta^2} D_\nu \boldsymbol{\psi} \wedge \boldsymbol{\psi} + D_\nu \boldsymbol{\phi} \wedge \boldsymbol{\phi} \ . \tag{2.9c}$$

Finite action requires the following behavior at infinity for the fields $\mathbf{A}_\mu$, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$:

$$\mathbf{F}_{\mu\nu} \underset{\rho\to\infty}{\longrightarrow} 0 \ , \tag{2.10a}$$

$$D_\mu \boldsymbol{\phi} \underset{\rho\to\infty}{\longrightarrow} 0 \ , \tag{2.10b}$$

$$D_\mu \boldsymbol{\psi} \underset{\rho\to\infty}{\longrightarrow} 0 \ , \tag{2.10c}$$

with $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ also satisfying

$$V(\boldsymbol{\phi},\boldsymbol{\psi}) \underset{\rho\to\infty}{\longrightarrow} 0 \ . \tag{2.11}$$

Minimal action SU(2) vortex configurations were found by taking one of the Higgs fields as a constant one everywhere,[15] i.e., playing no dynamical role and not contributing to the total action but ensuring complete symmetry breaking. It seems reasonable to expect that any other configuration with a nonconstant $\boldsymbol{\psi}$ should lead a greater action.

Concerning Eq. (2.10a) it implies that $\mathbf{A}_\mu$ is a pure gauge at infinity: i.e.,

$$\lim_{\rho\to\infty} \mathbf{A}_\mu = \frac{1}{i} g^{-1} \partial_\mu g \ . \tag{2.12}$$

The SU(2) vortex solution was obtained considering g in the form

$$g = e^{i\sigma_3 \Omega} \ . \tag{2.13}$$

Note then that (2.1) was achieved just by considering g in the Cartan subgroup.

B. The Bogomol'nyi equations

Inspired by the preceding discussion on the behavior of the vortex configuration we shall look for Bogomol'nyi equations under the following conditions (valid everywhere in space) on the Higgs fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$:

$$D_\mu \boldsymbol{\psi} = 0 \ , \tag{2.14a}$$

$$\boldsymbol{\psi}^2 = 1 \ , \tag{2.14b}$$

$$\boldsymbol{\phi} \cdot \boldsymbol{\psi} = 0 \ . \tag{2.14c}$$

We explained above that, at infinity, the gauge field has to be a pure gauge [see Eq. (2.12)] with g satisfying Eq. (2.1). In order to have topologically nontrivial configurations, condition (2.1) can be achieved just by taking g in the Cartain subgroup of the gauge group. In the present SU(2) case, this corresponds to

$$g = e^{i\sigma_3 \Omega(\theta)} \tag{2.15}$$

and

$$\Omega(2\pi) - \Omega(0) = (2k+n)\pi, \quad k \in Z, \ n = 0,1 \ . \tag{2.16}$$

More general configurations in the same topological sector can be obtained performing regular gauge rotations of $\mathbf{A}_\mu$.

Using conditions (2.14) we can rewrite the action in the following way:

$$S = e^2\eta^4 \int d^2x \left[\tfrac{1}{4}[\mathbf{F}_{\mu\nu} + a\epsilon_{\mu\nu}(\boldsymbol{\phi}^2 - 1)\hat{\boldsymbol{\psi}}]^2 + \frac{1}{2(1+b^2)}(D_\mu\boldsymbol{\phi} + b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}} \wedge D_\nu \boldsymbol{\phi})^2 \right.$$
$$\left. + (\lambda_1 - \tfrac{1}{2}a^2)(\boldsymbol{\phi}^2-1)^2 - \tfrac{1}{2}a\epsilon_{\mu\nu}\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}}(\boldsymbol{\phi}^2-1) - \frac{b}{1+b^2}\epsilon_{\mu\nu}D_\mu\boldsymbol{\phi}\cdot(\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi}) + \frac{b^2}{2(1+b^2)}(\hat{\boldsymbol{\psi}}\cdot D_\nu\boldsymbol{\phi})^2 \right] \ . \tag{2.17}$$

Here a and b are two *a priori* arbitrary parameters which will be conveniently fixed in what follows. Using conditions (2.14a) and (2.14c) one can see that the last term in Eq. (2.17) vanishes:

$$\hat{\boldsymbol{\psi}} \cdot D_\nu \boldsymbol{\phi} = 0 \ . \tag{2.18}$$

In the Appendix we demonstrate that under conditions (2.14) and working in any fixed gauge such that $\epsilon_{\mu\nu}\mathbf{A}_\mu \wedge \mathbf{A}_\nu = 0$ the action can be written as

$$S = e^2\eta^4 \int d^2x \left[\frac{1}{4}\left[\mathbf{F}_{\mu\nu} - \frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}}\right]^2 + \frac{1}{2(1+b^2)}(D_\mu\boldsymbol{\phi} + b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi})^2 + \left[\lambda_1 - \frac{b^2}{2(1+b^2)^2}\right](\boldsymbol{\phi}^2-1)^2 + \partial_\mu\left[-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\hat{\boldsymbol{\psi}}\cdot\mathbf{A}_\nu)\right]\right] \ . \tag{2.19}$$

In obtaining (2.19) we have related one parameter to the other:

$$a = -\frac{b}{1+b^2} \ . \tag{2.20}$$

The last term in Eq. (2.19) can be transformed into an integral over the border, a large circle surrounding $\mathbb{R}^2$:

$$\int \epsilon_{\mu\nu}\partial_\mu(\hat{\boldsymbol{\psi}}\cdot\mathbf{A}_\nu)d^2x = \oint (\hat{\boldsymbol{\psi}}\cdot\mathbf{A}_\theta)d\theta \ . \tag{2.21}$$

From Eqs. (2.12) and (2.13) $\mathbf{A}_\theta$ over the border takes the form

$$A_\theta^1 = 0, \quad A_\theta^2 = 0, \quad A_\theta^3 = \frac{d}{d\theta}\Omega^3(\theta) \tag{2.22}$$

and then

$$\hat{\boldsymbol{\psi}}\cdot\mathbf{A}_\theta = \hat{\psi}^3 A_\theta^3 \ . \tag{2.23}$$

On the other hand, condition (2.14a) at infinity reads

$$\frac{d}{d\theta}\hat{\boldsymbol{\psi}} + \mathbf{A}_\theta \wedge \hat{\boldsymbol{\psi}} = 0 \ ; \tag{2.24}$$

i.e.,

$$\frac{d\psi^3}{d\theta}=0 \ . \tag{2.25}$$

With this

$$\int \epsilon_{\mu\nu}\partial_\mu(\hat{\boldsymbol{\psi}}\cdot \mathbf{A}_\nu)d^2x=\psi^3_\infty \oint A^3_\theta d\theta \ . \tag{2.26}$$

From Eqs. (2.22) and (2.16),

$$\int \epsilon_{\mu\nu}\partial_\mu(\hat{\boldsymbol{\psi}}\cdot \mathbf{A}_\nu)d^2x=(2k+n)\pi\psi^3_\infty \ . \tag{2.27}$$

Finally, using Eq. (2.27) the action reads

$$S=e^2\eta^4\left\{-\frac{b}{1+b^2}\pi(2k+n)\psi^3_\infty+\int d^2x\left[\frac{1}{4}\left[\mathbf{F}_{\mu\nu}-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}}\right]^2+\frac{1}{2(1+b^2)}(D_\mu\boldsymbol{\phi}+b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi})^2 + \left[\lambda_1-\frac{1}{2}\frac{b^2}{(1+b^2)^2}\right](\boldsymbol{\phi}^2-1)^2\right]\right\} . \tag{2.28}$$

Up to now, the parameter b has not been fixed. We shall choose it as

$$|b|=1 \quad \text{if } \lambda_1\geq\tfrac{1}{8} \ , \qquad \frac{1}{2}\frac{b^2}{(1+b^2)^2}=\lambda_1 \quad \text{if } \lambda_1\leq\tfrac{1}{8} \ , \tag{2.29}$$

so that the last term in (2.28) vanishes.

According to Eqs. (2.28) and (2.29) the action is bounded:

$$S\geq\sqrt{2\lambda_1}\pi e^2\eta^4|(2k+n)\psi^3_\infty| \quad \text{if } \lambda_1\leq\tfrac{1}{8} \tag{2.30a}$$

and

$$S\geq\tfrac{1}{2}\pi e^2\eta^4|(2k+n)\psi^3_\infty| \quad \text{if } \lambda_1\geq\tfrac{1}{8} \ . \tag{2.30b}$$

The equality holds whenever

$$\mathbf{F}_{\mu\nu}-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}}=0 \ , \tag{2.31}$$

$$D_\mu\boldsymbol{\phi}+b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi}=0 \ . \tag{2.32}$$

The compatibility condition for this system fixes b:

$$b=\pm 1 \ . \tag{2.33}$$

These conditions on b can also be obtained if we demand that the solutions to Eqs. (2.14a), (2.31), and (2.32) satisfy the second-order Lagrangian equations of motion. In fact, taking the covariant derivative of Eq. (2.31) and using Eq. (2.32) we obtain

$$D_\mu\mathbf{F}_{\mu\nu}+\frac{2b^2}{1+b^2}D_\nu\boldsymbol{\phi}\wedge\boldsymbol{\phi}=0 \ , \tag{2.34}$$

which coincides with (2.9a) if $b=\pm 1$.

Summarizing, we have found that for $\lambda_1=\frac{1}{8}$ a bound for the action is attained when the three following equations hold:

$$\mathbf{F}_{\mu\nu}\mp\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}(\boldsymbol{\phi}^2-1)=0 \ , \tag{2.35}$$

$$D_\mu\boldsymbol{\phi}\pm\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi}=0 \ , \tag{2.36}$$

$$D_\mu\boldsymbol{\psi}=0 \ , \tag{2.37}$$

and

$$S=\tfrac{1}{2}\pi e^2\eta^4|(2k+n)\psi^3_\infty| \ , \tag{2.38}$$

where $n=0,1$, $k\in Z$.

Equations (2.35)–(2.37) are the Bogomol'nyi equations for SU(2) vortices and, as in the Abelian case, they are valid for a fixed value of the ϕ^4 coupling constant λ_1.

As stated in the Introduction, the action of a non-Abelian Bogomol'nyi vortex is not a topologically invariant quantity as it is in the Abelian case. The presence of k in Eq. (2.38) is a manifestation of topological noninvariance since one obtains different values for the action for different elements belonging to the same class. This result should not be surprising. In fact, two solutions belonging to the same class are gauge equivalent at infinity. But when these two solutions correspond to different values of k the corresponding gauge transformation cannot be well-defined everywhere in space. Then they are not gauge equivalent everywhere and their value for the action should differ.

For the class labeled by $n=0$ the most stable solution is that one associated with $k=0$, i.e., the vacuum. For the class labeled by $n=1$ the most stable solutions are those associated with $k=-1$ and 0.

It is interesting to note that if one interchanges roles between $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ instead of Eqs. (2.30) one gets

$$S\geq\sqrt{2\lambda_2}\pi e^2\eta^2\eta'^2|(2k+n)\phi^3_\infty| \quad \text{if } \lambda_2\leq\tfrac{1}{8} \ , \qquad S\geq\tfrac{1}{2}\pi e^2\eta^2\eta'^2|(2k+n)\phi^3_\infty| \quad \text{if } \lambda_2>2\tfrac{1}{8} \ . \tag{2.39}$$

The bound is now attained for $\lambda_2=\frac{1}{8}$ and Bogomol'nyi equations are

$$\mathbf{F}_{\mu\nu}\mp\epsilon_{\mu\nu}\hat{\boldsymbol{\phi}}(\boldsymbol{\psi}^2-1)=0 \ , \tag{2.40}$$

$$D_\mu\boldsymbol{\psi}\pm\epsilon_{\mu\nu}\hat{\boldsymbol{\phi}}\wedge D_\nu\boldsymbol{\psi}=0 \ , \tag{2.41}$$

$$D_\mu\boldsymbol{\phi}=0 \ . \tag{2.42}$$

At this point one may wonder if there exist solutions to Bogomol'nyi equations (2.35)–(2.37). Two *Ansätze* solving the equations of motion (2.9) for any value of the parameters $\lambda_1,\lambda_2,\lambda_3$ (and hence for the particular values for which Bogomol'nyi equations hold) are known.[15,16] Those solutions, found in Ref. 15, correspond to conditions (2.14) and also solve the Bogomol'nyi equations (2.35)–(2.37). Those solutions presented in Ref. 16 do not satisfy conditions (2.14). In our approach they correspond to the conditions (valid everywhere in space)

$$D_\mu(\boldsymbol{\phi}\wedge\boldsymbol{\psi})=0 \ , \tag{2.43a}$$

$$\phi^2=\psi^2 \ . \tag{2.43b}$$

One can find a new bound corresponding to these conditions and the associated Bogomol'nyi equations:

$$S\geq\left[\frac{1}{2}+\frac{1}{2}\frac{\eta'^2}{\eta^2}\right]e^2\eta^4\left|\frac{\pi}{2}(2k+n)(\hat{\phi}\wedge\hat{\psi})^3_\infty\right| \ , \tag{2.44}$$

$$\mathbf{F}_{\mu\nu}-\tfrac{1}{2}\epsilon_{\mu\nu}(\phi^2-1)\left[1+\frac{\eta'^2}{\eta^2}\right]\hat{\phi}\wedge\hat{\psi}=0 \ , \tag{2.45}$$

$$D_\mu\phi\pm\epsilon_{\mu\nu}(\hat{\phi}\wedge\hat{\psi})\wedge D_\nu\phi=0 \ , \tag{2.46}$$

$$D_\mu\psi\pm\epsilon_{\mu\nu}(\hat{\phi}\wedge\hat{\psi})\wedge D_\nu\psi=0 \ , \tag{2.47}$$

with

$$\lambda_1+\lambda_2\frac{\eta'^4}{\eta^4}=\frac{1}{8}\left[1+\frac{\eta'^2}{\eta^2}\right]^2 \ . \tag{2.48}$$

Note that the Bogomol'nyi equations and the bound for the action have been obtained using the asymptotic behavior for the gauge field given by Eq. (2.22). Although more general types of behavior can be accepted (with g taking values in G and not only in the Cartan subgroup) we were not able to obtain a bound for the action in the general case (the same difficulty arises in Bogomol'nyi analysis for monopoles). Of course, Bogomol'nyi equations (2.35)–(2.37) are valid for a general asymptotic behavior; i.e., any solution to these equations is a solution to the equations of motion as can be trivially checked.

III. THE SU(N) CASE

In this section we shall briefly describe the generalization of the analysis developed in Sec. II to the SU(N) case.

The SU(N) action reads

$$S=\int d^2x\left[\tfrac{1}{4}\,\mathrm{tr}F^2_{\mu\nu}+\tfrac{1}{2}\sum_{A=1}^{N-1}\mathrm{tr}(D_\mu\psi^A)^2+\tfrac{1}{2}\,\mathrm{tr}(D_\mu\phi)^2+V(\psi^A,\phi)\right] \ , \tag{3.1}$$

where ψ^A ($A=1,\dots,N-1$) and ϕ are N scalar fields taking values in the adjoint representation of SU(N). (As we stated in Sec. I at least N Higgs fields are necessary to have maximum symmetry breaking.)

The Higgs potential has been chosen in the form

$$V(\phi,\psi_A)=\sum_{A=1}^{N-1}\lambda_A\,\mathrm{tr}(\psi_A^2-\eta'^2_A)^2+\lambda\,\mathrm{tr}(\phi^2-\eta^2)^2+\sum_{A=1}^{N-1}\beta_A\,\mathrm{tr}(\psi_A\phi)^2+\sum_{\substack{A,B=1\\A\neq B}}^{N-1}\gamma_{AB}\,\mathrm{tr}(\psi_A\psi_B)^2 \ . \tag{3.2}$$

In order to have finite action solutions the fields must satisfy at infinity adequate conditions, analogous to those discussed for the SU(2) case [Eqs. (2.10)].

The SU(N) generalization of conditions (2.14) is

$$D_\mu\psi^A=0 \ , \tag{3.3a}$$

$$\mathrm{tr}\psi^A\psi^B=0 \ , \tag{3.3b}$$

$$\mathrm{tr}\psi^A\phi=0 \ , \tag{3.3c}$$

$A,B=1,\dots,N$. Defining

$$M=\sum_{A=1}^{N-1}a_A\psi_A \tag{3.4}$$

$$\mathrm{tr}M^2=1 \tag{3.5}$$

and proceeding in the same way as we did in Sec. II, one can rewrite the action (3.1) in the form

$$S=e^2\eta^4\int d^2x\left[\tfrac{1}{4}\,\mathrm{tr}\left[F_{\mu\nu}-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\phi^2-1)M\right]^2+\frac{1}{4}\frac{1}{1+b^2}\,\mathrm{tr}(D_\mu\phi+b\epsilon_{\mu\nu}[M,D_\nu\phi])^2+\left[\lambda_1-\frac{b^2}{2(1+b^2)^2}\right]\mathrm{tr}(\phi^2-1)^2\right]+\left[\frac{-b}{1+b^2}\right]e^2\eta^4\oint_{\rho\to\infty}\mathrm{tr}(MA_\nu)dx^\nu \ . \tag{3.6}$$

The last term in (3.6) is

$$\oint_{\rho\to\infty}\mathrm{tr}(MA_\nu)dx^\nu=\mathrm{tr}M\sum_{A=1}^{N-1}[\Omega_A(2\pi)-\Omega_A(0)]H_A \ . \tag{3.7}$$

Now, condition (2.1) implies

$$\sum_{A=1}^{N-1}[\Omega_A(2\pi)-\Omega_A(0)]H_A-\frac{2\pi n}{N}=2\pi K \ , \tag{3.8}$$

where K is an $N\times N$ diagonal matrix, $k_{ij}=m_i\delta_{ij}$ with $m_i\in Z$ (m_i corresponds to the magnetic weights introduced by Goddard, Nuyts, and Olive[18]). It can be easily seen that $\mathrm{tr}K=-n$. Then Eq. (3.7) takes the form

$$\mathrm{tr}M\oint A_\nu dx^\nu=2\pi\,\mathrm{tr}(MK) \ . \tag{3.9}$$

Choosing the same values for b as in the SU(2) case, see Eq. (2.29), we obtain the bound for the action:

$$S\geq\sqrt{2\lambda_1}\pi e^2\eta^4|\,\mathrm{tr}KM| \quad \text{if } \lambda_1\leq\tfrac{1}{8} \ , \tag{3.10}$$

$$S\geq\tfrac{1}{2}\pi e^2\eta^4|\,\mathrm{tr}KM| \quad \text{if } \lambda_1\geq\tfrac{1}{8} \ . \tag{3.11}$$

This bound is saturated when the following Bogomol'nyi equations hold (again the parameter b gets fixed to $b=\pm1$):

$$F_{\mu\nu}\mp\epsilon_{\mu\nu}(\phi^2-1)M=0 \ , \tag{3.12}$$

$$D_\mu \phi \pm \epsilon_{\mu\nu}[M, D_\nu \phi]=0 , \tag{3.13}$$

$$D_\mu \psi_A = 0 , \tag{3.14}$$

with

$$\lambda_1 = \tfrac{1}{8} .$$

Note that for different choices of M one has in principle different sets of Bogomol'nyi equations. However, solutions corresponding to different sets are gauge equivalents.

Any solution of system (3.12)–(3.14) satisfies the Euler-Lagrange equations of motion. Moreover, the SU(N) vortex solutions found in Ref. 15 satisfy these Bogomol'nyi equations.

Note added. After this work was finished, we received a paper by Chapline and Grossman[19] where self-dual vortices are also considered in the context of a topological field theory.

APPENDIX

We discuss in this appendix how the action for the SU(2) Higgs model can be written in the form (2.19),

$$S=e^2\eta^4 \int d^2x \left[\frac{1}{4} \left[\mathbf{F}_{\mu\nu} - \frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}} \right]^2 + \frac{1}{2(1+b^2)}(D_\mu \boldsymbol{\phi} + b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}} \wedge D_\nu \boldsymbol{\phi})^2 + \left[\lambda_1 - \frac{b^2}{2(1+b^2)^2} \right] (\boldsymbol{\phi}^2-1)^2 + \partial_\mu \left[-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\hat{\boldsymbol{\psi}}\cdot \mathbf{A}_\nu) \right] \right] , \tag{A1}$$

given the expression for the action (2.17) and using Eq. (2.18):

$$S=e^2\eta^4 \int d^2x \left[\tfrac{1}{4}[\mathbf{F}_{\mu\nu}+a\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}}]^2 + \frac{1}{2(1+b^2)}(D_\mu \boldsymbol{\phi} + b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu \boldsymbol{\phi})^2 + (\lambda_1 - \tfrac{1}{2}a^2)(\boldsymbol{\phi}^2-1)^2 - \tfrac{1}{2}a\epsilon_{\mu\nu}\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}}(\boldsymbol{\phi}^2-1) - \frac{b}{1+b^2}\epsilon_{\mu\nu}D_\mu\boldsymbol{\phi}\cdot(\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi}) \right] . \tag{A2}$$

We first rewrite the last term in (A2) in the form

$$\epsilon_{\mu\nu}D_\mu\boldsymbol{\phi}\cdot(\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi}) = \epsilon_{\mu\nu}\{\hat{\boldsymbol{\psi}}\cdot D_\nu(\boldsymbol{\phi}\wedge D_\mu\boldsymbol{\phi}) + \tfrac{1}{2}[\boldsymbol{\phi}^2\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}} - (\hat{\boldsymbol{\psi}}\cdot\boldsymbol{\phi})(\mathbf{F}_{\mu\nu}\cdot\boldsymbol{\phi})]\} \tag{A3}$$

using the orthogonality between $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$:

$$\epsilon_{\mu\nu}D_\mu\boldsymbol{\phi}\cdot(\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi}) = \epsilon_{\mu\nu}[\hat{\boldsymbol{\psi}}\cdot D_\nu(\boldsymbol{\phi}\wedge D_\mu\boldsymbol{\phi}) + \tfrac{1}{2}\boldsymbol{\phi}^2(\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}})] . \tag{A4}$$

With this S reads

$$S=e^2\eta^4 \int d^2x \left[\tfrac{1}{4}[\mathbf{F}_{\mu\nu}+a\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}(\boldsymbol{\phi}^2-1)]^2 + \frac{1}{2(1+b^2)}(D_\mu\boldsymbol{\phi}+b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi})^2 + (\lambda_1-\tfrac{1}{2}a^2)(\boldsymbol{\phi}^2-1)^2 - \frac{1}{2}\left[a + \frac{b}{1+b^2} \right]\epsilon_{\mu\nu}\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}}\boldsymbol{\phi}^2 - \frac{b}{1+b^2}\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot D_\nu(\boldsymbol{\phi}\wedge D_\mu\boldsymbol{\phi}) + \tfrac{1}{2}a\epsilon_{\mu\nu}\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}} \right] . \tag{A5}$$

Exploiting the arbitrariness in a and b we choose

$$a = \frac{-b}{1+b^2} .$$

Thus,

$$S=e^2\eta^4 \int d^2x \left[\frac{1}{4}\left[\mathbf{F}_{\mu\nu} - \frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}} \right]^2 + \frac{1}{2(1+b^2)}(D_\mu\boldsymbol{\phi}+b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi})^2 + \left[\lambda_1 - \frac{1}{2}\frac{b^2}{(1+b^2)^2} \right](\boldsymbol{\phi}^2-1)^2 - \frac{b}{1+b^2}\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot D_\nu(\boldsymbol{\phi}\wedge D_\mu\boldsymbol{\phi}) - \frac{b}{2(1+b^2)}\epsilon_{\mu\nu}\mathbf{F}_{\mu\nu}\cdot\hat{\boldsymbol{\psi}} \right] . \tag{A6}$$

In addition

$$\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot D_\nu(\boldsymbol{\phi}\wedge D_\mu\boldsymbol{\phi})=\epsilon_{\mu\nu}\partial_\nu[\hat{\boldsymbol{\psi}}\cdot(\boldsymbol{\phi}\wedge D_\mu\boldsymbol{\phi})] \ . \tag{A7}$$

Since this term does not contribute to the action because of the boundary condition imposed on $\boldsymbol{\phi}$ [see Eq. (2.10b)], the action is

$$S=e^2\eta^4\int d^2x\left[\frac{1}{4}\left[\mathbf{F}_{\mu\nu}-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}}\right]^2 +\frac{1}{2(1+b^{2})}(D_\mu\boldsymbol{\phi}+b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi})^2 +\left[\lambda_1-\frac{b^2}{2(1+b^2)^2}\right](\boldsymbol{\phi}^2-1)^2 -\frac{1}{2}\frac{b}{1+b^2}\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot\mathbf{F}_{\mu\nu}\right] \ . \tag{A8}$$

We shall focus on the last term in Eq. (A8):

$$\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot\mathbf{F}_{\mu\nu}=\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot(2D_\mu\mathbf{A}_\nu-\mathbf{A}_\mu\wedge\mathbf{A}_\nu) =2\epsilon_{\mu\nu}\partial_\mu(\hat{\boldsymbol{\psi}}\cdot\mathbf{A}_\nu)-\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\cdot\mathbf{A}_\mu\wedge\mathbf{A}_\nu \ . \tag{A9}$$

At this step, we shall choose a gauge in such a way that condition

$$\epsilon_{\mu\nu}\mathbf{A}_\mu\wedge\mathbf{A}_\nu=0 \tag{A10}$$

holds.

Then,

$$S=e^2\eta^4\int d^2x\left[\frac{1}{4}\left[\mathbf{F}_{\mu\nu}-\frac{b}{1+b^2}\epsilon_{\mu\nu}(\boldsymbol{\phi}^2-1)\hat{\boldsymbol{\psi}}\right]^2 +\frac{1}{2(1+b^2)}(D_\mu\boldsymbol{\phi}+b\epsilon_{\mu\nu}\hat{\boldsymbol{\psi}}\wedge D_\nu\boldsymbol{\phi})^2 +\left[\lambda_1-\frac{b^2}{2(1+b^2)^2}\right](\boldsymbol{\phi}^2-1)^2\right]$$

and the action takes the form (A1).

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