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Hidden BRS invariance in classical mechanics. II

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In this paper we develop a path-integral formulation of *classical* Hamiltonian dynamics, that means we give a functional-integral representation of *classical* transition probabilities. This is done by giving weight "one" to the classical paths and weight "zero" to all the others. With the help of anticommuting ghosts this measure can be rewritten as the exponential of a certain action $\widetilde{S}$. Associated with this path integral there is an operatorial formalism that turns out to be an extension of the well-known operatorial approach of Liouville, Koopman, and von Neumann. The new formalism describes the evolution of scalar probability densities and of p-form densities on phase space in a unified framework. In this work we provide an interpretation for the ghost fields as being the well-known Jacobi fields of classical mechanics. With this interpretation the Hamiltonian $\widetilde{H}$, derived from the action $\widetilde{S}$, turns out to be the Lie derivative associated with the Hamiltonian flow. We also find that the action $\widetilde{S}$ presents a set of Becchi-Rouet-Stora- (BRS-)type invariances mixing the original phase-space variables with the ghosts. Together with a Sp(2) symmetry of the pure ghosts sector, they form a *universal* invariance group ISp(2) which is present in any Hamiltonian system. The physical and geometrical meaning of the ISp(2) generators is discussed in detail: in particular the conservation of one of the generators is shown to be equivalent to the Liouville theorem. The ISp(2) algebra is then used to give a modern operatorial reformulation of the old Cartan calculus on symplectic manifolds.

I. INTRODUCTION

Since its formulation almost half a century ago[1] the path-integral approach to quantum mechanics has proved to be one of the most powerful tools for the study and "visualization" of quantum phenomena, both in the perturbative (Feynman diagrams) and nonperturbative (instantons, monopoles, etc.) domain. Virtually all aspects of field theory have been revitalized and refreshed by the path-integral approach. Nevertheless, no attempts have been made (at least to our knowledge) of giving an analogous path-integral formulation of *classical mechanics.* At first sight this might seem to be a strange way of making simple things complicated but, in fact, there are several motivations for developing such a theory. First, a formulation of this sort would help in better understanding the interplay between classical mechanics (CM) and quantum mechanics (QM): being both written in terms of path integrals now, it should be easier to see the differences and the similarities. Second, long ago Koopman[2] and von Neumann,[3] influenced by the invention of quantum mechanics, gave an *operatorial* formulation of CM. Therefore, it is plausible to suspect that this theory (CM) should have a path-integral counterpart. Third, a classical path integral might help in visualizing some phenomena in the classical regime which, even today, are only poorly understood: the example we have in mind is that of deterministic chaos in Hamiltonian systems.[4] Fourth, the operational formulation, equivalent to the classical path integral, turns out to be related to the differential geometry on symplectic manifolds in a way similar to the relation between Witten's supersymmetric quantum mechanics[5] and the differential geometry on Riemannian manifolds. Thus the path integral not only "knows" about the *dynamics,* but also about the *geometry* and the *topology* of phase space. There are also striking similarities between our classical path integral and the topological field theories recently discovered.[6]

A first attempt to provide such a path integral formulation of classical mechanics was made in Ref. 7 and we will refer to it as I. There it was suggested to use a Dirac δ function $\widetilde{\delta}[\phi-\phi_{\rm cl}]$ for the measure which gives weight "one" to the classical paths $\phi_{\rm cl}$ and weight "zero" to all the others, i.e., to those which are not solutions of the equations of motion. In Ref. 7 it also was shown how this measure can be rewritten in the standard form as the exponential of an action $\widetilde{S}$. The action $\widetilde{S}$ involves a set of anticommuting ghosts which could be understood as the Jacobi fields of classical dynamics.[8] The action $\widetilde{S}$ possesses a Becchi-Rouet-Stora (BRS) and anti-BRS symmetry which mixes the ghosts with the original bosonic variables. It is also invariant under a set of ghost-charge and ghost-conjugation-like transformations which, together with the (anti-)BRS operator, form an ISp(2) symmetry group. In Ref. 7 the conservation of one of its generators has been shown to be equivalent to the Liouville theorem, which states that the phase-space volume is invariant under the Hamiltonian flow. Some further analysis of these invariances, in particular of the anti-BRS symmetry, was performed by Kraenkel.[9]

In the present paper we continue the analysis of the classical path integral, in particular, of its Hamiltonian form, by interpreting the remaining ISp(2) generators in terms of more familiar (geometric) objects. For example, the BRS operator will be identified with the exterior derivative on the space of classical orbits, which, in turn, can be identified with phase space. Similarly, from the action $\tilde{S}$ we can derive a "super-Hamiltonian" $\tilde{\mathcal{H}}$ which turns out to be the Lie derivative associated with the Hamiltonian flow. Here lies the real surprise and the power of the path-integral approach: *it naturally generates geometrical objects, such as the exterior derivative or the Lie derivative, which do not have to be introduced in an abstract manner as it is usually done in the standard formulation of* CM. In this way the standard *Cartan calculus* on phase space[10] will be translated into a set of simple *operatorial* rules derivable from the operatorial content of the theory described by our path integral. The crucial elements of the operatorial formalism mentioned above are the "classical commutation relations" which follow from the classical path integral. This formalism naturally embeds the standard operator approach to CM pioneered by Liouville, Koopman[2] and von Neumann.[3] The operator method discussed here goes beyond the standard one in that it not only deals with scalar probability density functions on phase space, but also includes the dynamics of p-form fields on phase space. The appearance of these higher fields is a consequence of the ghosts, and since these are related to the Jacobi fields, the p-forms contain information about the behavior of nearby trajectories, for instance. This is the kind of information which is important for the study of chaotic phenomena.

This paper is organized as follows. In Sec. II we briefly review the configuration-space path integral introduced in Ref. 7. In Sec. III we write down an analogous path integral on phase space and derive its operatorial content. In Sec. IV we discuss the ISp(2)-symmetry algebra and various interesting representations of it. The following two sections are devoted to a detailed interpretation of its generators. In Sec. V we show that the ghosts may be identified with differential forms on phase space and in Sec. VI we further study the differential geometric meaning of the ISp(2) charges.

II. THE CLASSICAL PATH INTEGRAL: LAGRANGE FORMALISM

In this section we briefly review the main points of Ref. 7, where a Lagrangian formulation has been used. We shall be very sketchy here because we only want to convey the general ideas. A more detailed analysis (in a Hamiltonian formulation, however) will be given in the following sections.

In Ref. 7 a proposal was made for a path-integral formulation of classical mechanics along the lines of Feynman's path-integral approach to quantum mechanics. While the quantum generating functional (for simplicity we consider a system with one degree of freedom only and K and K' are normalization constants which we will omit in the rest of the paper)

$$Z_{QM}[J]=K\int\mathcal{D}\phi\,\exp\left[iS[\phi]+i\int_0^T dt\,J(t)\phi(t)\right] \tag{2.1}$$

gives weight $\exp(iS)$ to each path, the classical one could be built as giving weight "one" to the paths classically allowed and weight "zero" to all the others. It is given by

$$Z_{CM}[J]=K'\int\mathcal{D}\phi\,\tilde{\delta}[\phi-\phi_{cl}]\exp\left[\int_0^T dt\,J(t)\phi(t)\right] \tag{2.2}$$

with the measure

$$\mathcal{D}\phi\equiv\prod_{i=0}^{N}d\phi(i)\equiv d\phi(0)\mathcal{D}''\phi\,d\phi(N) \tag{2.3}$$

and the delta functional

$$\tilde{\delta}[\phi-\phi_{cl}]\equiv\prod_{i=1}^{N}\delta(\phi(i)-\phi_{cl}(i))\ , \tag{2.4}$$

where $\phi_{cl}(t)$ is a solution of the classical equation of motion $\delta S/\delta\phi(t)=0$, starting at $\phi(0)$ and ending at $\phi(N)$. Here we have sliced the time interval $[0,T]$ in N intervals of equal length T/N. Since ϕ_{cl} is a solution of the equation of motion, it is possible to rewrite Z_{CM} as

$$Z_{CM}[J]=\int\mathcal{D}\phi\,\tilde{\delta}\left[\frac{\delta S}{\delta\phi}\right]\det\left[\frac{\delta^2 S}{\delta\phi^2}\right]\exp\left[\int_0^T dt\,J\phi\right]\ . \tag{2.5}$$

In principle we should take the absolute value of the determinant, but for the moment we will suppose that it is positive. The delta functional can be represented as

$$\tilde{\delta}\left[\frac{\delta S}{\delta\phi}\right]=\int\mathcal{D}\lambda\,\exp\left[i\int_0^T dt\,\lambda(t)\frac{\delta S}{\delta\phi(t)}\right] \tag{2.6}$$

and the determinant is conveniently reexpressed as a functional integral over two Grassmann variables $c(t)$ and $\bar{c}(t)$:

$$\det\left[\frac{\delta^2 S}{\delta\phi^2}\right]=\int\mathcal{D}c\,\mathcal{D}\bar{c}\exp\left[\int_0^T dt\int_0^T dt'\bar{c}(t)\frac{\delta^2 S}{\delta\phi(t)\delta\phi(t')}c(t')\right]\ . \tag{2.7}$$

Inserting expressions (2.6) and (2.7) into (2.5), we obtain

$$Z_{CM}[J]=\int\mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left[i\tilde{S}+\int dt\,J\phi\right] \tag{2.8}$$

with the action $\tilde{S}$ given by

$$\tilde{S}=\int_0^T dt\,\lambda(t)\frac{\delta S}{\delta\phi(t)}-i\int_0^T dt\int_0^T dt'\bar{c}(t)\frac{\delta^2 S}{\delta\phi(t)\delta\phi(t')}c(t')\ . \tag{2.9}$$

From its construction it is clear that the path integral with weight $\tilde{S}$ is completely equivalent to classical mechanics; there are no quantum fluctuations driving the system away from the classical paths. The remarkable point about the action (2.9) is that it exhibits an *unexpected* BRS- and anti-BRS-type invariance. In fact, $\tilde{S}$ is invariant under the following two sets of transformations (our Hermiticity conventions are such that c and $\bar{c}$ are real, and ϵ, $\bar{\epsilon}$, θ, and $\bar{\theta}$ are purely imaginary Grassmann variables):

$$\delta\phi=\epsilon c, \quad \delta\bar{c}=-i\epsilon\lambda, \quad \delta c=0, \quad \delta\lambda=0 , \tag{2.10}$$

and

$$\delta\phi=-\bar{\epsilon}\bar{c}, \quad \delta c=-i\bar{\epsilon}\lambda, \quad \delta\bar{c}=0, \quad \delta\lambda=0 . \tag{2.11}$$

Here ϵ and $\bar{\epsilon}$ are anticommuting parameters. It is easy to see that both of the above transformations are nilpotent: i.e., $\delta^2=0$. Therefore, we may address them as BRS and anti-BRS transformations, respectively. The presence of these symmetries motivates[11] the use of the superspace formalism. Let us extend t space (the real line) to a superspace $(t,\theta,\bar{\theta})$ [it is easy to see that the transformations (2.10) and (2.11) correspond to translations of θ and $\bar{\theta}$] and let us define the following superfield on this space:

$$\Phi(t,\theta,\bar{\theta})=\phi(t)+i\theta c(t)-i\bar{c}(t)\bar{\theta}+i\bar{\theta}\theta\lambda(t) . \tag{2.12}$$

In terms of Φ the classical path integral assumes a very simple form

$$\mathbf{Z}_{\mathrm{CM}}[0]=\int\mathcal{D}\Phi \exp\left[i\int S[\Phi]d\theta\, d\bar{\theta}\right] . \tag{2.13}$$

Note the striking similarities between this path integral (2.13) and the quantum-mechanical one (2.1). The weight factor has the same structure in both cases. It is constructed from the same function S, with different arguments however: ϕ in QM and Φ in CM. This similarity reveals the *unique role* played by the action at both the classical and quantum level. Another point worth being noticed is the following: if we expand the $S[\Phi]$ appearing in (2.13) in θ and $\bar{\theta}$, we find

$$S[\Phi]=S[\phi]+\theta B_1[\bar{c},\phi]+\bar{\theta}B_2[c,\phi]+\theta\bar{\theta}\tilde{S}[\phi,\lambda,c,\bar{c}] , \tag{2.14}$$

where B_1 and B_2 are certain functionals of c, $\bar{c}$, and ϕ whose precise form is not important here. Note that the lowest component of (2.14) is the *quantum* weight $S[\phi]$ of (2.1), while the highest component is the *classical* weight $\tilde{S}[\phi,\lambda,c,\bar{c}]$ of (2.8). Thus the classical and the quantum weight can be unified in a unique "superaction" $S[\Phi]$. This fact cannot be a mere formal coincidence, it indicates that something *profound* is behind. There exists a kind of "covariance" between the classical and the quantum regime: by some combination of BRS and anti-BRS transformations we can rotate $S[\phi]$ into $\tilde{S}[\phi,\lambda,c,\bar{c}]$, i.e., the classical regime into the quantum regime. The geometrical meaning of this fact will be further explored in Ref. 12. At this point one might ask why one needs more fields at the classical level than at the quantum-mechanical level. There are various possible answers: one is that also at the quantum-mechanical level additional ghost fields could be introduced and they will illuminate the understanding of its geometrical structure (see Ref. 12). Another answer is that at the classical level the ghosts are needed to cut off the propagation "perpendicular" to the classical orbits. This is somehow like Yang-Mills theory, for instance, where the ghosts are needed to cancel the effect of the unphysical modes of the gauge field.

In the above construction the ghosts entered in a purely formal way as a tool to exponentiate the Hessian of the action. It turns out, however, that they play a role much more important than just that of some auxiliary fields. To see this, we consider a simple system with action

$$S=\int dt[\tfrac{1}{2}\dot{\phi}^2-V(\phi)] \tag{2.15}$$

and write down the equations of motion obtained by varying $\tilde{S}$ of Eq. (2.9):

$$\partial_t^2\phi+V'(\phi)=0 , \tag{2.16}$$

$$[\partial_t^2+V''(\phi)]c(t)=0 , \tag{2.17}$$

$$[\partial_t^2+V''(\phi)]\bar{c}(t)=0 , \tag{2.18}$$

$$[\partial_t^2+V''(\phi)]\lambda(t)=i\bar{c}(t)V'''(\phi)c(t) . \tag{2.19}$$

Using the superfield (2.12), these equations can be combined into the "super-Newton" equation:

$$\partial_t^2\Phi=-V'(\Phi) . \tag{2.20}$$

Equation (2.16) is the standard equation of motion of ϕ, which also could be derived from $S[\Phi]$ directly. (Lagrange could have used $\tilde{S}$ instead of S to give us his equations.) The equation for c and $\bar{c}$ is also well known: it is the equation of the first variation $\delta\phi(t)$ of the trajectory $\phi(t)$, which is usually referred to as the *Jacobi field*[8] or the geodesic deviation.[13] If $\phi(t)$ and $\phi(t)+\delta\phi(t)$ are two nearby solutions of Eq. (2.16), then the Jacobi field $\delta\phi(t)$ has to satisfy the equation

$$[\partial_t^2+V''(\phi)]\delta\phi(t)=0 . \tag{2.21}$$

Obviously, this is the same as the ghost and antighost equation of motion. So $\tilde{S}$ gives us not only the standard equation of motion but also the equation of motion of the Jacobi fields and it is for this that it is a much more *useful* and *important* object to use than S. Looking back at Eq. (2.10) we observe that the BRS transformation, when applied to a classical path $\phi(t)$, maps it into another classical path $\phi(t)+\epsilon c(t)$. It thus acts like a "translation" or "derivative" on the space of classical orbits.

Jacobi fields and their correlators $\langle\delta\phi(t)\delta\phi(0)\rangle$, respectively, are extremely important to detect possible chaotic behavior of a dynamical system.[14,15] In particular, if (in realistic applications ϕ is, of course, a multicomponent variable)

$$\lim_{T\to\infty}\frac{1}{T}\ln\langle\delta\phi(T)\delta\phi(0)\rangle\neq 0 \tag{2.22}$$

then the system is a so-called C system,[4,15] and neighboring trajectories fly apart exponentially. Clearly, with regard to such applications, it is useful to have a formalism in which the Jacobi fields appear explicitly at the same logical level as $\phi(t)$ itself, rather than just being derived objects (the "difference" of two ϕ's). Eventually, in fact, we would like to compute correlations such as (2.22) as ghost two-point functions of the path integral (2.8).

Having both ϕ and c in the same formalism means *geometrically* that we are not just working on the manifold of classical solutions $\phi(t)$ but rather on its cotangent bundle. We will see that this gives rise to a much richer structure than usually considered in classical dynamics. In particular, as will be seen in the next section, it leads to form-valued density functions on phase space.

III. THE CLASSICAL PATH INTEGRAL: HAMILTON FORMALISM

In this section we derive the classical path integral for the Hamiltonian formalism of classical mechanics. Because of its first-order character, the relevant integrals can be easily evaluated and the problem of finding the correct boundary conditions is easier to handle than in the Lagrangian formalism. We start from Hamilton's equations

$$\dot{\phi}^a(t)=\omega^{ab}\partial_b H[\phi(t)] \ , \tag{3.1}$$

where $\phi^a \equiv (q^1,\dots,q^n,p_1,\dots,p_n)$, $a=1,\dots,2n$, is a coordinate on a $2n$-dimensional phase space $\mathcal{M}_{2n}$, H is the Hamiltonian, and $\omega^{ab}=-\omega^{ba}$ is the symplectic two-form. (In this paper we make the simplifying assumption that ω^{ab} and its inverse ω_{ab} have their standard ϕ-independent form everywhere. Topologically nontrivial cases will be discussed elsewhere.) The standard Poisson brackets of two observables $A_1(\phi^a)$ and $A_2(\phi^a)$ is given by

$$\{A_1,A_2\}=\partial_a A_1 \omega^{ab}\partial_b A_2 \ . \tag{3.2}$$

Using (3.1) one obtains the usual equation for the time evolution of $A[\phi(t)]$:

$$\frac{d}{dt}A[\phi(t)]=\partial_a A\dot{\phi}^a=\{A,H\} \ . \tag{3.3}$$

Another important concept we shall need is that of a probability density function $\rho(\phi^a,t)$ on phase space. Typical examples include the standard microcanonical, canonical, and grandcanonical ensembles, respectively:

$$\begin{aligned}&\rho_{mc} \propto \delta(H-E) \ ,\\ &\rho_c \propto \exp(-\beta H) \ ,\\ &\rho_{gc} \propto \exp[-\beta(H-\mu N)] \ .\end{aligned} \tag{3.4}$$

Similarly, the distribution indicating the presence of a single particle reads

$$\rho(\phi^a,t)=\delta^{(2n)}(\phi^a-\phi^a_{\rm cl}(t;\phi^a_i)) \ . \tag{3.5}$$

Here and in the following $\phi^a_{\rm cl}(t;\phi^a_i)$ denotes the classical solution of the equation of motion (3.1) with initial condition $\phi^a_i=\phi^a_{\rm cl}(t=t_i,\phi^a_i)$. The time evolution of these distributions is given by

$$\frac{\partial}{\partial t}\rho(\phi_a,t)=-\{\rho,H\}\equiv-\hat{L}\rho(\phi^a,t) \ . \tag{3.6}$$

Here we have introduced the Liouville operator

$$\hat{L}=-\partial_a H\omega^{ab}\partial_b \tag{3.7}$$

which is the central element of the operatorial approach to classical dynamics.[2] Comparing this to the formalism of QM and thinking of ρ as the analogue of the density matrix, $\hat{L}$ is the analogue of the Hamilton operator. Keeping this analogy in mind, we define the *classical average* ("expectation value") for the variable ϕ^a at time t:

$$\begin{aligned}\langle\phi^a\rangle_t&=\int d^{2n}\phi\,\phi^a\rho(\phi,t)\\ &=\int d^{2n}\phi\, d^{2n}\phi_i\phi^a P(\phi,t|\phi_i,t_i)\rho(\phi_i,t_i) \ .\end{aligned} \tag{3.8}$$

In the second line of (3.8) we have expressed the distribution at time t, $\rho(\phi,t)$, in terms of the initial distribution $\rho(\phi_i,t_i)$ specified at some earlier time $t_i\le t$. Furthermore, $P(\phi,t|\phi_i,t_i)$ denotes the classical probability for a particle to be at a point ϕ at time t, if it was at ϕ_i at time t_i. Clearly P is nonzero only if the two points are connected by a classical path:

$$P(\phi,t|\phi_i,t_i)=\delta^{(2n)}(\phi^a-\phi^a_{\rm cl}(t;\phi_i)) \ . \tag{3.9}$$

Again, $\phi^a_{\rm cl}$ is a classical path with appropriate initial condition.

We now turn to the path-integral representation of averages such as (3.8). Starting from the Hamiltonian equations of motion (3.1), we now repeat the steps which led to Eq. (2.8). The delta functional forcing the system on its classical trajectory reads

$$\tilde{\delta}[\phi^a-\phi^a_{\rm cl}]=\tilde{\delta}[\dot{\phi}^a-\omega^{ab}\partial_b H]\det(\partial_t\delta^a_b-\omega^{ac}\partial_c\partial_b H) \ . \tag{3.10}$$

Exponentiating the right-hand side (RHS) of Eq. (3.10) as in Sec. II we arrive at [see the Appendix for details of the discretization procedure used in the definition of the path integral and for the reason of having external currents coupled to the ghosts c, $\bar{c}$, and λ in the "source terms" in Eq. (3.11)]

$$\begin{aligned}Z_{\rm CM}=\int&\mathcal{D}\phi^a\mathcal{D}\lambda_a\mathcal{D}c^a\mathcal{D}\bar{c}_a\\ &\times\exp\left[i\int dt(\tilde{\mathcal{L}}+\text{source terms})\right]\end{aligned} \tag{3.11}$$

with the Lagrangian

$$\begin{aligned}\tilde{\mathcal{L}}=&\lambda_a[\dot{\phi}^a-\omega^{ab}\partial_b H(\phi)]\\ &+i\bar{c}_a[\partial_t\delta^a_b-\omega^{ac}\partial_c\partial_b H(\phi)]c^b \ .\end{aligned} \tag{3.12}$$

We start the discussion of (3.11) by looking at the Euler-Lagrange equations of $\tilde{\mathcal{L}}$. Varying λ_a, $\bar{c}_a$, c^a, and ϕ^a, respectively, we find, from Eq. (3.12),

$$\dot{\phi}^a-\omega^{ab}\partial_b H=0 \ , \tag{3.13}$$

$$(\partial_t\delta^a_b-\omega^{ac}\partial_c\partial_b H)c^b=0 \ , \tag{3.14}$$

$$\bar{c}_a(\overleftarrow{\partial}_t\delta^a_b+\omega^{ac}\partial_c\partial_b H)=0 \ , \tag{3.15}$$

$$(\partial_t\delta^a_b+\omega^{ac}\partial_c\partial_b H)\lambda_a=i\bar{c}_a\omega^{ac}\partial_c\partial_d\partial_b Hc^d \ . \tag{3.16}$$

Again the first equation is the usual equation of motion and the ghost equation of motion coincides with the Jacobi equation. Note that $\tilde{\mathcal{L}}$ contains only first-order derivatives. Therefore, writing

$$\widetilde{\mathcal{L}}=\lambda_a\dot{\phi}^a+i\bar{c}_a\dot{c}^a-\widetilde{\mathcal{H}} \tag{3.17}$$

we can read off a "Hamilton" function

$$\widetilde{\mathcal{H}}=\lambda_a\omega^{ab}\partial_b H+i\bar{c}_a\omega^{ac}\partial_c\partial_b H c^b \, . \tag{3.18}$$

For the interpretation of $\widetilde{\mathcal{H}}$ as a Hamiltonian to be correct, we have to make sure that λ_a and ϕ^a, as well as $\bar{c}_a$ and c^a, are (in a generalized sense) pairs of canonically conjugate variables. To show that this is indeed the case, we now compute the equal-time (anti)commutators of ϕ^a, λ_a, c^a, and $\bar{c}_a$ from the path integral (3.11). We define

$$\langle [A_1(t),A_2(t)]\rangle=\lim_{\epsilon\to 0}\langle A_1(t-\epsilon)A_2(t) \pm A_2(t-\epsilon)A_1(t)\rangle \, , \tag{3.19}$$

where A_1 and A_2 are any function of $\phi^a(t)$, $\lambda_a(t)$, $c^a(t)$, and $\bar{c}_a(t)$. The symbol [ ] denotes the Z_2-graded commutator; if both entries are anticommuting, it means the anticommutator, in all other cases it is the ordinary commutator. Finally $\langle\ \rangle$ denotes the expectation value with respect to (3.11). The RHS of Eq. (3.19) has to be evaluated in a discretized version of the theory. Using standard techniques[16] we find that

$$\langle [\phi^a,\lambda_b]\rangle=i\delta^a_b \, , \tag{3.20}$$

$$\langle [\bar{c}_b,c^a]\rangle=\delta^a_b \tag{3.21}$$

and that all other commutators vanish. In particular, ϕ^a and ϕ^b commute for all values of a and b. In terms of the q's and p's (which were combined into ϕ^a) this means

$$\langle [q^i,p_j]\rangle=0 \tag{3.22}$$

for all i and j. This shows very clearly that we are doing classical mechanics and not quantum mechanics. The commutators (3.20) and (3.21) suggest the following interpretation of the variables $\phi^a,\lambda_a,c^a,\bar{c}_a$: we may consider them as coordinates for a $4\times(2n)$-dimensional extended phase space on which the following (graded) Poisson structure is defined:

$$\{\phi^a,\lambda_b\}=\delta^a_b \, , \tag{3.23}$$

$$\{\bar{c}_b,c^a\}=-i\delta^a_b \, . \tag{3.24}$$

[Another manner to derive the Poisson structure (3.23) and (3.24) is the following. The Lagrangian $\widetilde{\mathcal{L}}$ gives rise to the constraints $\Pi_{\phi^a}=\lambda_a$ and $\Pi_{c^a}=i\bar{c}_a$ where Π_{ϕ^a} and Π_{c^a} are the momenta conjugate to ϕ^a and c^a. Because of these constraints, in going to the Hamiltonian formalism we have to apply the Dirac procedure.[17] The resulting Dirac brackets would be Eqs. (3.23) and (3.24).] Note that this Poisson structure is not the one of Eq. (3.2). Using (3.23) and (3.24), it is easy to verify that the equations of motion (3.13)–(3.16) can be reproduced in Hamiltonian form:

$$\frac{d}{dt}A=\{A,\widetilde{\mathcal{H}}\} \, , \tag{3.25}$$

where A is ϕ^a, λ_a, c^a, or $\bar{c}_a$, or any function of them. Here $\widetilde{\mathcal{H}}$ is the "super-Hamiltonian" defined in Eq. (3.18). Furthermore, as in quantum mechanics, we can go over from the Poisson-brackets formalism to an operatorial formalism via the replacement

$$\{\ ,\ \}\to -i[\ ,\ ] \, . \tag{3.26}$$

This is what one usually would call "quantization," but it does not have this meaning in our case. We stress that only the extended $8n$-dimensional phase space leads to the appearance of noncommuting operators, but not the original $2n$-dimensional one. From the $8n$-dimensional point of view the operators ϕ^a are mutually commuting position variables (with conjugate momenta λ_a). Therefore, the q^i and p_i variables of the original $2n$-dimensional phase space still commute with each other, as they should in classical mechanics.

Equation (3.26) is exactly what transforms (3.23) and (3.24) into (3.20) and (3.21), respectively. In the *operatorial* formalism the time evolution of a certain observable $A=A(\phi^a(t),\lambda_a(t),c^a(t),\bar{c}_a(t))$ is given by a Heisenberg-type equation

$$i\frac{d}{dt}A=[A,\widetilde{\mathcal{H}}] \, . \tag{3.27}$$

In particular, using the basic brackets (3.20) and (3.21), we can easily show that (3.27) reproduces (3.13)–(3.16). [In going from the c-number observables A to its operatorial version there might be ordering ambiguities because of the noncommutativity of ϕ with λ and of $\bar{c}$ with c; however, these ambiguities are not present for the restricted class of observables $A=A(\phi)$ considered in the conventional operatorial approach.]

Let us now use the path integral (3.11) to compute transition probabilities and classical averages. Let us look at the integral (the symbol $\mathcal{D}''$ means that we omit the integration over the initial and final points which are kept fixed)

$$K(\phi_f,c_f,t_f|\phi_i,c_i,t_i) =\int \mathcal{D}''\phi\,\mathcal{D}\lambda\,\mathcal{D}''c\,\mathcal{D}\bar{c}\,\exp\left[i\int_{t_i}^{t_f}dt\,\widetilde{\mathcal{L}}\right] \tag{3.28}$$

subject to the boundary conditions

$$\begin{aligned}&\phi^a(t_i)=\phi^a_i,\quad \phi^a(t_f)=\phi^a_f \, ,\\ &c^a(t_i)=c^a_i,\quad c^a(t_f)=c^a_f \, ,\\ &\lambda_a(t_i),\ \lambda_a(t_f),\ \bar{c}_a(t_i),\ \bar{c}_a(t_f)\ \text{arbitrary} \, .\end{aligned} \tag{3.29}$$

These boundary conditions are motivated by our experience with phase-space path integrals in QM. There, to compute a transition amplitude, one has to fix the position variables of the end points, but one integrates over all values of the momenta at the end points. Since, according to (3.20) and (3.21), (ϕ^a,λ_a) and $(c^a,\bar{c}_a)$ are conjugate pairs, we try the same prescription here. In fact, the integral (3.28) can be done exactly[18,19] and the result reads

$$K(\phi^a_f,c^a_f,t_f|\phi^a_i,c^a_i,t_i)=\delta^{(2n)}(\phi^a_f-\phi^a_{\rm cl}(t_f,\phi_i)) \times\delta^{(2n)}(c^a_f-C^a_{\rm cl}(t_f,c_i;[\phi_{\rm cl}])) \, . \tag{3.30}$$

Here $\phi^a_{\rm cl}$ and $C^a_{\rm cl}$ denote solutions of the equations of motion (3.13) and (3.14) with initial conditions ϕ_i^a and c_i^a, respectively. Note that the solution $C_{\rm cl}$ functionally depends on the path $\phi_{\rm cl}(t)$. The result (3.30) should not come as a surprise. [It essentially tells us that only the zero modes of the ghosts kinetic operator contribute to the path integral. That means that the Jacobi fields saturate the path integral (3.28).] Let us look at the ghost piece in (3.12), for instance. It is a first-order fermionic action, very similar to the usual Dirac action. For actions of this type it is well known[17,18] that the quantum-mechanical transition amplitude is given by a δ function containing a solution of the classical equation of motion. This is due to the fact that for any Lagrangian of the form $\bar{\psi} D_t \psi$, where D_t is some first-order differential operator, the integral over $\bar{\psi}$ gives rise to a δ function $\delta(D_t\psi)$. Thus only ψ paths with $D_t\psi=0$, i.e., solutions of the equation of motion, contribute to the path integral. This is what gives rise to the second δ function on the RHS of (3.30). What is unusual about (3.12) is that also the bosonic part of the action is of the form "momentum times equation of motion" which means that the λ integration yields a δ function with the ϕ equation of motion. Performing the ϕ integration we then obtain the first δ function on the RHS of (3.30). The transition function (3.30) contains information both about the paths (via ϕ) and the Jacobi fields (via C). If we are not interested in the information about the Jacobi fields we can integrate over c_f to get back the probability P introduced earlier:

$$\int d^{2n}c_f K(\phi_f,c_f,t_f|\phi_i,c_i,t_i)=\delta^{(2n)}(\phi_f-\phi_{\rm cl}(t_f,\phi_i))$$
$$=P(\phi_f,t_f|\phi_i,t_i)\ . \qquad (3.31)$$

However, in the spirit of the discussion at the end of Sec. II, we would like to keep the information about the Jacobi fields since they are indispensable tools for the study of chaos, for instance. [Note that integrating in (3.31) over both c_i and c_f yields zero. We recall that for transition probabilities we only have to integrate over either the initial or the final position to get its renormalization and not over both. That is what we did here by integrating K only over c_f. The full normalization is obtained after a further integration over ϕ_f.]

In Sec. II we used a one-particle language, i.e., we considered some trajectory $\phi(t)$ and the associated Jacobi field $c(t)$ describing nearby trajectories. Now we would like to go over to a statistical, or many-particle, description in which we do not consider individual trajectories $\phi(t)$ but rather the evolution of density distributions $\rho(\phi^a,t)$, where ϕ^a is considered merely as a phase-space coordinate and not as function of t. Including the ghosts into this picture amounts to a "Schrödinger-type" formulation (but let us remember that we are not doing any kind of quantum mechanics) of our theory. The operator algebra (3.20) and (3.21) can be realized by differential operators

$$\lambda_a=-i\frac{\partial}{\partial\phi^a}\ , \qquad (3.32)$$

$$\bar{c}_a=\frac{\partial}{\partial c^a}\ , \qquad (3.33)$$

and multiplicative operators ϕ^a and c^a acting on functions $\tilde{\rho}(\phi^a,c^a,t)$. These functions can be thought of as a generalization of the ordinary density $\rho(\phi^a,t)$. Inserting the differential operators (3.32) and (3.33) into the Hamiltonian (3.18), it is easy to verify that the evolution kernel (3.30) obeys the following diffusion (or "Schrödinger-type") equation

$$(i\partial_t-\tilde{\mathcal{H}})K(\phi^a,c^a,t|\phi_i^a,c_i^a,t_i)=0 \qquad (3.34)$$

with the operatorial form of $\tilde{\mathcal{H}}$ given by

$$\tilde{\mathcal{H}}=-i\omega^{ab}\partial_b H\partial_a+i\frac{\partial}{\partial c^a}\omega^{ac}\partial_c\partial_b H c^b\ . \qquad (3.35)$$

(Note that there is no ordering problem in $\tilde{\mathcal{H}}$ because of the antisymmetric nature of ω^{ab}.) Therefore, a generalized density distribution depending on the ghosts and defined in terms of some given initial distribution $\tilde{\rho}(t_i)$ via

$$\tilde{\rho}(\phi^a,c^a,t)=\int d^{2n}\phi_i d^{2n}c_i$$
$$\times K(\phi^a,c^a,t|\phi_i^a,c_i^a,t_i)\tilde{\rho}(\phi_i^a,c_i^a,t_i) \qquad (3.36)$$

solves a diffusion equation such as (3.34):

$$i\partial_t\tilde{\rho}(\phi^a,c^a,t)=\tilde{\mathcal{H}}\tilde{\rho}(\phi^a,c^a,t)\ . \qquad (3.37)$$

Expanding the c^a dependence of $\tilde{\rho}$ in the form

$$\tilde{\rho}(\phi^a,c^a,t)=\sum_{p=0}^{2n}\frac{1}{p!}\rho^{(p)}_{a_1\cdots a_p}(\phi,t)c^{a_1}\cdots c^{a_p} \qquad (3.38)$$

we see that (3.37) is equivalent to a set of $2n+1$ real equations for the components $\rho^{(p)}_{a_1\cdots a_p}(\phi,t)$. In Sec. V we shall interpret these components as differential forms on phase space and the Hamiltonian $\tilde{\mathcal{H}}$ as a Lie derivative acting on them. From a physical point of view the coefficient functions $\rho^{(p)}_{a_1\cdots a_p}$ are important because they encode information about the behavior of nearby trajectories (see, for instance, Ref. 15). Here we only would like to point out that the equation for $\rho^{(p=0)}(\phi,t)\equiv\rho(\phi,t)$ is nothing else than the Liouville equation (3.6). In fact, looking at (3.18) with (3.32) and (3.33), it is clear that the fermionic part of $\tilde{\mathcal{H}}$ is absent in the $p=0$ sector and that the bosonic part is essentially the same as the Liouvillian:

$$\tilde{\mathcal{H}}|_{p=0}=-i\hat{L}\ . \qquad (3.39)$$

We can now use the generalized density functions $\tilde{\rho}(\phi^a,c^a,t)$ to define "expectation values" of observables $A=A(\phi^a,c^a)$ which contain the ghosts:

$$\langle A\rangle_t=\int d^{2n}\phi\, d^{2n}c\ A(\phi^a,c^a)\tilde{\rho}(\phi^a,c^a,t)\ . \qquad (3.40)$$

Writing $\tilde{\rho}$ as in (3.36) and taking the explicit form of K (3.30) into account, $\langle A\rangle_t$ becomes

$$\langle A\rangle_t=\int d^{2n}\phi_i d^{2n}c_i\, A(\phi^a_{\rm cl}(t;\phi_i),C^a(t;c_i,[\phi_{\rm cl}]))$$
$$\times\tilde{\rho}(\phi_i^a,c_i^a,t_i)\ . \qquad (3.41)$$

These equations illustrate once more that, depending on the choice of A, all kinds of information about both trajectories and Jacobi fields can be extracted from

$\tilde{\rho}(\phi^a,c^a,t)$. Another interesting class of observable is the time-dependent correlation functions (for their physical interpretation see, for instance, Parisi[20])

$$\langle A_1(t_1)A_2(t_2)\cdots A_N(t_N)\rangle_{\tilde{\rho}} \equiv \int d^{2n}\phi\, d^{2n}c\, A_1(\phi^a,c^a;t_1) \times A_2(\phi^a,c^a;t_2)\cdots A_N(\phi^a,c^a;t_N)\tilde{\rho}(\phi^a,c^a) . \tag{3.42}$$

Here $A_j(\phi^a,c^a;t)$ denotes the time-evolved observables and $\tilde{\rho}(\phi^a,c^a)$ is some initial density distribution specified at $t=0$. The equation of motion of the A_j's is (3.27) (a "Heisenberg-type" equation), which has the formal solution

$$A(t)=e^{i\tilde{\mathcal{H}}t}A(0)e^{-i\tilde{\mathcal{H}}t} . \tag{3.43}$$

We shall now derive a path-integral representation of the correlation functions (3.42). Assuming that its time arguments are ordered according to

$$t_1>t_2>\cdots>t_N>0 \tag{3.44}$$

and using (3.43), we obtain

$$\langle A_1(t_1)A_2(t_2)\cdots A_N(t_N)\rangle_{\tilde{\rho}} =\int dx\, e^{i\tilde{\mathcal{H}}t_1}A_1(x;0)e^{-i\tilde{\mathcal{H}}(t_1-t_2)}A_2(x;0)e^{-i\tilde{\mathcal{H}}(t_2-t_3)}\cdots e^{-i\tilde{\mathcal{H}}(t_{N-1}-t_N)}A_N(x;0)e^{-i\tilde{\mathcal{H}}t_N}\tilde{\rho}(x) . \tag{3.45}$$

We have introduced here the abbreviation $x\equiv(\phi^a,c^a)$. We may omit the operator $\exp(i\tilde{\mathcal{H}}t_1)$ from the RHS of (3.45) since it is easy to show that, for any function G vanishing on $\partial\mathcal{M}_{2n}$, we have

$$\int d^{2n}\phi\, d^{2n}c\, e^{it\tilde{\mathcal{H}}}G(\phi^a,c^a)=\int d^{2n}\phi\, d^{2n}c\, G(\phi^a,c^a) . \tag{3.46}$$

(This expresses a kind of "superprobability" conservation.) With the help of the time evolution kernel K, Eq. (3.45) can be written as

$$\langle A_1(t_1)A_2(t_2)\cdots A_N(t_N)\rangle_{\tilde{\rho}}=\int\prod_{l=1}^{N}dx_l\int dx_0\, A_1(x_1;0)K(x_1,t_1|x_2,t_2)A_2(x_2;0) \times K(x_2,t_2|x_3,t_3)\cdots K(x_{N-1},t_{N-1}|x_N,t_N)A_N(x_N;0)K(x_N,t_N|x_0,0)\tilde{\rho}(x_0) . \tag{3.47}$$

Each of the K's has a path-integral representation of the form (3.28), but we can also combine the RHS of (3.47) into a single path integral: this is well known in the quantum case[18] and can easily be understood by looking at the discretized form of the path integral presented in the Appendix. So (3.47) can be written in the continuum notation

$$\langle \mathbf{T}A_1(t_1)A_2(t_2)\cdots A_N(t_N)\rangle_{\tilde{\rho}}=\int\mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}\prod_{j=1}^{N}A_j(\phi^a(t_j),c^a(t_j))\tilde{\rho}(\phi^a(0),c^a(0))\exp\left[i\int_0^{t_{\max}}dt\,\tilde{\mathcal{L}}\right] . \tag{3.48}$$

The measures such as $\mathcal{D}\phi$ denote the integration over all $\phi(t)$ including the end-point variables $\phi(0)$ and $\phi(t_{\max})$. In writing down (3.48) we allowed for an arbitrary ordering of the time arguments t_j (the largest of them is denoted by $t_{\max}$) because then, by the usual arguments, the path integral computes the average of the time-ordered product of the A_j. The main difference between (3.48) and a quantum-mechanical path integral (apart from the different time-evolution operators H for QM and $\tilde{\mathcal{H}}$ for CM and the different measures) is that the time-evolution kernels and observables are convoluted with only one "wave function" $\tilde{\rho}$, whereas in QM one has a factor of ψ^* coming from the final state and a factor ψ from the initial state, respectively. This is another reflection of the simple fact that the QM analogue of $\tilde{\rho}$ is a density matrix $\sum_n p_n|n\rangle\langle n|$, and not a Hilbert space vector $|n\rangle$. We will come back to these analogies in Ref. 12.

Similarly as in quantum field theory the correlation functions (3.48) can be obtained taking derivatives of a generating functional with respect to external currents. Coupling the fields to external sources through the Lagrangian $\mathcal{L}_s$ (the subscript s is for sources)

$$\mathcal{L}_s\equiv J_a(t)\phi^a(t)+\Lambda^a(t)\lambda_a(t) +\bar{\eta}_a(t)c^a(t)+\bar{c}_a(t)\eta^a(t) \tag{3.49}$$

the generating functional is given by

$$\mathbf{Z}_{\tilde{\rho}}[J,\Lambda,\bar{\eta},\eta]\equiv\left\langle \mathbf{T}\exp\left[i\int_0^{\infty}dt\,\mathcal{L}_s\right]\right\rangle_{\tilde{\rho}} =\int\mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\tilde{\rho}(\phi^a(0),c^a(0)) \times\exp\left[i\int_0^{\infty}dt(\tilde{\mathcal{L}}+\mathcal{L}_s)\right] . \tag{3.50}$$

Before closing this section we should mention an important difference between the Lagrange and Hamilton formalism as far as the determinants in Eqs. (2.7) and (3.10), respectively, are concerned. In the Lagrangian version of the classical path integral the determinant of $\delta^2S/\delta\phi^2$ is a highly nontrivial nonlocal functional of ϕ and, therefore, we need the ghosts to exponentiate this determinant. On the other hand, the corresponding determinant in the Hamiltonian version [see Eq. (3.10)] can be formally proven to be equal to unity, i.e., independent of ϕ. The proof as follows:

$$\det[\partial_t \delta^a_b - \omega^{ac}\partial_c\partial_b H(\phi)] \propto \exp\{\mathrm{Tr}\ln[\delta^a_b - \partial_t^{-1}\omega^{ac}\partial_c\partial_b H(\phi)]\} = \exp\left[\tfrac{1}{2}\int dt\, \mathrm{Tr}\,\omega^{ac}\partial_c\partial_b H\right] = 1 \ . \quad (3.51)$$

The first equality in the third line of Eq. (3.51) is due to the fact that the Green's function of ∂_t is $\theta(t)$ and that, consequently, the power series of the logarithm terminates after the first term. The second step follows from the antisymmetry of ω^{ab}. This proof indicates that we could in principle omit the ghost sector in the Hamiltonian formalism, because the determinant is 1. The reader might then ask why we do not do so: the reason is that we are interested in the dynamics of the ghosts because they are the Jacobi fields and to study chaos[15] we need to calculate their correlation functions [see Eq. (2.22)]. To determine these correlations we have to couple the ghosts to external currents as we did in (3.49) and (3.50):

$$\mathbf{Z}_{\tilde\rho}[J,\Lambda,\bar\eta,\eta] = \int \mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar c\,\tilde\rho \exp\left[i\int dt(\tilde{\mathcal{L}} + \mathcal{L}_s)\right] \ . \quad (3.52)$$

Integrating out the ghosts in (3.52) we are left with

$$\mathbf{Z}_{\tilde\rho}[J,\Lambda,\bar\eta,\eta] = \int \mathcal{D}\phi\,\mathcal{D}\lambda\,\tilde\rho \times \exp\left[i\int dt(\tilde{\mathcal{L}}_B + J_a\phi^a + \Lambda^a\lambda_a + \bar\eta M^{-1}\eta)\right] , \quad (3.53)$$

where $\tilde{\mathcal{L}}_B$ is the bosonic part of $\tilde{\mathcal{L}}$ and M is the matrix

$$M^a_b(t,t') \equiv (\partial_t\delta^a_b - \omega^{ac}\partial_c\partial_b H)\delta(t-t') \ . \quad (3.54)$$

So, even if the determinant of M is one, it is the matrix $M^a_b(t-t')$ that is nontrivial and that will give us nontrivial information on the ghost (or Jacobi-field) correlations.

IV. THE ISp(2) ALGEBRA

In this section we discuss various BRS-type symmetries of the action $\tilde S = \int dt\,\tilde{\mathcal{L}}$ with $\tilde{\mathcal{L}}$ given by (3.12). First of all, it is easy to verify that $\tilde S$ has the same kind of BRS and anti-BRS invariance as its Lagrangian counterpart. It is invariant under the BRS transformation

$$\delta\phi^a = \epsilon c^a, \quad \delta\bar c_a = i\epsilon\lambda_a, \quad \delta c^a = \delta\lambda_a = 0 \quad (4.1)$$

and the anti-BRS transformation

$$\delta\phi^a = -\bar\epsilon\omega^{ab}\bar c_b, \quad \delta c^a = i\bar\epsilon\omega^{ab}\lambda_b, \quad \delta\bar c_a = \delta\lambda_a = 0 \ . \quad (4.2)$$

They are generated by the charges

$$Q = ic^a\lambda_a, \quad \bar Q = i\bar c_a\omega^{ab}\lambda_b \quad (4.3)$$

according to the rule

$$\delta X = [\epsilon Q + \bar\epsilon\bar Q, X] \ . \quad (4.4)$$

[The graded commutator rules were given in Eqs. (3.20) and (3.21) but only in expectation value. So the commutators below and all the following are intended in expectation value.] Using the equations of motion (3.13)–(3.16), it can be checked that the charges Q and $\bar Q$ are conserved, and it is also easily seen that they commute with $\tilde{\mathcal{H}}$:

$$[Q,\tilde{\mathcal{H}}] = [\bar Q,\tilde{\mathcal{H}}] = 0 \ . \quad (4.5)$$

Moreover they anticommute among themselves and are nilpotent:

$$[Q,Q] = [\bar Q,\bar Q] = [Q,\bar Q] = 0 \ . \quad (4.6)$$

Using (3.13)–(3.16) one can show that also the following ghost bilinears are conserved:

$$Q_g = c^a\bar c_a, \quad K = \tfrac{1}{2}\omega_{ab}c^ac^b, \quad \bar K = \tfrac{1}{2}\omega^{ab}\bar c_a\bar c_b \ . \quad (4.7)$$

Here we introduced the inverse of ω^{ab}:

$$\omega_{ac}\omega^{cb} = \delta^b_a \ . \quad (4.8)$$

Of course, the quantities (4.7) commute with the Hamiltonian [all the *classical commutation relations* that appear in the following hold (modulo some $\pm i$) also in the *classical Poisson-brackets* formalism of Eqs. (3.23) and (3.24)]:

$$[Q_g,\tilde{\mathcal{H}}] = [K,\tilde{\mathcal{H}}] = [\bar K,\tilde{\mathcal{H}}] = 0 \ . \quad (4.9)$$

The charges (4.7) have the following commutators with the ghosts

$$\begin{aligned} &[Q_g,c^a] = c^a, \quad [Q_g,\bar c_a] = -\bar c_a \ , \\ &[K,c^a] = 0, \quad [K,\bar c_a] = c^b\omega_{ba} \ , \\ &[\bar K,c^a] = \bar c_b\omega^{ba}, \quad [\bar K,\bar c_a] = 0 \ . \end{aligned} \quad (4.10)$$

This identifies Q_g as the ghost charge operator; it assigns ghost charge $+1$ to c^a and -1 to $\bar c_a$. On the other hand, K and $\bar K$ act like a kind of charge-conjugation operator. These charges have the following algebra among themselves:

$$[Q_g,K] = 2K, \quad [Q_g,\bar K] = -2\bar K, \quad [K,\bar K] = Q_g \ . \quad (4.11)$$

This is the Lie algebra of Sp(2). The commutators of Q and $\bar Q$ with the new charges are

$$\begin{aligned} &[Q_g,Q] = +Q, \quad [Q_g,\bar Q] = -\bar Q \ , \\ &[K,Q] = 0, \quad [K,\bar Q] = +Q \ , \\ &[\bar K,Q] = \bar Q, \quad [\bar K,\bar Q] = 0 \ . \end{aligned} \quad (4.12)$$

These relations together with (4.11) make up the Lie algebra of the inhomogeneous symplectic group ISp(2) (this will become clear in the following). The above symmetries can be conveniently represented in a superspace formalism. We combine the different fields of the theory into the following real superfield:

$$\Phi^a(t,\theta,\bar\theta) = \phi^a + \theta c^a(t) + \bar\theta\omega^{ab}\bar c_b(t) + i\bar\theta\theta\omega^{ab}\lambda_b(t) \ . \quad (4.13)$$

In terms of Φ^a the Hamiltonian and the Lagrangian, respectively, assume a remarkably simple form:

$$\tilde{\mathcal{H}}=i\int d\theta\, d\bar{\theta}\, H(\Phi^a)\ , \quad (4.14)$$

$$\tilde{\mathcal{L}}=i\int d\theta\, d\bar{\theta}[\tfrac{1}{2}\Phi^a\omega_{ab}\dot{\Phi}^b-H(\Phi)]\ . \quad (4.15)$$

[In writing down (4.15) we have omitted a surface term.] The ISp(2) generators can be represented by the superspace differential operators

$$\hat{Q}=\frac{\partial}{\partial\theta},\quad \hat{\bar{Q}}=-\frac{\partial}{\partial\bar{\theta}},\quad \hat{Q}_g=\bar{\theta}\frac{\partial}{\partial\bar{\theta}}-\theta\frac{\partial}{\partial\theta}\ ,$$
$$\hat{K}=\bar{\theta}\frac{\partial}{\partial\theta},\quad \hat{\bar{K}}=\theta\frac{\partial}{\partial\bar{\theta}}\ . \quad (4.16)$$

They generate the ISp(2) transformations according to

$$\delta\Phi^a\equiv[\Omega,\Phi^a]=\hat{\Omega}\Phi^a\ , \quad (4.17)$$

where Ω is any of the operators $Q,\bar{Q},Q_g,K,\bar{K}$. It can be checked that the algebra of the $\hat{\Omega}$'s coincides with (4.11) and (4.12). We observe that $\hat{Q}_g$, $\hat{K}$, and $\hat{\bar{K}}$ leave the bilinear $\bar{\theta}\theta$ invariant. By definition, this means that they generate the group Sp(2). On the other hand, $\hat{Q}$ and $\hat{\bar{Q}}$ perform translations on θ and $\bar{\theta}$:

$$\theta\rightarrow\theta+\epsilon,\quad \bar{\theta}\rightarrow\bar{\theta}-\bar{\epsilon}\ . \quad (4.18)$$

This shows that the BRS and anti-BRS operators provide the inhomogeneous part of ISp(2).

There is still another interesting representation of the ISp(2) algebra. In Sec. III we introduced generalized density distributions $\tilde{\rho}(\phi^a,c^a)$ which do not only depend on a point in phase space, but also on the ghosts. These functions $\tilde{\rho}$ provide a representation space for the operatorial formulation of the theory defined by the path integral (3.11). In this formulation, which is the analogue of the quantum-mechanical Schrödinger picture, the momenta λ_a and $\bar{c}_a$ conjugate to ϕ^a and c^a, respectively, are represented by the differential operators $-i\partial_a$ and $\partial/\partial c^a$ in order to satisfy the commutation relations (3.20) and (3.21). Acting on the functions $\tilde{\rho}(\phi^a,c^a)$ the ISp(2) generators have the representation

$$Q=c^a\partial_a,\quad \bar{Q}=\frac{\partial}{\partial c^a}\omega^{ab}\partial_b,\quad Q_g=c^a\frac{\partial}{\partial c^a}\ ,$$
$$K=\tfrac{1}{2}\omega_{ab}c^ac^b,\quad \bar{K}=\tfrac{1}{2}\omega^{ab}\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b}\ . \quad (4.19)$$

A general function $\tilde{\rho}(\phi^a,c^a)$ has an expansion of the form

$$\tilde{\rho}(\phi^a,c^a)=\sum_{p=0}^{2n}\frac{1}{p!}\rho^{(p)}_{a_1\cdots a_p}(\phi^a)c^{a_1}\cdots c^{a_p}\ . \quad (4.20)$$

Let us now consider "homogeneous forms of degree p," i.e., functions for which $\rho^{(p)}_{a_1\cdots a_p}$ is nonzero only for one value of p:

$$\tilde{\rho}_p(\phi^a,c^a)=\frac{1}{p!}\rho^{(p)}_{a_1\cdots a_p}(\phi^a)c^{a_1}\cdots c^{a_p}\ . \quad (4.21)$$

The action of the ISp(2) operators on such functions is

$$Q\tilde{\rho}_p=\frac{1}{p!}\partial_b\rho^{(p)}_{a_1\cdots a_p}c^bc^{a_1}\cdots c^{a_p}\ ,$$
$$\bar{Q}\tilde{\rho}_p=\frac{1}{(p-1)!}\omega^{a_1b}\partial_b\rho^{(p)}_{a_1a_2\cdots a_p}c^{a_2}\cdots c^{a_p}\ ,$$
$$Q_g\tilde{\rho}_p=p\tilde{\rho}_p\ , \quad (4.22)$$
$$K\tilde{\rho}_p=\frac{1}{2p!}\omega_{b_1b_2}\rho^{(p)}_{a_1\cdots a_p}c^{b_1}c^{b_2}c^{a_1}\cdots c^{a_p}\ ,$$
$$\bar{K}\tilde{\rho}_p=\frac{1}{2(p-2)!}\omega^{a_2a_1}\rho^{(p)}_{a_1a_2a_3\cdots a_p}c^{a_3}\cdots c^{a_p}\ .$$

We will soon return to these expressions and give them a differential geometric interpretation.

We shall not treat here the physical state condition[21] and the cohomology problem associated with these BRS operators but defer this to Ref. 22. The reason for not discussing it here is that the problem is not trivial and more on the line of the *basic-cohomology* formulation of topological field theory.[6,23] To attack this problem we first need a further understanding of the *differential geometric structure* of the operators (4.22) and of our whole construction.

V. THE GHOSTS AS PHASE-SPACE DIFFERENTIAL FORMS

In the following two sections we shall give an interpretation of the five ISp(2) charges and relate them to more familiar differential geometric objects. These charges are *universal,* they are present in *any* dynamical system. Hence, they should represent something we know already, but under a different name perhaps. Let us recall our interpretation of the ghosts $c^a(t)$: they are the Jacobi fields $\delta\phi^a(t)$, i.e., the infinitesimal displacement between two classical trajectories. For the charge K, for example, this correspondence reads

$$K\equiv\tfrac{1}{2}\omega_{ab}c^a(t)c^b(t)\leftrightarrow\tfrac{1}{2}\omega_{ab}\delta\phi^a(t)\delta\phi^b(t)\ . \quad (5.1)$$

The next point to be remembered is that the space of classical trajectories, which we will denote by $\mathcal{P}$, is in one-to-one correspondence with the phase space $\mathcal{M}_{2n}$ (Ref. 24). The origin of this correspondence is that to each classical path we can associate that point in phase space from which it starts. (Because the equations of motion are first order, the initial point uniquely specifies the trajectory.) Thus we have

$$\mathcal{P}\leftrightarrow\mathcal{M}_{2n}\ . \quad (5.2)$$

The Jacobi fields $\delta\phi^a(t)$ could be thought of as elements of $T^*\mathcal{P}$, the cotangent bundle over $\mathcal{P}$. (Recall that the Jacobi fields depend on the point in the "base space" $\mathcal{P}$, i.e., on the trajectory around which they describe small fluctuations.) The correspondence (5.2) induces a similar correspondence for the cotangent bundles:

$$T^*\mathcal{P}\leftrightarrow T^*\mathcal{M}_{2n}\ . \quad (5.3)$$

The relation $\delta\phi^a(t)\leftrightarrow d\phi^a$ between Jacobi fields and differentials $d\phi^a\in T^*_\phi\mathcal{M}_{2n}$ can be understood in the following way. We know that the time evolution generated by the Hamiltonian is a special canonical transformation, i.e., a special symplectic diffeomorphism on $\mathcal{M}_{2n}$. Usually this diffeomorphism is interpreted in an "active" way in the sense that a coordinate change corresponds to a physical displacement of a particle. We also could look at it in a "passive" way: we stay at a fixed point ϕ on $\mathcal{M}_{2n}$ and study how the time evolution manifests itself in the cotangent space $T^*_\phi\mathcal{M}_{2n}$. It will map a basis $d\phi^a|_{t=0}$ used at $t=0$ onto a new basis $d\phi^a|_t$ at $t>0$ according to the usual tensorial transformation law. We easily can derive an evolution equation for $d\phi^a|_t$; it is nothing else than the Jacobi equation. Therefore we may identify $d\phi^a|_t \hat{=} \delta\phi^a(t)$ and $d\phi^a|_{t=0}=d\phi^a$. This shows that the Jacobi fields $\delta\phi^a$ "grow" out of the differentials $d\phi^a$ in the course of the time evolution which establishes the one-to-one correspondence (5.3). Because of this identification we may now complete (5.1) as

$$K\equiv\tfrac{1}{2}\omega_{ab}c^a(t)c^b(t)\leftrightarrow\tfrac{1}{2}\omega_{ab}\delta\phi^a(t)\delta\phi^b(t)$$
$$\leftrightarrow\tfrac{1}{2}\omega_{ab}d\phi^a\wedge d\phi^b\equiv\omega\ . \tag{5.4}$$

What we obtained on the RHS of (5.4) is the symplectic two-form ω, which is known to be invariant under the Hamiltonian flow.[10] The same is true for all its exterior powers $\omega\wedge\omega\wedge\cdots\wedge\omega$ (note that the exterior powers ω^k give rise to the Poincaré-Cartan integral invariants), and in particular for the phase-space volume form vol$\equiv\omega^n$. The conservation of the latter is the statement of the well-known Liouville theorem. We conclude that the conservation of K is a consequence of, or actually the same thing as, the conservation of the symplectic two-form.

Above we identified the ghosts c^a with the basis elements $d\phi^a$ of the cotangent space. For this identification to be possible we have to make sure, however, that c^a and $d\phi^a$ transform in the same way under any symplectic diffeomorphism of $\mathcal{M}_{2n}$ (not just under time evolution). It is easy to see that this is indeed the case. Let us consider an arbitrary time-independent infinitesimal coordinate transformation:

$$\phi'^a=\phi^a-\varepsilon^a(\phi^c)\ . \tag{5.5}$$

(We also could consider time-dependent transformations; the only difference is the usual $\partial G/\partial t$ term in the transformation of the Hamiltonian. In order for this transformation to be symplectic, i.e., to preserve ω_{ab}, the vector field ε^a has to be (at least locally) of the form

$$\varepsilon^a(\phi^c)=\omega^{ab}\partial_b G(\phi^c) \tag{5.6}$$

with some generating function G. Equations (5.5) and (5.6) imply that $\dot\phi^a$ and $\omega_{ab}\partial_b H$ transform like vectors. Now we determine the transformation law of λ_a, c^a, and $\bar c_a$ in such a way that $\tilde{\mathcal{L}}$ of Eq. (3.12) behaves as a scalar. It is easy to see that we must require

$$\lambda'_a=\frac{\partial\phi^b}{\partial\phi'^a}\lambda_b,\quad c'^a=\frac{\partial\phi'^a}{\partial\phi^b}c^b,\quad \bar c'_a=\frac{\partial\phi^b}{\partial\phi'^a}\bar c_b\ . \tag{5.7}$$

We see that our theory is canonically covariant, as it should be, if we transform λ_a and $\bar c_a$ as $\partial/\partial\phi^a$ and c^a as This confirms the interpretation that the ghosts c^a "live" in cotangent space. Similarly, the antighosts $\bar c_a$ behave like elements of the tangent space.

Since in symplectic geometry the two-form ω provides a natural isomorphism between tangent and cotangent space, it is clear that the conservation of $K=\frac{1}{2}\omega_{ab}c^ac^b$ must have an analogue in tangent space. Obviously this "dual Liouville theorem" is just the conservation of $\bar K=\frac{1}{2}\omega^{ab}\bar c_a\bar c_b$. Now we have explained why there are the universal charges K and $\bar K$ in any Hamiltonian system. However, K and $\bar K$ being conserved implies that also their commutator is conserved. In view of the relations (4.11) this means that we automatically get a third conserved charge, namely Q_g. [In Ref. 7 Q_g was identified with the symplectic two-form ω. In view of the analysis presented here that is wrong. See, however, the parenthetical remark before Eq. (6.19). It was also stated that the conservation of Q_g was essentially due to the Liouville theorem (or the invariance of ω) and that conclusion is basically right.] This completes the discussion of the Sp(2) generators and we now turn to Q and $\bar Q$.

VI. CARTAN CALCULUS WITH GHOSTS

The BRS operator Q is most easily understood in the "Schrödinger-type" representation introduced in Eq. (4.19):

$$Q=c^a\partial_a\ . \tag{6.1}$$

If we again identify the ghosts with $d\phi^a$ we have

$$Q=c^a\partial_a\leftrightarrow d\phi^a\partial_a\equiv d\ ; \tag{6.2}$$

i.e., *the BRS operator can be put in correspondence with the exterior derivative on phase space.* (Remember that, contrary to the previous section, c^a now denotes an anticommuting *number* [the argument of $\tilde\rho(\phi^a,c^a)$] and not a function of t.) Under this identification the function $\tilde\rho_p(\phi^a,c^a)$ of Eq. (4.21) becomes an ordinary p-form on phase space:

$$\tilde\rho_p(\phi^a)=\frac{1}{p!}\rho^{(p)}_{a_1\cdots a_p}(\phi^a)d\phi^{a_1}\wedge\cdots\wedge d\phi^{a_p}\ . \tag{6.3}$$

Looking at the first of Eq. (4.22), we recognize that Q acts on differential forms like the standard exterior derivative d. We should note at this point that Q plays a remarkable double role: first, according to the BRS transformations rules (4.1), it maps a classical trajectory $\phi^a(t)$ onto a new classical trajectory $\phi^a+\epsilon c^a(t)$ by adding a Jacobi field to it. In this sense it is a translation or derivative operator on the infinite-dimensional space of classical trajectories $\mathcal{P}$. Second, in Eq. (6.2) it appears as an ordinary exterior derivative on phase space. Clearly these two roles are reconciled by the identifications (5.2) and (5.3), i.e., by tracing back all trajectories to their starting point. This identification is certainly possible for integrable systems, but it might break down for nonintegrable systems. In fact the correspondence (5.2) does not imply that the topologies of the two spaces are the same. Let us remember that the space $\mathcal{P}$ feels the dynamics through H, while $\mathcal{M}_{2n}$ does not. This will imply that also the homological and cohomological properties of the two spaces are different. So the cohomology of Q seen as an operator in $\mathcal{P}$ will be different from the cohomology of Q seen as an exterior derivative d on $\mathcal{M}_{2n}$. This analysis will be reported elsewhere.[21]

The second equation of (4.22) suggests that the anti-BRS operator $\overline{Q}$ is a kind of exterior coderivative. It reduces the rank of the form by one unit and takes the "divergence" with respect to the free index. In Riemannian geometry this contraction would be performed with the help of the Riemannian metric, which enters the coderivative via the definition of the Hodge star operator. In our case no Riemannian metric is available and the contraction is done with the symplectic structure instead. (In analogy with the Riemannian case one could be tempted to define a Laplacian as the anticommutator of Q and $\overline{Q}$. This would yield $\omega^{ab}\partial_a\partial_b$, which vanishes identically, however.) Returning to the relations (4.22) we observe that the ghost charge operator Q_g simply counts the rank of differential forms and that K is just a multiplication of the respective form with the symplectic two-form. Finally, $\overline{K}$ performs a kind of trace operation. It reduces the rank of the form by two units and contracts the free indices with ω^{ab}.

In order to better understand why the BRS charge is conserved, we recall that the time evolution of the generalized density functions $\tilde{\rho}(\phi^a,c^a,t)$ is given by the "Schrödinger-type" equation (3.37) with the Hamiltonian $\tilde{\mathcal{H}}$ defined in (3.35). This Hamiltonian $\tilde{\mathcal{H}}$ has a very simple differential geometric interpretation. Introducing the Hamiltonian vector field[10]

$$h^a(\phi^c)=\omega^{ab}\partial_b H(\phi^c) \tag{6.4}$$

we have

$$\tilde{\mathcal{H}}=-il_h \;, \tag{6.5}$$

where

$$l_h=h^a\partial_a+c^b(\partial_b h^a)\frac{\partial}{\partial c^a} \tag{6.6}$$

is the Lie-derivative operator along the vector field h^a. In fact, applying l_h to an expansion such as (3.38), we find the standard action of a Lie derivative on a covariant tensor:

$$\begin{aligned} l_h\rho^{(p)}_{a_1\cdots a_p}=&\,h^b\partial_b\rho^{(p)}_{a_1\cdots a_p}+\partial_{a_1}h^b\rho^{(p)}_{ba_2\cdots a_p}\\ &+\partial_{a_2}h^b\rho^{(p)}_{a_1ba_3\cdots a_p}+\cdots \;. \end{aligned} \tag{6.7}$$

Hence, the evolution equation (3.37) for the generalized densities can be written as

$$\partial_t\rho^{(p)}_{a_1\cdots a_p}(\phi^a,t)=-l_h\rho^{(p)}_{a_1\cdots a_p}(\phi^a,t) \;. \tag{6.8}$$

This equation is manifestly real. It says that the tensor components of $\tilde{\rho}$ are time evolved by Lie transporting them along the Hamiltonian vector field h^a. This is a consequence of the fact that, in classical mechanics, the dynamics is implemented by a special coordinate transformation. Equation (6.8) is nothing but a rule for a series of successive infinitesimal coordinate changes parametrized by t. Note that in the $p=0$ sector Eq. (6.8) reduces to the standard Liouville equation (3.6) since in that case l_h coincides with the Liouvillian $\hat{L}=h^a\partial_a$. The conservation of the BRS charge is easy to understand now if we remember that the exterior derivative d commutes with any Lie derivative. Since Q can be identified with d and $\tilde{\mathcal{H}}$ with $-il_h$, respectively, Q is then conserved as a consequence of the well-known differential geometric property that we mentioned above: $[d,l]=0$. Recalling that $l_h\omega^{ab}=0$ for any Hamiltonian vector field h, the conservation of $\overline{Q}\equiv\bar{c}_a\omega^{ab}\partial_b$ can be understood analogously.

In the above discussion we have represented the ISp(2) generators by differential operators acting on functions $\tilde{\rho}(\phi^a,c^a)$, which we interpreted as (inhomogeneous) differential forms. Returning to the analogy with quantum mechanics, this kind of representation would be the analogue of the Schrödinger picture. In the rest of this section we describe the relation between the ISp(2) generators and standard differential geometry in what would be the analogue of the "Heisenberg picture." We shall not consider explicit realizations of "states" and their transformation laws, but rather equal-time commutators of certain composite operators. We start from the geometric objects

$$\begin{aligned} &v=v^a\partial_a,\quad \alpha=\alpha_a d\phi^a,\\ &F^{(p)}=\frac{1}{p!}F_{a_1\cdots a_p}d\phi^{a_1}\wedge\cdots\wedge d\phi^{a_p}\;,\\ &V^{(p)}=\frac{1}{p!}V^{a_1\cdots a_p}\partial_{a_1}\wedge\cdots\wedge\partial_{a_p}\;. \end{aligned} \tag{6.9}$$

Obviously, v is a vector, α a covector, $F^{(p)}$ a general p form, and $V^{(p)}$ a completely antisymmetric contravariant tensor. To these objects we associate operators according to

$$\begin{aligned} &\hat{v}=v^a\bar{c}_a,\quad \hat{\alpha}=\alpha_a c^a\;,\\ &\hat{F}^{(p)}=\frac{1}{p!}F_{a_1\cdots a_p}c^{a_1}\cdots c^{a_p}\;,\\ &\hat{V}^{(p)}=\frac{1}{p!}V^{a_1\cdots a_p}\bar{c}_{a_1}\cdots\bar{c}_{a_p}\;. \end{aligned} \tag{6.10}$$

Here the superscript caret means that we are dealing with operators containing ghosts. Because of the basic rules (3.20) and (3.21) the objects (6.10) have well-defined (graded) commutators among themselves and with the ISp(2) generators (4.3) and (4.7). It turns out that all the tensor manipulations on symplectic manifolds, sometimes referred to as the Cartan calculus, can be reformulated in terms of such commutators. Let us give a few examples. (For the reader not familiar with the terminology of symplectic differential geometry we refer to Ref. 10 or 25 or, for a compact review, to Appendix A of Ref. 26.) The fact that Q acts like the exterior derivative d is now expressed by

$$[Q,\hat{F}^{(p)}]=(dF^{(p)})^\wedge \;. \tag{6.11}$$

There is an analogous relation between $\bar{Q}$ and the antisymmetric contravariant tensor fields:

$$[\bar{Q},\hat{V}^{(p)}]=\frac{1}{p!}\omega^{ab}\partial_b V^{a_1\cdots a_p}\bar{c}_a\bar{c}_{a_1}\cdots\bar{c}_{a_p} . \quad (6.12)$$

In particular for $p=0$, i.e., for functions, this reads

$$[\bar{Q},f]=[(df)^{\#}]^{\wedge} , \quad (6.13)$$

where we introduced the map # which associates a vector field $\alpha^{\#}$ to any one-form α (see Refs. 25 and 26) according to

$$(\alpha^{\#})^a=\omega^{ab}\alpha_b . \quad (6.14)$$

Furthermore we find that the interior contraction has the representation

$$[\hat{v},\hat{F}^{(p)}]=(\iota_v F^{(p)})^{\wedge} , \quad [\hat{\alpha},\hat{V}^{(p)}]=(\iota_\alpha V^{(p)})^{\wedge} \quad (6.15)$$

with the following two types of interior contractions:

$$(\iota_v F^{(p)})_{a_1\cdots a_{p-1}}=v^b F_{ba_1\cdots a_{p-1}} ,$$
$$(\iota_\alpha V^{(p)})^{a_1\cdots a_{p-1}}=\alpha_b V^{ba_1\cdots a_{p-1}} . \quad (6.16)$$

The charges K and $\bar{K}$ can be used to express the correspondence that exists in symplectic geometry[25,26] between vector fields and one-forms: they can be used to transform vector fields into one-forms and vice versa (this correspondence is a sort of analogue to the Riemannian geometry operation of pulling indices up and down):

$$[K,\hat{v}]=(v^{\flat})^{\wedge}, \quad [\bar{K},\hat{\alpha}]=(\alpha^{\#})^{\wedge} . \quad (6.17)$$

Here $v^{\flat}$ denotes[26] the form associated to the vector field n or (in simple words) it indicates the lowering of indices according to the rule

$$(v^{\flat})_a=\omega_{ac}v^c . \quad (6.18)$$

The raising operation # has been defined already in (6.14). Equation (6.17) again identifies K as the symplectic two-form and $\bar{K}$ as its analogue in tangent space. [Going back to Ref. 7 where we identified the Q_g as the symplectic two-form, we could interpret that result in the following sense: let us interpret K as a one-form $\hat{\alpha}$ with values in the "forms." Then inserting it in the second equation in (6.17) and using the last equation in (4.11) we obtain that the $(\alpha^{\#})^{\wedge}$ is nothing else than the ghost charge Q_g. So we can say the ghost charge (once interpreted as a vector field with values in the forms) is the image of the symplectic form K under the correspondence generated by the map #.] Recalling the standard identity for the Lie derivative,

$$l_v=d\iota_v+\iota_v d , \quad (6.19)$$

and combining (6.11) with (6.16) we obtain the following representation of the Lie derivative:

$$(l_v F^{(p)})^{\wedge}=[[Q,\hat{v}],\hat{F}^{p}] . \quad (6.20)$$

As an example let us look at the special case $v=(dH)^{\#}$ corresponding to the Hamiltonian vector field $h^a=\omega^{ab}\partial_b H$ of (6.4):

$$(l_{(dH)^{\#}}F^{(p)})^{\wedge}=[[Q,(dH)^{\#\wedge}],\hat{F}^{(p)}]$$
$$=[[Q,[\bar{Q},H]],\hat{F}^{(p)}] . \quad (6.21)$$

In the second line we used Eq. (6.13). It is a simple exercise to check that

$$[Q,[\bar{Q},H]]=i\tilde{\mathcal{H}} \quad (6.22)$$

so that

$$(l_{(dH)^{\#}}F^{(p)})^{\wedge}=[i\tilde{\mathcal{H}},\hat{F}^{(p)}] . \quad (6.23)$$

This is in complete agreement with (6.5) where we had identified l_h with $i\tilde{\mathcal{H}}$. It is amusing to note that even the Poisson brackets (PB) of the original ($2n$-dimensional) phase space can be expressed in terms of our commutators. For any two functions $f(\phi^a)$ and $g(\phi^a)$, the PB is given by

$$\{f,g\}=[[[f,Q],\bar{K}],[[[g,Q],\bar{K}],K]] . \quad (6.24)$$

Even if the RHS of (6.24) looks complicated, it is a straightforward exercise, based upon the rules given above and the definition of the PB (Ref. 26), to derive it. This rewriting of the PB { , } in terms of a chain of *classical* commutators [,] will be helpful, once we go over to QM (Ref. 12), in better understanding the approach to quantization based on the "deformation"[27] of the Poisson brackets into the *quantum* commutators.

To summarize the content of this section we can say that, once the five basic operators $Q,\bar{Q},Q_g,K,\bar{K}$ are given, all the Cartan calculus rules (interior products, Lie derivative, etc.) reduce to simple commutators. But, on the other side, the price to pay is that the Poisson brackets become a complicated chain of commutators. This might be anyhow a price worth paying if it helps in throwing new light on the geometry of phase space.[12,27] With these remarks we have completed our discussion of the ISp(2) charges and their geometrical meaning.

The reader might have realized that much of the discussion is very similar to Witten's work on supersymmetric quantum mechanics.[5] This theory has been used to give proof of the (holomorphic) Morse inequalities[28] and of the Atiyah-Hirzebruch theorem,[29] for instance, which heavily relied on the techniques developed for supersymmetric field theories. Apart from the fact that we are doing classical mechanics, the main difference between Witten's theory and ours is that supersymmetric quantum mechanics "lives" on a Riemannian manifold, whereas we are working on a symplectic manifold. Nevertheless, in both cases the operatorial formalism derived from a certain path integral turns out to be clearly related to the standard (exterior) calculus on the respective type of manifold. In our case anyhow this nice piece of mathematics came out of a *physical problem:* namely, the problem of rewriting classical mechanics using path integrals. We did not approach the problem the other way around, as it is mostly done nowadays, by seeing which piece of mathematics the path integral might simulate.

Before concluding let us remark that also Crnkovic and Witten,[24] and Zuckerman[24] had noted that in any variational problem there exists a universal conserved charge similar to our $\bar{K}$. In a sense we have completed their work finding four other universal charges that give the full ISp(2) structure.

VII. CONCLUSIONS

In this paper we have described the general framework of a path integral and operatorial approach to classical Hamiltonian dynamics. Its basic ingredients are not only the conventional density distribution over phase space, but also a set of p-form fields reflecting the dynamics of the ghosts or, equivalently, of the Jacobi fields. So far we have only presented the general ideas, but we have not discussed the potential applications of the formalism. Since these applications go in various different directions they will be described separately elsewhere.

As mentioned already, the theory presented here is a very natural framework to study chaotic phenomena in Hamiltonian systems. The essential observation is that the ghosts, and therefore also the p-form fields, contain information about the behavior of nearby trajectories. Further work will have to concentrate on how to extract interesting observables from the path integral, such as the Lyapunov exponents, the various entropies,[15] and the order parameters for the different degrees of "chaoticity" (ergodic, mixing, C systems, etc.). A first step in this direction is contained in Ref. 14, where we use our formalism to algebraically characterize ergodic systems. To do this we exploit the fact that, in addition to the ISp(2) symmetry discussed here, $\tilde{\mathcal{L}}$ is also invariant under a true supersymmetry. Contrary to the ISp(2) generators, the supersymmetry generators not only "know" about the geometry (and topology) of the phase space, but also about the dynamics, i.e., they explicitly depend on the Hamiltonian. It can be shown that ergodic systems have this supersymmetry unbroken, while in KAM or integrable systems it is spontaneously broken.[14] These hidden symmetries help in completing the old program[30] of algebraically characterizing the equilibrium states of a system. The Kubo-Martin-Schwinger (KMS) nature of these states is deeply related to the supersymmetry mentioned above and this symmetry might be useful in unveiling the real physical meaning of the KMS condition. There are also indications that the abstract modular automorphism[30] discussed in the algebraic approach to dynamical systems is related to a simpler modular invariance manifest in any classical systems once a proper reformulation of the Lagrangian (in the spirit of the nonlinear sigma model) is given.[31]

Another interesting line of investigation has its origin in the identity (6.22), showing that the Hamiltonian $\tilde{\mathcal{H}}$ is BRS exact. It can be shown[22] that the Lagrangian $\tilde{\mathcal{L}}$ is a pure BRS commutator as well. This means that our theory might fall into the class of topological field theories advocated by Witten.[6] It is in fact possible to characterize various topological feature of $\mathcal{M}_{2n}$ using this path integral. To clearly set this problem, and to disentangle the geometrical features from the dynamical ones it is crucial to properly understand the cohomology of the BRS operator Q. The space on which Q acts is the infinite-dimensional space $\mathcal{P}$ of classical trajectories and the study of its (co)homological properties is not an easy task. Once completed[22] it will give us information about the topological properties of the correspondences (5.2) and (5.3). This correspondence in turn will throw light on the *integrability* of the system, and maybe it will allow for an algebraic characterization of integrability similar to the algebraic characterization of ergodicity.[14] So it seems that the tools we have developed in this paper might be the right ones to address various interesting questions related to the ergodic, KAM, or integrable nature of dynamical systems.

Last but not least there is a relation between our formalism and quantum mechanics to be studied. We would like to see if this path integral for classical mechanics can provide some new understanding of quantum mechanics. Of course, it is true that the ghosts were introduced to suppress quantum fluctuations and that therefore they seem to be superfluous in quantum mechanics. However, we suspect that there also exists a quantum-mechanical path integral generalizing the usual one, in the sense that it describes not only scalar probability amplitudes but also form-valued amplitudes. They would not be forms on phase space but rather on the N-dimensional Lagrangian submanifold[10] introduced to quantize the theory. The weight in this path integral would no longer be the Feynman one but a generalization that allows the propagation of higher forms. The old Feynman weight would be, with respect to the new one, in the same relation as the Liouville operator $\hat{L}$ is to the Hamiltonian $\tilde{\mathcal{H}}$. There are indications[12] that this Feynman weight is still a Lie derivative of the Hamiltonian flow but on a space different than the phase space (that means the exterior derivative d would be different, and so also the BRS charge and the symmetries of this weight). This space seems to be the Grassmannian Lagrangian introduced in Ref. 10. The new Feynman path integral will also be useful (contrary to the usual one) to calculate geometrical features of this Grassmannian Lagrangian. In fact, if the new weight is a Lie derivative it can be put in the form of a BRS variation of something and so it will also fall into the class of topological field theories. What is more important, anyhow, is that the formalism of Ref. 12 will help in understanding the different nature of the modular invariance present in quantum mechanics with respect to the one of classical mechanics:[31] this difference seems to be at the heart of quantum mechanics.

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APPENDIX

In this appendix we present a *discretized* version of Eqs. (3.10) and (3.11) and analyze various associated problems. Let us start from Eq. (3.10); in an obvious notation, its discretized version reads

$$\tilde{\delta}[\phi^a-\phi^a_{\rm cl}]\equiv\prod_{j=1}^{m}\delta[\phi^a(j)-\phi^a_{\rm cl}(j;\phi^a(0))]$$

$$=\prod_{j=1}^{m}\delta\left[\frac{\phi^a(j-1)-\phi^a(j)}{\Delta}-\omega^{ab}\partial_b H(j)\right]\left|\left|\frac{i}{\Delta}\delta^a_b-\omega^{ab}\partial_c\partial_b H(j)\right|\right|$$

$$=\prod_{j=1}^{m}d\lambda_a(j)dc^a(j)d\bar{c}_a(j)\exp\left[i\lambda_a(j)\left[\frac{\phi^a(j-1)-\phi^a(j)}{\Delta}-\omega^{ab}\partial_b H(j)\right]\right.$$

$$\left.+\bar{c}_a(j)\left[\frac{1}{\Delta}\delta^a_b(c^b(j-1)-c^b(j))-\omega^{ac}\partial_c\partial_b H(j)c^b(j)\right]\right]. \quad \text{(A1)}$$

We divide the interval of time in m intervals of length Δ and label the instants of time with $j=0,\ldots,N$. It is clear that upon inserting this expression into the generating functional (3.11), we obtain a path integral of the form

$$Z_{\rm CM}=\int\mathcal{D}\phi\,\mathcal{D}'\lambda\,\mathcal{D}'c\,\mathcal{D}'\bar{c}\,\Gamma^{\{\phi(0)\}}[\phi(t),\lambda(t),c(t),\bar{c}(t)]\,. \quad \text{(A2)}$$

From now on we will omit the indices on ϕ, λ, c, and $\bar{c}$. The symbol $\mathcal{D}'$ indicates that the integration over the initial points for $\lambda(0),c(0),\bar{c}(0)$ is missing:

$$\mathcal{D}'\lambda=\prod_{j=1}^{m}d\lambda(j),\quad \mathcal{D}'c=\prod_{j=1}^{m}dc(j),\mathcal{D}'\bar{c}=\prod_{j=1}^{m}d\bar{c}(j)\,.$$

The integrand Γ is a function of the initial $\phi(0)$ and a functional of $\phi(t),\lambda(t),c(t),\bar{c}(t)$. Having one integration missing for $\lambda,c,\bar{c}$, it will be hard to figure out how to go to the continuum limit in the measure of (A2). Moreover it will be difficult to study the symmetries present in $Z_{\rm CM}$ because the measure $\mathcal{D}\phi$ and the measure $\mathcal{D}'\phi\,\mathcal{D}'c\,\mathcal{D}'\bar{c}$ do not appear on a completely equal footing: there is one integration less in $\lambda,c,\bar{c}$ than in ϕ. In particular, if we have a symmetry (such as Q or $\bar{Q}$) that mixes ϕ with $\bar{c}$, it would be desirable to have a more symmetric measure. A solution to this problem might be the following: insert in (A2) a function $N(\lambda(0),c(0),\bar{c}(0))$ such that

$$\int d\lambda(0)dc(0)d\bar{c}(0)N(\lambda(0),c(0),\bar{c}(0))=1\,. \quad \text{(A3)}$$

Then $Z_{\rm CM}$ in (A2) could be rewritten as

$$Z_{\rm CM}=\int\mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\Gamma^{\{\phi(0)\}}[\phi,\lambda,c,\bar{c}]\times N(\lambda(0),c(0),\bar{c}(0))\,. \quad \text{(A4)}$$

In (A4) the measure is perfectly matched now between $\phi,\lambda,c,\bar{c}$. It will not be a problem to go to the continuum limit. It yields

$$\Gamma^{\{\phi(0)\}}[\phi,\lambda,c,\bar{c}]\propto\exp\left[i\int dt\,\tilde{\mathcal{L}}\right]$$

so that (A4) becomes

$$Z_{\rm CM}=\int\mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}\exp\left[i\int dt\,\tilde{\mathcal{L}}\right]\times N(\lambda(0),c(0),\bar{c}(0))\,. \quad \text{(A5)}$$

[It might be that in choosing a midpoint slicing in (A1), in order to go to a perfect continuum limit the N function gets changed by further surface terms, but this is not important here. What is important is that a surface terms has to survive to guarantee a nonzero $Z_{\rm CM}$ (see the discussion below).] At first sight the function N in the integrand of (A5) seems to spoil the beauty of the path-integral representation of $Z_{\rm CM}$. However, there is a simple reason why this function has to be there: the path integral (A5) without N would vanish because of the zero modes of the kinetic operator of the ghosts. To see this we expand the ghost $c(t)$ in terms of eigenfunctions of its kinetic operator:

$$c(t)=\sum_{a_{(n)}}e^{i\alpha_{(n)}t}c_{(n)}\,,$$

where

$$(\partial_t\delta^a_b-\omega^{ac}\partial_c\partial_b H)c^b_{(n)}=\alpha_{(n)}c^a_{(n)}\,. \quad \text{(A6)}$$

Thus the path integral (A5) without the N function becomes

$$Z_{\rm CM}=\int\mathcal{D}\phi\,\mathcal{D}\lambda\exp\left[i\int dt\,\tilde{\mathcal{L}}_B\right]\times\int dc_{(0)}d\bar{c}_{(0)}\prod_{n=1}^{\infty}dc_{(n)}d\bar{c}_{(n)}e^{\bar{c}_{(n)}\alpha_n c_{(n)}}\,, \quad \text{(A7)}$$

where

$$\tilde{\mathcal{L}}_B=\lambda_a(\dot{\phi}^a-\omega^{ab}\partial_b H)$$

and where $c_{(0)}$ is the zero mode: $\alpha_{(0)}=0$. We assume that there exists only one of these zero modes. This functional integral is clearly zero because the $c_{(0)}$ factor is not contained in the integrand and so performing the integration over $c_{(0)}$ we would get $\int dc_{(0)}=0$. If, instead, we had used the $Z_{\rm CM}$ with the N function inserted, we would have had

$$Z_{CM}=\int \mathcal{D}\phi\,\mathcal{D}\lambda \exp\left[i\int \tilde{\mathcal{L}}_B\right]\int d\bar{c}_{(0)}dc_{(0)}\prod_{n=1}^{\infty} dc_{(n)}d\bar{c}_{(n)}e^{i\bar{c}_{(n)}\alpha_{(n)}c(n)}N\left[\lambda(0),c_{(0)}+\sum_{n=1}^{\infty} c_{(n)},\bar{c}_{(0)}+\sum_{n=1}^{\infty}\bar{c}_{(n)}\right] \tag{A8}$$

and here the $c_{(0)},\bar{c}_{(0)}$ integration is not zero because $c_{(0)}$ and $\bar{c}_{(0)}$ are also contained in the N function. So the function N has the role of regulating the zero modes. Inserting Eq. (A3) into Z_{CM} is analogous to the usual trick[32] of treating the zero modes by inserting a "gauge fixing" of the symmetry associated and the corresponding Faddeev-Popov term. The arbitrariness we have in N [in (A5) we can insert any function that satisfies (A3)] is the same arbitrariness we have in choosing the "gauge fixing" for the zero modes. In the actual applications of the path integral we can choose N as part of the source coupling $\exp(i\int\mathcal{L}_s)$ where $\mathcal{L}_s$ is given in Eq. (3.49). This term contains $\lambda(0)$, $\bar{c}(0)$, and $c(0)$ and so it acts like the N function above. That is the reason why in Eq. (3.11) we did not put any N function. We can conclude then by saying that coupling the ghosts and λ to external currents enables us to have a perfectly symmetric measure, and a nonvanishing Z_{CM}. Besides this, the coupling is also essential if we want to calculate ghost-ghost correlations which are the central elements necessary to detect chaos.

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