Solvable light-front model of the electromagnetic form factor of the relativistic two-body bound state in 1+1 dimensions

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Within a relativistically correct yet analytically solvable model of light-front quantum mechanics we construct the electromagnetic form factor of the two-body bound state and we study the validity of the static approximation to the full form factor. Upon comparison of full form factors calculated for different values of binding energy we observe an unexpected effect that for very strongly bound states further increase in binding leads to an increase in the size of the bound system. A similar effect is found for another quantum-mechanical model of relativistic dynamics.

I. INTRODUCTION

The goal of this paper is to study the electromagnetic form factor of a relativistic bound state in the framework of the light-front quantum mechanics (LFQM). We shall discuss the validity of the static approximation for calculations of such a form factor.

To this end we devised the simple model[1] based on a scalar field model quantized on the light-front surface in one time and one space dimension. The model is relativistically correct, yet simple enough to provide for analytical results for the form factor in question. In a nonrelativistic limit the model boils down to the two-body Schrödinger equation with a δ-type potential representing a contact interaction. We calculate the full electromagnetic form factor in a rigorous way and then compare our results to those based on a static approximation. It turns out that the static approximation works only for small momentum transfers, and its quality is controlled by a dimensionless parameter $\tilde{\beta}$ that measures a fractional mass defect of the bound state. Plotting the form factor as a function of momentum transfer Q^2 we observe that the form factor becomes more flat as the binding of the system is increased from $\tilde{\beta}=0$, but there exists a critical value of the binding $\tilde{\beta}_{cr}=0.4257$ for which the form factor is the flattest. Increasing the binding beyond $\tilde{\beta}_{cr}$ we observe just the opposite trend and the shape of the form factor for a very strongly bound system resembles that of a weakly bound one. However, if the form factors are plotted as functions of $(Q/M)^2$, then the effect mentioned above is not observed. This raises the question of the size of the bound system that we subsequently address. Finally, we point out that a similar effect exists also for another quantum-mechanical model[2] of the relativistic bound state.

II. THE MODEL

Our model is the light-front version of the Wick-Cutkosky model[3] in one time and one space dimension, and describes a relativistic system of two scalar particles of mass m, interacting via exchange of a heavy scalar boson with mass μ. The underlying interaction Lagrangian is $L=g\phi^2\phi_0$, where ϕ and ϕ_0 are scalar fields with mass m and μ, respectively. In the lowest order of perturbation theory the two-body wave function is given by the Weinberg equation.[4] To obtain a relativistic version of the contact interaction, we perform the limits $\mu\to\infty$, $g\to\infty$, keeping $\lambda\equiv g^2/4\pi\mu^2m^2$ constant. The Weinberg equation takes the form

$$\psi(x_1,x_2)=\frac{1}{\sqrt{x_1x_2}}\frac{1}{M^2-\dfrac{m^2}{x_1x_2}}(-m^2\lambda)\int_0^1 dy_1\int_0^1 dy_2\,\delta(1-y_1-y_2)\frac{1}{\sqrt{y_1y_2}}\psi(y_1,y_2)\ . \tag{1}$$

Here M is the mass of the bound state, x_1 (x_2) is the fraction of the total light-front momentum $P^+=P^0+P^3$ of the system carried by the first (second) constituent. Since the light-front variables x_i,y_i are invariant under the Lorentz boost, the wave function ψ is the Lorentz-invariant object. One has $0<x_i<1$, and $x_1+x_2=1$, and likewise for y_i, thus the solution of Eq. (1) is the function of the single variable $x=x_1-x_2$ only, i.e., $\psi=\psi(x)$, and reads

$$\psi(x)=N\frac{\sqrt{1-x^2}}{a^2+x^2}\ . \tag{2}$$

Here N is the normalization constant and we have defined the dimensionless parameters

$$a^2 \equiv \tilde{\beta}/(1-\tilde{\beta}) \ , \tag{3}$$

where the parameters $\tilde{\beta}$ and β measure the strength of the binding

$$\tilde{\beta} \equiv \beta - \beta^2/4 \ , \tag{4}$$

$$\beta \equiv B/m \ , \tag{5}$$

and B is the mass defect (binding energy) of the system:

$$B = 2m - M \ . \tag{6}$$

The relation between the coupling constant and the mass of the bound state is provided by the eigenvalue of Eq. (1), and is discussed in Ref. 1. Let us note some typical values of the parameter $\tilde{\beta}$. For positronium one has $\tilde{\beta} \approx 0.000\,015$, for a deuteron $\tilde{\beta} \approx 0.001$; for strongly bound nuclei an average binding per nucleon corresponds to $\tilde{\beta} \approx 0.01$. In a naive quark model with the quark mass $m_q = 330$ MeV we obtain $\tilde{\beta} \approx 0.15$ for a nucleon and $\tilde{\beta} \approx 0.96$ for a pion. In Fig. 1 we plot the bound-state wave function $\psi(x)$ of Eq. (2) for few representative values of the parameter $\tilde{\beta}$, ranging from weakly to strongly bound systems. One sees that for the weakly bound system the two-body wave function resembles the δ-type function centered at $x=0$. This indicates that for a weakly bound system each of constituents carries an equal amount of total light-front momentum P^+, irrespective

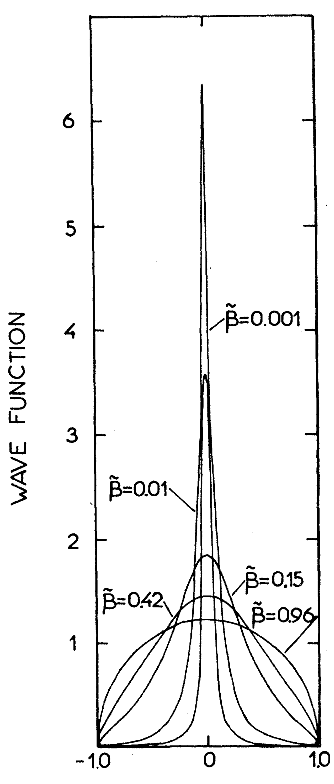

FIG. 1. The bound-state wave function ψ plotted vs the relative light-front momentum fraction x for different values of the binding parameter $\tilde{\beta}$. Larger values of $\tilde{\beta}$ correspond to stronger binding, cf. Eqs. (4)–(6).

of the frame of reference.

Our results for the wave function could be recast into a more traditional form. To this end we introduce the relativistic relative momentum p as the new variable, defining

$$x_{1,2} \equiv \tfrac{1}{2}[1 \pm p/\epsilon(p)], \quad -\infty \le p \le +\infty \ , \tag{7}$$

where

$$\epsilon(p) = (m^2 + p^2)^{1/2} \ . \tag{8}$$

Equation (1) could now be rewritten as

$$(p^2/m + m\tilde{\beta})\phi(p) = \frac{\lambda}{2}\int_{-\infty}^{+\infty} dp' \frac{m}{\epsilon(p')}\phi(p') \ , \tag{9}$$

where the new wave function $\phi(p) = (1-x^2)\psi(x)$ reads explicitly as

$$\phi(p) = N\frac{1}{1+a^2}\frac{1}{p^2 + m^2a^2/(1+a^2)} \ . \tag{10}$$

The structure of Eq. (9) strongly resembles that of the Lippmann-Schwinger equation

$$(p^2/m + B)\phi_{\mathrm{LS}}(p) = \frac{\lambda}{2}\int_{-\infty}^{+\infty} dp' \phi_{\mathrm{LS}}(p') \ , \tag{11}$$

describing a one-dimensional, nonrelativistic system of two particles of mass m bound by a contact potential, that in the position space has the form $V(z) = -\pi\lambda\delta(z)$, with z being the relative distance between both particles. For a weakly bound state (i.e., for $\tilde{\beta} \approx \beta \ll 1$) the kernels of Eqs. (9) and (11) coincide in the dominant low-momentum region (i.e., for $p'/m \ll 1$), and the solutions of both equations become identical.

III. FORM FACTORS

Suppose that the composite system with momentum P^ν absorbs a virtual photon and remains intact. Assume that only the constituent 1 is charged and couples to the photon, acquiring full the photon's momentum $q^\nu = (q^0, q^3)$, and yet both constituents are found in a final state representing the composite system with the overall momentum P'^ν. The probability of such a sequence is measured by the elastic electromagnetic form factor $F(Q^2)$, where

$$q^2 = (q^0)^2 - (q^3)^2 \equiv -Q^2 \tag{12}$$

so that $Q^2 > 0$.

In quantum mechanics one usually works within a static approximation that neglects the motion of the target and calculates the form factor as the Fourier transform of the static charge-density distribution in the target. In this case $Q = q^3$, and one has

$$F_{\mathrm{st}}(Q^2) = \int_{-\infty}^{+\infty} dz_1 e^{iQz_1}\phi^*(z_1)\phi(z_1) \ . \tag{13}$$

Here z_1 is the distance of the first particle from the center of mass of the target. The wave function ϕ is taken as the Fourier transform of the exact wave function given by Eq. (10). Keeping in mind that $z = 2z_1$, where z is the relative distance appearing as the argument of the Fourier transform, one easily obtains

$$F_{\rm st}(Q^2)=\frac{1}{1+Q^2/16m^2\tilde{\beta}} \ . \tag{14}$$

This is the static form factor based on the exact relativistic wave function of Eq. (10). If we took the nonrelativistic wave function $\phi_{\rm LS}$ of Eq. (11) instead, the only change would be the replacement $\tilde{\beta}\to\beta$ in Eq. (14). For weakly bound systems $\tilde{\beta}\approx\beta\approx 0$ and both results coincide. On the other hand, for an extreme binding one has $\tilde{\beta}\approx 1$, whereas $\beta\approx 2$, and the form factor based on $\phi_{\rm LS}$ takes values which for large Q could be two times larger than those given by Eq. (14). Since already the static form factor of Eq. (14) will be found overshooting systematically the true form factor (defined below), we hereafter disregard the nonrelativistic wave function in our discussion.

In actual scattering processes the deuteron does not remain at rest, but for large momentum transfers, suffers a huge recoil. This is rigorously accounted for upon constructing the electromagnetic current for the constituent field ϕ,

$$j^\nu=i:\phi(\partial^\nu\phi)-(\partial^\nu\phi)\phi: \ , \tag{15}$$

and defining the form factor by the relation

$$\langle\Psi_{P'^+}|j^\nu(0)|\Psi_{P^+}\rangle=\frac{1}{4\pi}(P+P')^\nu F(Q^2) \ , \tag{16}$$

where $P'^\nu=P^\nu+q^\nu$, and the state vectors of the initial and final deuterons are given as the two-body sectors of the light-front Fock space involving the wave function ψ. We extract the form factor upon calculating the longitudinal (i.e., $\nu=+$) component of Eq. (16), arriving at the following expression for the form factor:

$$F(\alpha)=\frac{1}{2+\alpha}\int_{-\infty}^{+\infty}dx\frac{1+x+\alpha}{\sqrt{(1+x)(1+x+2\alpha)}}\psi(x)\psi(y) \ , \tag{17}$$

where $y=(x+\alpha)/(1+\alpha)$, and the key parameter $\alpha\equiv q^+/P^+$ that measures the longitudinal-momentum transfer is given by

$$\alpha=\tfrac{1}{2}[Q^2/M^2+\sqrt{(Q^2/M^2)^2+4(Q^2/M^2)}] \ . \tag{18}$$

The integral in Eq. (17) is analytically calculable and full but a rather lengthy result is given in the Appendix of Ref. 1. Here we will discuss the characteristic features of the form factor $F(\alpha)$ for various strengths of the binding.

A. Weakly bound systems ($\tilde{\beta}\approx\beta\ll 1$)

For small values of the momentum transfer we could expand the integrand of Eq. (17) into the power series of the parameter α. The linear terms cancel exactly and the result reads

$$\begin{aligned}F(\alpha)&=1-\frac{\alpha^2}{4\tilde{\beta}}+O(\alpha^4)\\&=1-\frac{Q^2}{16m^2\tilde{\beta}}+O(Q^4/M^4) \ .\end{aligned} \tag{19}$$

This result is identical with the static result, cf. Eq. (14). Likewise, for large values of momentum transfer one expands the integrand into the power series of $1/\alpha$, obtaining

$$\begin{aligned}F(\alpha)&=4\tilde{\beta}/\alpha+O(1/\alpha^2)\\&=16m^2\tilde{\beta}/Q^2+O(M^4/Q^4) \ ;\end{aligned} \tag{20}$$

i.e., one recovers again the static result. However, there is an unexpected disagreement in the intermediate momentum-transfer region—this is visualized in Fig. 2(a), where the full form factor for the weakly bound system with $\tilde{\beta}=0.001$ (deuteronlike binding) is compared with its static counterpart. The static approximation works for $Q/m\le 1$, whereas for larger values of momentum transfer full form factor bends down and then only very slowly approaches its asymptotic (and static) limit, cf. Fig. 2(b). This unexpected bending of the form factor

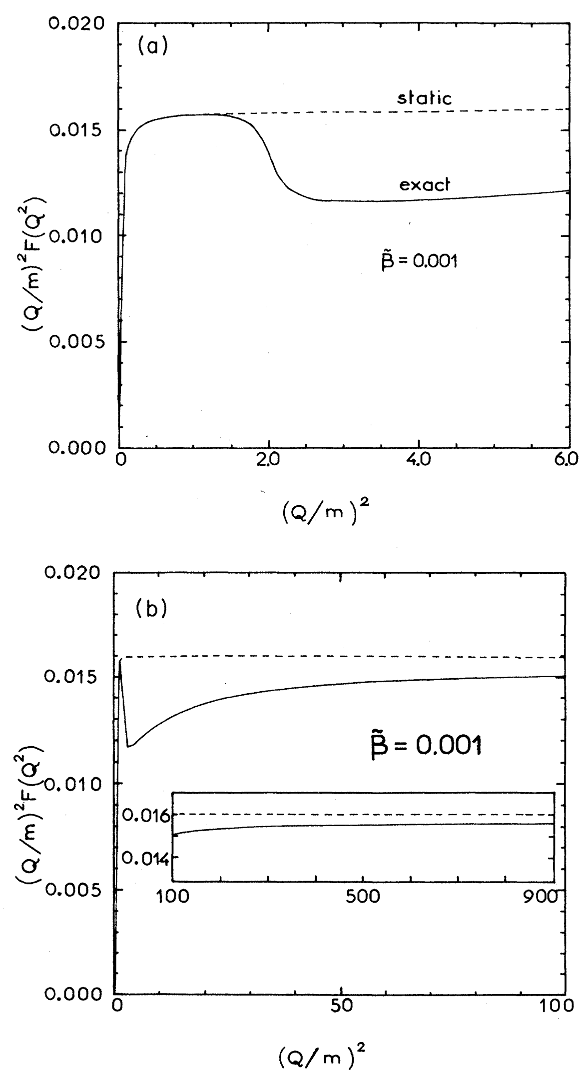

FIG. 2. (a) The exact electromagnetic form factor $(Q/m)^2F(Q^2)$ (solid line) and its static approximation (dashed line) plotted vs $(Q/m)^2$ for weakly bound system with $\tilde{\beta}=0.001$ (deuteronlike system). (b) The same as in (a), but for different scale of momentum transfers.

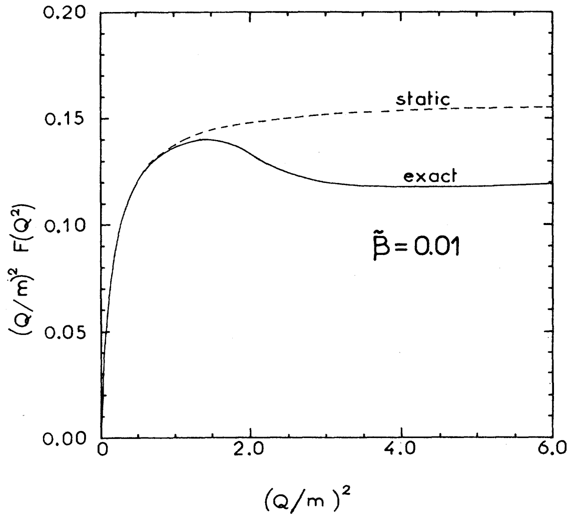

FIG. 3. The same as Fig. 2(a), but for $\tilde{\beta}=0.01$ (nuclear binding).

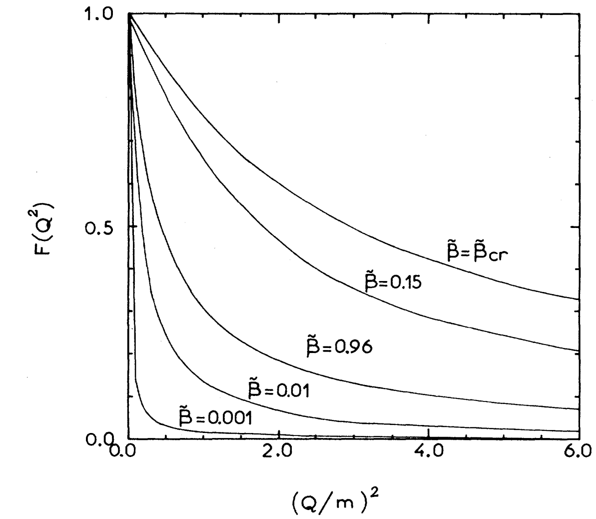

FIG. 5. Exact form factors for different values of binding parameter $\tilde{\beta}$, plotted vs $(Q/m)^2$.

is a characteristic of a weakly bound system, as it could be confirmed by Fig. 3, where similar behavior is found for $\tilde{\beta}=0.01$ (nuclear binding). This effect could be traced back to the particular, needle-shaped profile that the wave function ψ exhibits for weakly bound systems, cf. Fig. 1.

B. Intermediate binding ($0.1 \leq \tilde{\beta} \leq 0.4$)

As a representative binding for this class of systems we take $\tilde{\beta}=0.15$, corresponding to a three-quark picture of a nucleon. The results are presented in Fig. 4. For this strength of the binding the wave function ψ has already rather smooth a behavior as a function of the variable x, cf. Fig. 1, and the bending of the form factor disappears. The static approximation works only for $(Q/m)^2 \leq 0.5$, and there is a substantial difference between asymptotic values of full and static form factor.

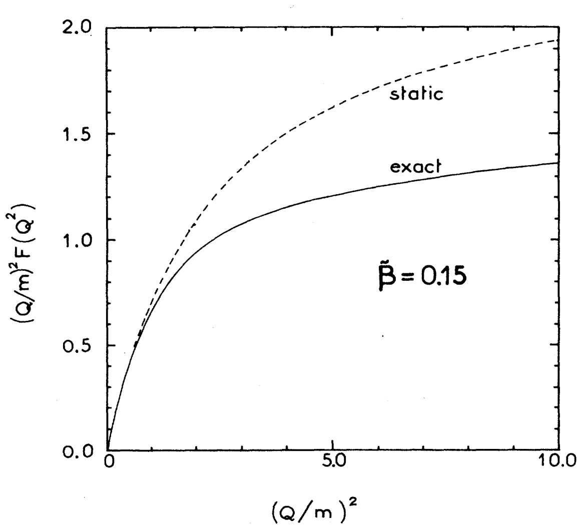

FIG. 4. The same as Fig. 2(a), but for $\tilde{\beta}=0.15$ (quark binding in nucleon).

C. Strong binding ($\tilde{\beta} \geq 0.4$)

Despite an unexpected bending of form factors for weakly bound systems, a clear trend could be observed upon comparison of Figs. 2–4. As the binding is increased, the full form factor becomes more flat, in agreement with an intuitive picture, where stronger binding results in shrinking of a system, thus leading to a flattening of a form factor.

However, when the strength of the binding is further increased and the parameter $\tilde{\beta}$ reaches the value $\tilde{\beta}_{cr}=0.4257$, we observe the onset of another unexpected behavior of our bound system—namely, the form factor starts to fall off more rapidly with Q/m, so in effect for

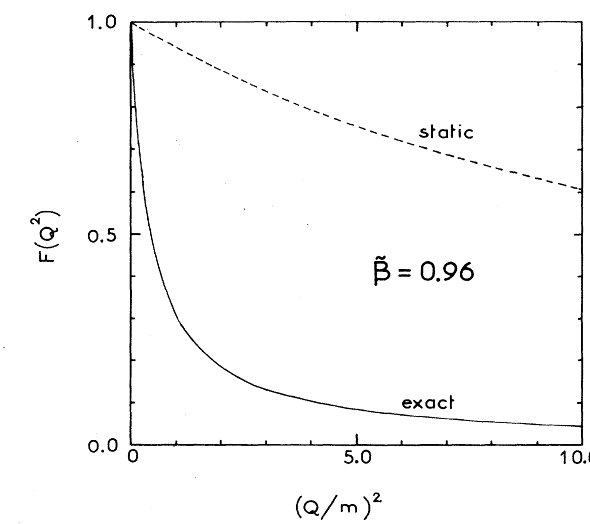

FIG. 6. Exact electromagnetic form factor $F(Q^2)$ (solid line) and its static approximation (dashed line) plotted vs (Q/m^2) for extremely strong binding with $\tilde{\beta}=0.96$ (quark binding in pion).

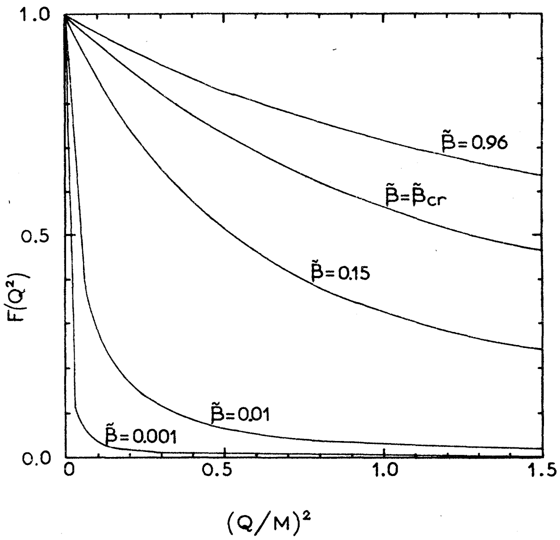

FIG. 7. The same as Fig. 5, but plotted vs $(Q/M)^2$, where M is the mass of the bound system.

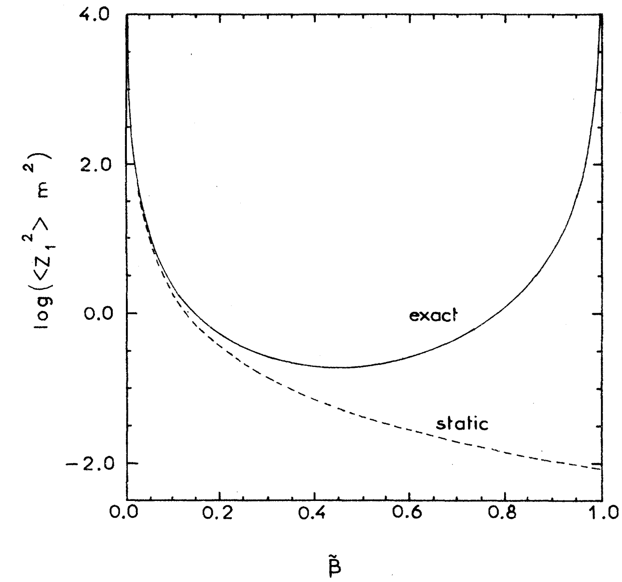

FIG. 8. The bound-state size $\langle z_1^2 \rangle$, in units of $1/m^2$, for different values of the binding parameter $\tilde{\beta}$. The solid line is the full result, Eq. (25), and the dashed line is the static result, Eq. (23).

large values of β (extreme binding) the shape of the form factor resembles that of weakly bound systems. This is visualized in Fig. 5, where the form factors for different strengths of the binding are drawn as functions of $(Q/m)^2$. The curve representing the extreme case of binding with $\tilde{\beta}=0.96$ (quark-antiquark binding in pion) lies somewhere between those for $\tilde{\beta}=0.01$ and $\tilde{\beta}=0.15$. Needless to say, the static approximation fails completely for such a strong binding, as is clear from Fig. 6.

Let us point out that the surprising effect presented in Fig. 5 could be hidden by an alternative presentation of our results. To this end we plot in Fig. 7 the same form factors rather as functions of $(Q/M)^2$; i.e., we adjust the unit of momentum transfer for each curve separately. The effect is gone, and the larger the binding, the flatter the form factor.

This raises the question—what is a size of the bound system under consideration? Conventionally the size could be defined by an average square of the distance of the particle 1 from the center of mass. In our one-dimensional model we write

$$\langle z_1^2 \rangle_{\rm st} = \int_{-\infty}^{+\infty} dz_1 \phi^*(z_1) z_1^2 \phi(z_1) \; . \tag{21}$$

Combining Eq. (21) with Eq. (13) we obtain

$$\langle z_1^2 \rangle_{\rm st} = -2 \frac{\partial F_{\rm st}(Q^2)}{\partial Q^2} \bigg|_{Q^2=0} \; , \tag{22}$$

and the result is

$$\langle z_1^2 \rangle_{\rm st} = 1/(8m^2 \tilde{\beta}) \; . \tag{23}$$

As the binding increases from $\tilde{\beta}=0$ to $\tilde{\beta}=1$, $\langle z_1^2 \rangle_{\rm st}$ decreases to the finite value $1/8m^2$. But the static form factor does not approximate the full form factor for a strongly bound system, so we better work with the latter and define the size of the system by

$$\langle z_1^2 \rangle = -2 \frac{\partial F(Q^2)}{\partial Q^2} \bigg|_{Q^2=0} \; . \tag{24}$$

Expanding the form factor $F(\alpha)$ given by Eq. (17) into the Taylor series and using Eq. (18) we obtain

$$\langle z_1^2 \rangle = -\frac{1}{M^2} \frac{d^2 F(\alpha)}{d\alpha^2} \bigg|_{\alpha=0} \; . \tag{25}$$

For weakly bound systems the results (25) and (23) coincide and it is the second term on the right-hand side of Eq. (25) that causes the size to grow infinitely for negligible binding. For a strongly bound system it is the vanishing bound state mass M in the denominator of Eq. (25) that makes the result diverge again. In Fig. 8 we plot the size $\langle z_1^2 \rangle$ of the bound state as defined by Eq. (25) for all values of the binding parameter $\tilde{\beta}$. The minimum at $\tilde{\beta}=\tilde{\beta}_{\rm cr}$ is in agreement with the discussion of Fig. 5. For comparison we show also the static result given by Eq. (23).

IV. SUMMARY

The electromagnetic form factor of a two-body relativistic composite system has been studied within the model of light-front quantum mechanics. The model interaction could be regarded as a relativistic version of a two-body contact interaction. As one would expect, the static approximation to the electromagnetic form factor has been found to work only for weakly bound systems, and only for small momentum transfers, i.e., for $(Q/m) < 1$.

Two genuine relativistic effects have been observed.

(i) For weakly bound systems the full form factor plotted as $(Q/m)^2 F(Q^2)$ exhibits a surprising deviation from its static counterpart at the momentum transfer (Q/m) of the order of 1. As the momentum transfer is further increased, the full form factor only very slowly regains its asymptotic $(Q/m)\rightarrow\infty$ value, where once again it coincides with static result.

(ii) When the binding of the system is increased, the size of the bound state, defined as the derivative of the form factor with respect to the momentum transfer, exhibits unexpected behavior. It turns out that a minimal size is obtained for the value of the mass defect parameter $\tilde{\beta}=0.4257$, i.e., for the mass defect of some 24%. Further increase of the binding leads to an increase in the size of the bound system.

The second property is not a particular feature of our model. Recently Glöckle, Nogami, and Fukui (GNF) constructed a relativistically covariant model[2] of a two-body bound state, based on the two-body Dirac equation with a contact interaction.[5] For weakly bound systems the exact GNF form factor coincides with the results of our work. The static form factors are identical in both models for all values of binding strength. Moreover, the GNF model yields an almost identical effect for the binding dependence of the exact form factor. In fact, in the GNF model the form factors calculated for $\tilde{\beta}$ and $1-\tilde{\beta}$ coincide exactly and the minimal size of the two-body system is obtained for $\tilde{\beta}_{\rm cr}=0.5$.

One may speculate whether such behavior of the form factor could be found also for other $(1+1)$-dimensional models. To shed light on this issue we are currently investigating a $(1+1)$-dimensional model based on the Bethe-Salpeter equation.

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[1]M. Sawicki and L. Mankiewicz, Phys. Rev. D **37**, 421 (1988).

[2]W. Glöckle, Y. Nogami, and I. Fukui, Phys. Rev. D **35**, 584 (1987).

[3]G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, *ibid.* **96**, 1135 (1954).

[4]S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[5]An alternative derivation of the GNF model of Ref. 2 has been recently provided by Y. Munakata, T. Ino, and F. Nagamura, Prog. Theor. Phys. **79**, 1404 (1988), who obtained the GNF model as the Breit version of the Bethe-Salpeter equation. Because of a different treatment of the δ-type potential, their energy eigenvalue differs from that of GNF, but coincides with it for weak binding. The results for the wave function remain unchanged.