Excited-state spectra of de Sitter-space scalar fields

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To illustrate gravitational effects on the dynamics of a quantized field, the spectrum of excited states available to a linear scalar field in de Sitter space is examined in detail. Explicit Schrödinger-picture wave functionals are obtained for the excitation-number eigenstates of the familiar Fock-space description. The field energies of these states are calculated from expectation values of the appropriate Hamiltonian. The Euclidean vacuum state is seen to be the lowest-energy de Sitter-invariant state, although for any massive or nonconformally coupled field instantaneous Hamiltonian diagonalization, breaking de Sitter invariance, yields states of lower energy. All other de Sitter-invariant vacua are characterized by uniform excitation, relative to the Euclidean vacuum, in all field modes. Associated with any vacuum state is a Fock-space basis of excited states. These have field energies in integral increments above the vacuum; i.e., they represent the quantized excitations of the field's normal modes. The energy increments—the (renormalization-independent) energies of individual "particles"—differ markedly from the classical normal-mode frequencies of the field. For fields with combined mass and curvature coupling above a certain threshold, "particle" energies oscillate at late and early times rather than approach a fixed limit; for fields with strong curvature coupling these energies can even become negative. They show exponential rather than oscillatory time dependence if the combined mass and curvature coupling is below the threshold. Excitation energies of a massless, conformally coupled field have the same time dependence as the energy of a classical relativistic particle, but for other fields with the same mass–curvature-coupling sum, "particle" energies have an amplified component, which grows as the radius of the space. The interaction between a field in an excited state and a monopole "detector" is also calculated: Each excitation gives, above the vacuum signal, a finite response with a finite width in energy. These effects on "particle" energies and interactions are due to modulation of the field normal modes in a time-dependent spacetime metric, so similar effects should occur in general spacetimes.

I. INTRODUCTION

Most work on quantum field theory in curved spacetime concerns vacuum states of the fields. Phenomena such as inflation[1] and black-hole evaporation,[2] consequences of curved-spacetime quantum field physics, are associated with fields in a vacuum state. But in ordinary quantum mechanics the dynamics of a system lies in the totality of states accessible to it, and in the transitions which the system can undergo between those states. Hence, this work examines the array of states available to a curved-spacetime quantized field. It treats a particular example, that of a linear (free) real scalar field in de Sitter space.

de Sitter-space scalar field theory is a convenient and useful example. After flat-spacetime theory it is the simplest and best studied quantum field theory, and it is of physical interest in connection with inflationary[1] cosmological models.

Field states are characterized here in several ways. The Fock-state description in terms of creation operators applied to a vacuum state is well known.[3] Explicit wave functionals for the states are obtained as well, utilizing the covariant functional Schrödinger formalism.[4,5] The expectation values of the appropriate Hamiltonian serve to classify states as energy levels. Finally, states are characterized by the response they induce in an idealized monopole "detector" coupled linearly to the field, an example of the quantum dynamics associated with field interactions. Each of these attributes of the field states exhibits distinct features arising from spacetime geometry, i.e., gravitational effects upon the field physics.

This paper is organized as follows. Section II contains the particulars of de Sitter-space scalar field theory, giving the notation and formalism to be used. In Sec. III this formalism is used to construct Fock-space bases of quantum states for the field; wave functionals, number-eigenstate descriptions, and energy levels for these states are calculated. The vacuum states from these bases are examined in Sec. IV, and their descriptions here are related to features of de Sitter-space vacua found in previous works.[6–8] The excited states are treated in Sec. V. Section VI treats "detector responses" to the field in these states. Section VII is a summary and discussion of the results.

Units with $\hbar=c=1$ are used throughout. Sign conventions and general notation follow those of Misner, Thorne, and Wheeler.[9]

II. de SITTER-SPACE SCALAR FIELD THEORY

A. Spacetime geometry

de Sitter space, of $N+1$ dimensions for generality, is the fixed background for the field theory. Coordinates with closed spatial sections will be used, covering the complete de Sitter manifold. Thus the metric is

$$ds^2=-dt^2+a^2\cosh^2(t/a)d\Omega_N^2$$
$$=a^2\sec^2\eta(-d\eta^2+d\Omega_N^2)\ , \qquad (2.1a)$$

where a is a positive constant and $d\Omega_N^2$ is the line element of the unit N-sphere. Comoving-observer proper time $t\in(-\infty,+\infty)$ and conformal time $\eta\in(-\pi/2,+\pi/2)$ are related by

$$\eta\equiv\int_0^{t/a}\frac{du}{\cosh u}=\arctan[\sinh(t/a)]\ . \qquad (2.1b)$$

The spatial coordinates are the angles $\Omega_N\equiv\{\theta_1,\ldots,\theta_{N-1},\phi\}$, with $\theta_i\in[0,\pi]$ for each i and $\phi\in[0,2\pi)$.

B. Scalar field theory

The field is a real scalar φ with no explicit nongravitational couplings. The theory is defined by the Lagrangian density

$$\mathcal{L}=-\tfrac{1}{2}(\nabla_\alpha\varphi\nabla^\alpha\varphi+\mu^2\varphi^2)\ . \qquad (2.2)$$

The corresponding classical field equation is

$$(\Box-\mu^2)\varphi=0\ , \qquad (2.3)$$

with $\Box$ the covariant d'Alembertian in de Sitter space. The constant

$$\mu^2=\mu_0^2+\xi R=\mu_0^2+\xi\frac{N(N+1)}{a^2} \qquad (2.4)$$

is the sum of mass (μ_0^2) and curvature-coupling (ξR) contributions, with $\xi=0$ for minimal and $\xi=(N-1)/(4N)$ for conformal coupling.

Canonical quantization of the field is represented in terms of an expansion in normal-mode solutions of Eq. (2.3). That equation is readily solved by separation of variables; the desired expansion may be written

$$\varphi(t,\Omega_N)=a^{-N/2}\sum_L[b_L\chi_L(t)\mathcal{Y}_L(\Omega_N)+\text{ H.c.}]\ . \qquad (2.5)$$

Here $\mathcal{Y}_L$ is a spherical harmonic in N dimensions, with L denoting the complete set of N angular-momentum quantum numbers, and χ_L is a complex, "positive frequency" solution of the separated field equation. These functions are detailed below. The amplitude operators b_L satisfy $[b_L,b_{L'}^\dagger]=\delta_{LL'}$, etc., implementing the canonical commutation relations.

The harmonics $\mathcal{Y}_L$ are eigenfunctions of the Laplacian on the unit N-sphere, with eigenvalues $-L^2=-l(l+N-1)$, where l is any non-negative integer.[10] Their form may be found by successive separation of that eigenvalue equation, yielding

$$\mathcal{Y}_L(\Omega_N)=\prod_{j=1}^{N-1}\left[\left[\lambda_j\frac{(\lambda_j-\frac{1}{2}+\lambda_{j+1})!}{(\lambda_j-\frac{1}{2}-\lambda_{j+1})!}\right]^{1/2}\times(\sin\theta_j)^{(j+1-N)/2}\mathrm{P}_{\lambda_j-(1/2)}^{-\lambda_{j+1}}(\cos\theta_j)\right]\times[\pi(1+\delta_{m0})]^{-1/2}\begin{cases}\cos(m\phi)\\ \sin(m\phi)\ ,\end{cases} \qquad (2.6a)$$

with

$$\lambda_j\equiv l_j+\frac{N-j}{2} \qquad (2.6b)$$

and

$$l\equiv l_1\geq l_2\geq\cdots\geq l_{N-1}\geq l_N\equiv m\geq 0\ , \qquad (2.6c)$$

the integers $l_1,\ldots,l_N$ (plus an index $I=\pm$ for the cosine or sine choice) constituting the set L. The P's are associated Legendre functions of the first kind. Real harmonics are chosen for convenience. These $\mathcal{Y}_L$ constitute a complete, orthonormal set on the N-sphere. The number of harmonics with a given l value is[10]

$$D_N(l)=\frac{(2l+N-1)(l+N-2)!}{(N-1)!l!}$$
$$=\frac{2\lambda[\lambda+(N-3)/2]!}{(N-1)![\lambda-(N-1)/2]!}\ , \qquad (2.7)$$

with $\lambda\equiv\lambda_1$; i.e., this is the degeneracy of modes with total angular momentum l.

The functions χ_L are solutions of the differential equation

$$\left[\frac{d^2}{dt^2}+\frac{N}{a}\tanh(t/a)\frac{d}{dt}+\omega_L^2(t)\right]\chi_L(t)=0\ , \qquad (2.8a)$$

with

$$\omega_L(t)\equiv\left[\mu^2+\frac{L^2}{a^2\cosh^2(t/a)}\right]^{1/2}, \qquad (2.8b)$$

satisfying the "positive frequency" normalization condition

$$i\cosh^N(t/a)\left[\chi_L^*\frac{\overleftrightarrow{\partial}}{\partial t}\chi_L\right]=1\ . \qquad (2.9)$$

They too can be given explicitly in terms of associated Legendre functions of the first kind:

$$\chi_L(t)=\left[\frac{a\pi/2}{\sinh(\pi qa)}\right]^{1/2}\cosh^{-N/2}(t/a)\times\{\kappa_L^{(+)}\mathrm{P}_{\lambda-(1/2)}^{-iqa}[\tanh(t/a)]+\kappa_L^{(-)}\mathrm{P}_{\lambda-(1/2)}^{+iqa}[\tanh(t/a)]\}\ , \qquad (2.10a)$$

with

$$q\equiv\left[\mu^2-\frac{N^2}{4a^2}\right]^{1/2} \qquad (2.10b)$$

and $\lambda=\lambda_1$ as above. The coefficients $\kappa_L^{(\pm)}$ are c numbers, obeying $|\kappa_L^{(+)}|^2-|\kappa_L^{(-)}|^2=1$ (for positive real q) or

$2\,\mathrm{Im}(\kappa_L^{(+)}\kappa_L^{(-)*})=\pm 1$ [for imaginary q, with the sign that of $\sin(\pi iqa)$] in consequence of condition (2.9). A choice of $\kappa_L^{(\pm)}$ values effects a choice of positive-frequency normal modes. Thus that mode choice involves one complex parameter per mode, the overall phase being immaterial. Equivalently, any choice of χ_L is completely (up to a phase) determined by, e.g., the value of $\dot{\chi}_L(0)/\chi_L(0)$, where the overdot denotes the derivative with respect to t.

It is also useful to express the χ_L in terms of real solutions of the separated field equation. They can be written

$$\chi_L(t)=\chi_L(0)f_L^{(1)}(\eta)+a\dot{\chi}_L(0)f_L^{(2)}(\eta) . \tag{2.11}$$

The real functions $f_L^{(1)}$ and $f_L^{(2)}$, functions of conformal time for convenience, are solutions of the differential equation

$$\left[\frac{d^2}{d\eta^2}+(N-1)\tan\eta\frac{d}{d\eta}+\tilde{\omega}_L^2(\eta)\right]f_L^{(1,2)}(\eta)=0 , \tag{2.12a}$$

with

$$\tilde{\omega}_L(\eta)=a\cosh(t/a)\omega_L(t)=(L^2+\mu^2\sec^2\eta)^{1/2} , \tag{2.12b}$$

satisfying the initial conditions

$$f_L^{(1)}(0)=1 \text{ and } \frac{df_L^{(1)}}{d\eta}(0)=0 \tag{2.13a}$$

and

$$f_L^{(2)}(0)=0 \text{ and } \frac{df_L^{(2)}}{d\eta}(0)=1 . \tag{2.13b}$$

They can be given explicitly in terms of hypergeometric functions:

$$f_L^{(1)}(\eta)=\cos^{N/2+iqa}\eta \times F\left[\frac{\lambda+\frac{1}{2}+iqa}{2},\frac{-\lambda+\frac{1}{2}+iqa}{2};\frac{1}{2};\sin^2\eta\right] \tag{2.14a}$$

and

$$f_L^{(2)}(\eta)=\cos^{N/2+iqa}\eta\sin\eta \times F\left[\frac{\lambda+\frac{3}{2}+iqa}{2},\frac{-\lambda+\frac{3}{2}+iqa}{2};\frac{3}{2};\sin^2\eta\right], \tag{2.14b}$$

with q and λ as above.

For a massless, conformally coupled field or any other with the same μ value, viz., $\mu^2=(N^2-1)/(4a^2)$, the χ_L take a simple form obtained via the conformal invariance of the field equation. In such a case they are

$$\chi_L(t)=\left[\frac{a}{2\lambda}\right]^{1/2}\cosh^{(1-N)/2}(t/a) \times(k_L^{(+)}e^{-i\lambda\eta(t)}+k_L^{(-)}e^{+i\lambda\eta(t)}) , \tag{2.15}$$

with $\eta(t)$ the function (2.1b) and the coefficients $k_L^{(\pm)}$, distinct from the $\kappa_L^{(\pm)}$ of Eq. (2.10a), normalized via $|k_L^{(+)}|^2-|k_L^{(-)}|^2=1$.

C. Functional Schrödinger formalism

In the functional Schrödinger formalism[4,5] quantum states of the field are represented by explicit wave functionals, giving probability amplitudes for the values of a complete set of commuting observables for the theory. The values of the field on a constant-time hypersurface constitute such a set, giving a "field-coordinate" representation. The evolution of any wave functional from one hypersurface to another is described by the functional Schrödinger equation

$$\left[i\frac{\partial}{\partial t}-H(t)\right]\Psi[\varphi(\Omega_N),t]=0 , \tag{2.16a}$$

or equivalently

$$\left[i\frac{\partial}{\partial\eta}-\tilde{H}(\eta)\right]\Psi[\varphi(\Omega_N),\eta]=0 . \tag{2.16b}$$

That is, the field Hamiltonian H generates evolution in proper time t, as does the Hamiltonian $\tilde{H}=a\cosh(t/a)H$ in conformal time η.

The appropriate Hamiltonian operators are obtained from integrals over a constant-time hypersurface of the field energy density:

$$H(t)=\int d\Omega_N a^N\cosh^N(t/a)|g_{tt}|^{-1/2}T_{tt} \tag{2.17a}$$

or

$$\tilde{H}(\eta)=\int d\Omega_N a^N\sec^N\eta|g_{\eta\eta}|^{-1/2}T_{\eta\eta} , \tag{2.17b}$$

where $d\Omega_N$ is the volume element on the unit N-sphere. The energy densities are components of the *canonical* stress-energy tensor

$$T_{\alpha\beta}=\nabla_\alpha\varphi\nabla_\beta\varphi+g_{\alpha\beta}\mathcal{L} , \tag{2.18}$$

with $\mathcal{L}$ from Eq. (2.2). The forms of H and $\tilde{H}$ as functional differential operators follow from the replacements

$$\Pi=\nabla_t\varphi=ig_{tt}|g|^{-1/2}\frac{\delta}{\delta\varphi(\Omega_N)} \tag{2.19a}$$

and

$$\tilde{\Pi}=\nabla_\eta\varphi=ig_{\eta\eta}|\tilde{g}|^{-1/2}\frac{\delta}{\delta\varphi(\Omega_N)} , \tag{2.19b}$$

where g is the determinant of the metric as given by the first line of Eq. (2.1a) and $\tilde{g}$ that of the second line. These implement the canonical commutation relations between the conjugate momentum Π or $\tilde{\Pi}$ and the field φ in the field-coordinate representation.[4] The field operators and their spatial derivatives (which commute with all the field operators on a constant-time hypersurface) are replaced by their values.

These Hamiltonians take more useful forms in terms of a normal-mode expansion. If the field on a constant-time hypersurface is expanded in spherical harmonics, thus

$$\varphi(\Omega_N)=a^{(1-N)/2}\sum_L y_L \mathcal{Y}_L(\Omega_N)\ ; \qquad (2.20)$$

the amplitudes y_L constitute a complete set of commuting observables for the theory, equivalent to the field values themselves. Using the corresponding expansion for the spatial derivatives of φ and with $\delta/\delta\varphi$ transformed via the chain rule, the Hamiltonians can be written

$$H(t)=\sum_L \left[-\frac{1}{2a}\,\mathrm{sech}^N(t/a)\frac{\delta^2}{\delta y_L^2} + \frac{a}{2}\cosh^N(t/a)\omega_L^2(t)y_L^2 \right] \qquad (2.21a)$$

and

$$\tilde{H}(\eta)=\sum_L \left[-\frac{1}{2}\cos^{N-1}\eta\frac{\delta^2}{\delta y_L^2} + \frac{1}{2}\sec^{N-1}\eta\tilde{\omega}_L^2(\eta)y_L^2 \right], \qquad (2.21b)$$

where ω_L and $\tilde{\omega}_L$ are given by Eqs. (2.8b) and (2.12b), respectively.

A general solution of this functional Schrödinger equation can be given, in the form of a propagator.[11] The propagator $G[\varphi_1,\eta_1;\varphi_0,\eta_0]$, the amplitude for evolution from field configuration φ_0 on the $\eta=\eta_0$ hypersurface to configuration φ_1 at $\eta=\eta_1$, is given by the field path integral

$$G[\varphi_1,\eta_1;\varphi_0,\eta_0] = \int e^{iS[\Phi]}\mathcal{D}\varphi = \int \exp\left[i\int_{\eta_0}^{\eta_1} d\eta \int d\Omega_N (a\sec\eta)^{N+1}\mathcal{L} \right]\mathcal{D}\varphi\ , \qquad (2.22)$$

where the action integral is taken over the spacetime region between the $\eta=\eta_0$ and η_1 hypersurfaces, and the functional integral is over all fields in that region taking configuration φ_0 on the initial hypersurface and configuration φ_1 on the final one. Since the action is quadratic in the field, the path integral can be evaluated exactly in terms of the action for the classical field with the given boundary conditions.[12] The result is conveniently expressed as a function of normal-mode amplitudes as in Eq. (2.20): With φ_0 thus expanded with coefficients x_L, and φ_1 likewise with coefficients y_L, and with $\eta_0=0$ for simplicity, the propagator takes the form

$$G[\{y_L\},\eta;\{x_L\},0] = \prod_L [2\pi i f_L^{(2)}(\eta)]^{-1/2} \times\exp\left[\frac{i}{2f_L^{(2)}(\eta)} \left[y_L^2\frac{df_L^{(2)}}{d\eta}(\eta)\sec^{N-1}\eta + x_L^2 f_L^{(1)}(\eta) - 2y_L x_L \right]\right], \qquad (2.23)$$

where $f_L^{(1)}$ and $f_L^{(2)}$ are as given by Eqs. (2.14). That this is the desired propagator may be seen directly, as it solves Eqs. (2.16) with Hamiltonians (2.21), and reduces in the limit $\eta\to 0$ to a product of delta functions in $y_L - x_L$. Any solution of the functional Schrödinger equation can be obtained through

$$\Psi[\{y_L\},\eta]=\int G[\{y_L\},\eta;\{x_L\},0]\Psi[\{x_L\},0]\prod_L dx_L \qquad (2.24)$$

from any initial values $\Psi[\{x_L\},0]$.

The canonical Hamiltonian H differs, in general, from the field energy obtained from the *gravitational* stress-energy tensor which appears in the Einstein field equations. That tensor, obtained by variation of the field action with respect to metric, is[13]

$$T_{\alpha\beta}^{(g)}=(1-2\xi)\partial_\alpha\varphi\partial_\beta\varphi + (2\xi-\tfrac{1}{2})g_{\alpha\beta}g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi - \tfrac{1}{2}g_{\alpha\beta}\mu_0^2\varphi^2 + 2\xi\varphi(g_{\alpha\beta}\Box\varphi - \nabla_\alpha\partial_\beta\varphi + \tfrac{1}{2}G_{\alpha\beta}\varphi)\ , \qquad (2.25)$$

with $G_{\alpha\beta}$ the Einstein curvature tensor. The "gravitational Hamiltonian" obtained using this in the integral (2.17a), with the same operator implementations as for H, is

$$H^{(g)}(t)=H(t)-\frac{N\xi}{a}\sum_L \left[\cosh^N(t/a)y_L^2 + i\tanh(t/a) \times\left[y_L\frac{\delta}{\delta y_L} + \frac{\delta}{\delta y_L}y_L \right]\right], \qquad (2.26)$$

where the factor ordering of the last two terms is chosen to give a Hermitian operator. The gravitational and canonical operators coincide only for minimally coupled fields. ("Gravitational" here refers to the gravitational stress-energy tensor; both H and $H^{(g)}$ neglect the gravitational *self*-energy of the field, necessarily since the spacetime geometry is fixed.) In any case, however, the canonical Hamiltonian is the correct time-evolution operator for use in the functional Schrödinger equation,[14,15] as is evident since the path integral (2.22) solves that equation with Hamiltonian H, not $H^{(g)}$.

The functional Schrödinger description of the theory and the canonical operator formulation are, of course, equivalent. The canonical amplitude operators b_L can be expressed as functional differential operators by inverting expansion (2.5), via a Klein-Gordon inner product, and using replacement (2.19a) and expansion (2.20) to obtain[11]

$$b_L=-ia^{1/2}\cosh^N(t/a)\dot{\chi}_L^*(t)y_L + a^{-1/2}\chi_L^*(t)\frac{\delta}{\delta y_L}\ . \qquad (2.27)$$

This form satisfies the appropriate commutation relations, as may be seen directly. Conversely, the Hamiltonians can be written in terms of the canonical expansion, e.g.,

$$H^{(g)}(t)=\frac{\cosh^N(t/a)}{2}\sum_L\left\{(b_L^\dagger b_L+b_L b_L^\dagger)\left[\dot\chi_L\dot\chi_L^*+\frac{2N\xi}{a}(\chi_L\dot\chi_L^*+\chi_L^*\dot\chi_L)\tanh(t/a)+\left[\omega_L^2-\frac{2N\xi}{a^2}\right]\chi_L\chi_L^*\right]\right.$$
$$+b_L^2\left[\dot\chi_L^2+\frac{4N\xi}{a}\tanh(t/a)\chi_L\dot\chi_L+\left[\omega_L^2-\frac{2N\xi}{a^2}\right]\chi_L^2\right]$$
$$\left.+b_L^{\dagger 2}\left[\dot\chi_L^{*2}+\frac{4N\xi}{a}\tanh(t/a)\chi_L^*\dot\chi_L^*+\left[\omega_L^2-\frac{2N\xi}{a^2}\right]\chi_L^{*2}\right]\right\}, \tag{2.28}$$

while H is written as the specialization of this to $\xi=0$ (with μ fixed), and $\tilde H$ similarly.

III. QUANTUM-STATE SPECTRA

Sets of wave functionals spanning the Fock spaces of states available to the field can be obtained by specifying convenient bases on, say, the $t=\eta=0$ hypersurface, and using formula (2.24). Vacuum-state wave functionals are products of Gaussians in the field normal-mode amplitudes;[5,7,11,15,16] this suggests the ansatz

$$\Psi_{\{n_L\}}[\{x_L\},0]=\prod_L\psi_{n_L}(x_L,0)=\prod_L\left[\frac{\mathrm{Re}\gamma_L}{\pi}\right]^{1/4}(2^{n_L}n_L!)^{-1/2}H_{n_L}[(\mathrm{Re}\gamma_L)^{1/2}x_L]\exp(-\tfrac{1}{2}\gamma_L x_L^2) \tag{3.1}$$

for initial wave functionals. Here the H_{n_L} are Hermite polynomials (not to be confused with the Hamiltonian H), and the γ_L are arbitrary c numbers with positive real parts. With the n_L ranging over all non-negative integers, these functionals form an orthonormal set, complete on a mode-by-mode basis. The propagator (2.23) yields the corresponding set of wave functionals at all times:

$$\Psi_{\{n_L\}}[\{y_L\},\eta]=\prod_L\psi_{n_L}(y_L,\eta)=\prod_L\left[\frac{\mathrm{Re}\gamma_L}{\pi}\right]^{1/4}Q_L^{-1/2}(\eta)\left[\frac{Q_L^*(\eta)}{Q_L(\eta)}\right]^{n_L/2}(2^{n_L}n_L!)^{-1/2}$$
$$\times H_{n_L}[\Delta_L^{1/2}(\eta)y_L]\exp\{-\tfrac{1}{2}[\Delta_L(\eta)-i\bar\Delta_L(\eta)]y_L^2\}, \tag{3.2a}$$

with

$$Q_L(\eta)\equiv f_L^{(1)}(\eta)+i\gamma_L f_L^{(2)}(\eta), \tag{3.2b}$$

$$\Delta_L(\eta)\equiv\frac{\mathrm{Re}\gamma_L}{|Q_L(\eta)|^2}, \tag{3.2c}$$

and

$$\bar\Delta_L(\eta)\equiv\frac{\sec^{N-1}\eta\frac{d}{d\eta}[|Q_L(\eta)|^2]}{2|Q_L(\eta)|^2}. \tag{3.2d}$$

The indices n_L are constants of the wave-functional evolution.

The wave functionals $\Psi_{\{n_L\}}$ correspond to Fock-space number eigenstates, with the choice of initial widths γ_L equivalent to the choice of positive-frequency modes. The action of an operator b_L on the single-mode wave function ψ_{n_L} is, from form (2.27) and the Hermite-polynomial recursion relations,

$$b\psi_n=\left[-\Delta^{-1/2}[ia^{1/2}\sec^N\eta\dot\chi^*+a^{-1/2}\chi^*(\Delta-i\bar\Delta)]\times\frac{H_{n+1}+2nH_{n-1}}{2H_n}+2a^{-1/2}\chi^*\Delta^{1/2}\frac{nH_{n-1}}{H_n}\right]\psi_n, \tag{3.3}$$

where the mode label L is suppressed. The particular choice of initial width

$$\gamma_L=-ia\frac{\dot\chi_L^*(0)}{\chi_L^*(0)}, \tag{3.4}$$

given the positive-frequency function χ_L, or equivalently, the choice of χ_L thus defined, given γ_L, implies $\chi_L/\chi_L(0)=Q_L^*$. With this Eq. (3.3) can be reduced, using condition (2.9) and the Wronskian relation for Eq. (2.12a), to $b\psi_n=n^{1/2}\psi_{n-1}$, provided only that $\chi_L(0)$ is real (as may be assumed without loss of generality). The choice (3.4) similarly implies $b^\dagger\psi_n=(n+1)^{1/2}\psi_{n+1}$. The relations $b^\dagger b\psi_n=n\psi_n$ and $\psi_n=(n!)^{-1/2}b^{\dagger n}\psi_0$, characterizing number eigenstates, then follow.

The field energies of the states described by the $\Psi_{\{n_L\}}$ can be measured by the expectation values of $H^{(g)}$ in those states. These may be calculated directly, using the operators (2.26) and (2.21a):

$$\langle H^{(g)}\rangle_{\{n_L\}}=\int\Psi_{\{n_L\}}^*H^{(g)}\Psi_{\{n_L\}}\prod_L dy_L$$
$$=\sum_L(n_L+\tfrac{1}{2})\omega_L(1+\sigma_L), \tag{3.5a}$$

with

$$\sigma_L \equiv \frac{(\tilde{\omega}_L \sec^{N-1}\eta - \Delta_L)^2 + \bar{\Delta}_L^2}{2\Delta_L \tilde{\omega}_L \sec^{N-1}\eta} + \frac{N\xi}{\Delta_L \tilde{\omega}_L}(2\bar{\Delta}_L \tan\eta - \sec^{N+1}\eta) ; \tag{3.5b}$$

the same result is obtained from form (2.28), using mode choice (3.4) and the associated number-eigenstate properties. These expectation values are, of course, infinite, and a subtraction procedure is required to extract meaningful energies from them. This is treated further in Secs. IV and V below.

The wave functionals $\Psi_{\{n_L\}}$ are not in general eigenfunctions of the gravitational or canonical Hamiltonians. Except in one case, they can at most be chosen to be either (not both) at a particular instant, effecting "instantaneous Hamiltonian diagonalization."[17] The diagonalization of $H^{(g)}$, at time t_D or η_D, is achieved via the positive-frequency-mode choice defined by

$$\dot{\chi}_L(t_D) = -i\nu_L(t_D)\chi_L(t_D) \tag{3.6a}$$

with

$$\nu_L(t_D) \equiv + \left[\omega_L^2(t_D) - \frac{2N\xi}{a^2} - \frac{4N^2\xi^2}{a^2}\tanh^2(t_D/a)\right]^{1/2} - i\frac{2N\xi}{a}\tanh(t_D/a) , \tag{3.6b}$$

as follows from form (2.28). For the corresponding eigenfunctions this implies

$$\frac{dQ_L}{d\eta}(\eta_D) = +i\tilde{\nu}_L^*(\eta_D)Q_L(\eta_D) ; \tag{3.7a}$$

hence

$$\gamma_L^{(D)} = i\left.\frac{\dfrac{df_L^{(1)}}{d\eta} - i\tilde{\nu}_L^* f_L^{(1)}}{\dfrac{df_L^{(2)}}{d\eta} - i\tilde{\nu}_L^* f_L^{(2)}}\right|_{\eta=\eta_D} , \tag{3.7b}$$

with $\tilde{\nu}_L(\eta_D) \equiv a\cosh(t_D/a)\nu_L(t_D)$, and conversely. Conditions (3.6) define functions satisfying Eq. (2.9)—equivalently, the $\gamma_L^{(D)}$ have positive real parts—only if the argument of the square root in Eq. (3.6b) is positive, requiring

$$\mu_0^2 a^2 + N(N-1)\xi\left[1 - \frac{4N\xi}{N-1}\tanh^2(t_D/a)\right] + L^2\operatorname{sech}^2(t_D/a) > 0 . \tag{3.8}$$

If this inequality is violated $H^{(g)}(t_D)$ cannot be diagonalized by Fock-space number eigenstates. It is satisfied, however, by fields with any positive mass and any curvature coupling between the minimal and conformal values inclusively, for all modes at any time. When it is satisfied, condition (3.6) or (3.7) also minimizes $\langle H^{(g)}\rangle_{\{n_L\}}(t_D)$ on a mode-by-mode basis. The instantaneous diagonalization of H and $\tilde{H}$ is effected by conditions such as (3.6) and (3.7), with ω_L replacing ν_L and $\tilde{\omega}_L$ replacing $\tilde{\nu}_L$, and can be achieved for any field with $\mu > 0$. But none of these diagonalization conditions is preserved by Eq. (2.8a) or (2.12a)—no single positive-frequency-mode choice or wave-functional choice diagonalizes any of the Hamiltonians for any interval of time—unless the field is massless and conformally coupled, in which case the $\Psi_{\{n_L\}}$ with $\gamma_L = \lambda$ are always eigenfunctions of $H^{(g)}$.

IV. VACUUM STATES

The $\Psi_{\{n_L\}}$ with $n_L = 0$ for all modes L describe ostensible vacuum states of the field. These include the one-parameter family of de Sitter-space vacua found in the literature,[6–8] and more: They form an infinite-complex-parameter family, each defined by a choice of γ_L for each mode. Invariance requirements, however, restrict the range of suitable vacuum states. The vacua can also be ordered by energy using result (3.5), in accord with the flat-spacetime notion of the vacuum as lowest-energy state.

A. de Sitter invariance

Requiring invariance under the de Sitter group imposes several restrictions on the choice of vacuum state. Invariance under the proper or connected de Sitter group entails invariance under the subgroup $SO(N+1)$ of spatial rotations, and under boosts (i.e., transformations equivalent to boosts in the higher-dimensional Minkowski space in which the de Sitter space may be embedded as a timelike hyperboloid). Invariance under the full de Sitter group requires, in addition, invariance under the discrete transformations spatial (parity) inversion and time reversal.

The wave functional of a rotationally invariant vacuum must assign the same probability to field configurations, on any constant-time hypersurface, related by a rotation. This is equivalent to the condition

$$\gamma_L = \gamma_l , \tag{4.1}$$

for all L, i.e., the width of the wave functional in each mode must depend only on the mode's total angular momentum. Then the wave functional depends on the field only through sums of y_L^2 over modes with the same l value. Since such modes give rise to irreducible, unitary representations of the rotation group, these sums are rotationally invariant.

The requirement of boost invariance, in addition to rotational invariance, restricts the range of vacuum states to a one-complex-parameter family. A boost tilts the constant-time surfaces; equivalently, boosted field configurations on fixed hypersurfaces are linear combinations of configurations of the field and its time derivatives. Thus eigenkets of an infinitesimally boosted field are eigenkets of a linear combination of the original field and its conjugate momentum operator. The corresponding change in the wave functional i.e., the difference between the projection of the state on these and its projection on the original field eigenkets, is given by the operation of a boost generator on the state. This must be zero for boost-invariant states. Such a generator is obtained from an integral analogous to those in Eqs. (2.17) for the generators of time translations. The generator of a boost in the direction of the θ_1 polar axis is

$$M(\eta)=\int d\Omega_N(a\sec\eta)^{N-1}T_{\eta\alpha}\times(\cos\eta\cos\theta_1\delta^\alpha_\eta-\sin\eta\sin\theta_1\delta^\alpha_{\theta_1}) . \quad (4.2)$$

The vector in the last set of parentheses in the integrand is the generator on the spacetime of boosts in the desired direction. The canonical stress-energy tensor (2.18) is used; thus M has the appropriate commutation relations with the field. Using the formalism of Sec. II C and the properties of the $\mathcal{Y}_L$ from Eq. (2.6a), M can be written

$$M=\frac{1}{2}\sum_L\left[\frac{\left[l+\frac{N}{2}\right]^2-\left[l_2+\frac{N-2}{2}\right]^2}{\left[l+\frac{N}{2}\right]^2-\frac{1}{4}}\right]^{1/2}\left[-\cos^N\eta\frac{\delta}{\delta y_L}\frac{\delta}{\delta y_{L^+}}+\frac{1}{2}\cos^{2-N}\eta(\varpi_L^2+\varpi_{L^+}^2-N)y_Ly_{L^+}+i\sin\eta\left[ly_L\frac{\delta}{\delta y_{L^+}}-(l+N)y_{L^+}\frac{\delta}{\delta y_L}\right]\right] , \quad (4.3)$$

with $L^+=\{l+1,l_2,\ldots,l_N,I\}$, given $L=\{l,l_2,\ldots,l_N,I\}$. The invariance condition $M\Psi_{\{0\}}=0$, for a vacuum satisfying the rotational-invariance condition (4.1), is equivalent to

$$\mathcal{M}_l(\eta)\equiv\cos^N\eta\Gamma_l\Gamma_{l+1}-\frac{1}{2}\cos^{2-N}\eta(\varpi_l^2+\varpi_{l+1}^2-N)+i\sin\eta[l\Gamma_{l+1}-(l+N)\Gamma_l]=0 \quad (4.4)$$

for all l. Here Γ_l denotes $\Delta_l-i\bar{\Delta}_l$, and the notation reflects the dependence of ω_L, Δ_L, and $\bar{\Delta}_L$ on l alone. The functions $\mathcal{M}_l$ satisfy

$$\frac{d\mathcal{M}_l}{d\eta}=-i\cos^{N-1}\eta(\Gamma_l+\Gamma_{l+1})\mathcal{M}_l \quad (4.5)$$

and hence vanish at all η if and only if they vanish at $\eta=0$. Condition (4.4) is thus equivalent to

$$\gamma_l\gamma_{l+1}=l(l+N)+\mu^2a^2=\left[l+\frac{N}{2}\right]^2+q^2a^2 \quad (4.6)$$

for all l. (This is the boost-invariance condition obtained by Burges,[7] expressed in the notation used here. With the representations of de Sitter-group generators appropriate to the coordinate system used here, subtractions such as those used by Floreanini, Hill, and Jackiw[16] are not needed, and the associated phases, i.e., generator eigenvalues, do not appear.) Rotational invariance plus invariance under boosts in one direction are equivalent to invariance under all boosts. Any rotation- and boost-invariant vacuum is characterized by conditions (4.1) and (4.6), and completely determined by the choice of a single γ_L for any mode.

Requiring spatial-inversion invariance imposes no constraints on the choice of vacuum. The spherical harmonics (2.6a) are eigenfunctions of the parity-inversion transformation $\theta_i\to\pi-\theta_i$, for i from 1 to $N-1$, and $\phi\to\phi+\pi$, with eigenvalues $(-1)^l$. Hence, the inversion of a field configuration with mode amplitudes y_L has amplitudes $(-1)^ly_L$. All the vacuum wave functionals $\Psi_{\{0\}}$ are invariant under this transformation.

Imposing time-reversal invariance, as well as rotational and boost invariance, reduces the range of vacua to a one-*real*-parameter family. The wave functional of a time-reversal-invariant vacuum must satisfy $\Psi_{\{0\}}[\{y_L\},\eta]=\Psi^*_{\{0\}}[\{y_L\},-\eta]$ for all η. This is equivalent to the condition $\gamma_L=\gamma_L^*$ for all modes, i.e., the initial widths defining the state must be real.

The Chernikov-Tagirov[6] or Euclidean[8] vacuum has a wave functional $\Psi^{(E)}_{\{0\}}$ characterized by the initial widths[11]

$$\gamma_L^{(E)}=2\frac{\Gamma\left[\frac{\frac{3}{2}+\lambda+iqa}{2}\right]\Gamma\left[\frac{\frac{3}{2}+\lambda-iqa}{2}\right]}{\Gamma\left[\frac{\frac{1}{2}+\lambda-iqa}{2}\right]\Gamma\left[\frac{\frac{1}{2}+\lambda+iqa}{2}\right]} , \quad (4.7)$$

with λ and q as in Sec. II B. These satisfy conditions (4.1) and (4.6), and are real; the Euclidean vacuum is invariant under the full de Sitter group. Any other vacuum state can be described in relation to the Euclidean vacuum: its initial widths given by $\gamma_L=r_L\gamma_L^{(E)}$, where the r_L are c numbers with positive real parts. The rotational-invariance condition is then $r_L=r_l$, the boost-invariance condition $r_lr_{l+1}=1$. Vacua invariant under the connected de Sitter group are thus identified by the single parameter $r\equiv r_0$. For fully de Sitter-invariant vacua this parameter is real. [The parameter λ used by Chernikov and Tagirov[6] and Burges[7] to label de Sitter-invariant vacua, not to be confused with angular-momentum index λ used here, is simply $(1-r)/(1+r)$.]

"In" and "out" vacua of de Sitter-space fields are sometimes employed.[18] These may be defined for fields with $\mu>N/(2a)$, i.e., with positive real q, and are invariant under the connected de Sitter group. The "in" vacuum is that defined by the choice of positive-frequency functions $\chi_L^{(\text{in})}$ satisfying

$$\lim_{t\to-\infty}[\cosh^{N/2}(t/a)\chi_L^{(\text{in})}(t)]\sim\frac{e^{-iqt}}{(2q)^{1/2}} . \quad (4.8)$$

These are given by Eq. (2.10a), with coefficients

$$\kappa_L^{(+)}=\frac{\sin[\pi(\lambda-\frac{1}{2}+iqa)]}{i\sinh(\pi qa)}\frac{\Gamma(\lambda+\frac{1}{2}+iqa)}{\Gamma(\lambda+\frac{1}{2}-iqa)}\times\exp[-i\arg\Gamma(1+iqa)] \quad (4.9a)$$

and

$$\kappa_L^{(-)}=\frac{i\sin[\pi(\lambda-\frac{1}{2})]}{\sinh(\pi qa)}\exp[-i\arg\Gamma(1+iqa)] \ . \quad (4.9b)$$

The "out" vacuum is defined by the positive-frequency mode choice with functions $\chi_L^{(\text{out})}$, satisfying

$$\lim_{t\to+\infty}[\cosh^{N/2}(t/a)\chi_L^{(\text{out})}(t)]\sim\frac{e^{-iqt}}{(2q)^{1/2}} \ . \quad (4.10)$$

These too are given by Eq. (2.10a), with coefficients

$$\kappa_L^{(+)}=\frac{\Gamma(1+iqa)}{[\pi qa/\sinh(\pi qa)]^{1/2}}=\exp[i\arg\Gamma(1+iqa)] \quad (4.11a)$$

and

$$\kappa_L^{(-)}=0 \ . \quad (4.11b)$$

The initial widths for the "in"- and "out"-vacuum wave functionals follow from Eq. (3.4):

$$\gamma_L^{(\text{in})}=+i\tan\left[\frac{\pi}{2}(\lambda-\tfrac{1}{2}-iqa)\right]\gamma_L^{(E)} \ , \quad (4.12a)$$

and

$$\gamma_L^{(\text{out})}=-i\tan\left[\frac{\pi}{2}(\lambda-\tfrac{1}{2}+iqa)\right]\gamma_L^{(E)}=\gamma_L^{(\text{in})*} \ . \quad (4.12b)$$

These satisfy conditions (4.1) and (4.6). They are real if N is even; in that case the "in" and "out" vacua are the same state, which is time-reversal invariant. If N is odd the "in" and "out" vacua are distinct, and each is the time reversal of the other.

With a single exception, the Hamiltonian-diagonalizing vacua are *not* de Sitter invariant.[19] The $\gamma_L^{(D)}$ of Eq. (3.7b) do satisfy condition (4.1). But condition (3.7a) implies $\Gamma_l(\eta_D)=\tilde{v}_l^*(\eta_D)\sec^{N-1}\eta_D$; given this, $\mathcal{M}_l$ vanishes at η_D, and hence always, for all l only for a massless, conformally coupled field. In that case the $H^{(g)}$-diagonalizing and Euclidean vacua coincide. For H- and $\tilde{H}$-diagonalizing vacua, with $\Gamma_l(\eta_D)=\tilde{\omega}_l(\eta_D)\sec^{N-1}\eta_D$, $\mathcal{M}_l(\eta_D)$ is nonzero for any field with $\mu>0$.

For fields with $\mu=0$, e.g., a massless, minimally coupled field, condition (4.6) cannot be satisfied by any choice of γ_l's such that γ_0 has positive real part. Such a field has no de Sitter-invariant vacuum state.[8]

B. Vacuum energies

The vacuum states can be characterized by their energies, as given by Eqs. (3.5) with all n_L zero. That sum of mode energies diverges for any state; some subtraction procedure must be used to obtain a finite result. This means, however, that energy *differences* between states given by Eqs. (3.5) are significant. That is, any meaningful regularized or renormalized energy will give the same values for such energy differences as the formal expressions.[20,21]

The Euclidean vacuum is the lowest-energy de Sitter-invariant state. The difference in energy between any other rotationally invariant vacuum and the Euclidean vacuum is

$$\langle H^{(g)}\rangle_{\{0\}}-\langle H^{(g)}\rangle_{\{0\}}^{(E)}=\frac{1}{2}\sum_{l=0}^{\infty}D_N(l)\omega_l(\sigma_l-\sigma_l^{(E)}) \ , \quad (4.13)$$

with $D_N(l)$ from Eq. (2.7). The limiting behavior of σ_l is[22]

$$\sigma_l=\frac{(1-\text{Re}r_l)^2+(\text{Im}r_l)^2}{2\,\text{Re}r_l}[1+O(\lambda^{-1})] \quad (4.14)$$

for $\lambda\gg1$ and $\mu a\sec\eta$ (at fixed η), with $r_l\equiv\gamma_l/\gamma_l^{(E)}$ as above. For a boost-invariant state r_l is either $r=r_0$ for even l or $1/r$ for odd l. Hence, for any de Sitter-invariant state except the Euclidean vacuum, σ_l is of order unity at large l, not decreasing with increasing l. For the Euclidean vacuum $\sigma_l^{(E)}$ is at most of order λ^{-2}. The summand in Eq. (4.13) is positive for large l, varying as λ^N. Hence, any other de Sitter-invariant vacuum has infinitely higher energy than the Euclidean vacuum.

The other de Sitter-invariant vacua can be described as states of "uniform excitation " with respect to the Euclidean vacuum. The boost-invariance condition $r_{l+1}=1/r_l$ implies $\sigma_{l+1}=\sigma_l$, to leading order in λ or l. This means the "excitation number" $\frac{1}{2}(\sigma_l-\sigma_l^{(E)})$ is mode independent (as well as time independent) to that order. This accords with the intuitive notion of a boost-invariant, i.e., flat, excitation spectrum.

The Euclidean vacuum is not the lowest-energy state of all, however, unless the field is massless and conformally coupled. Condition (3.6) or (3.7), given condition (3.8), minimizes $\langle H^{(g)}\rangle_{\{0\}}(t_D)$ mode by mode. In the massless, conformally coupled case this corresponds to the Euclidean vacuum. But for any other fields an infinite number of states of lower energy than the Euclidean vacuum at any given time can be constructed, e.g., by taking $\gamma_L=\gamma_L^{(D)}$ in some modes, $\gamma_L=\gamma_L^{(E)}$ in the others. This reduces $\langle H^{(g)}\rangle_{\{0\}}-\langle H^{(g)}\rangle_{\{0\}}^{(E)}$ to a sum of negative terms, breaking de Sitter invariance in the process. The complete $H^{(g)}$-diagonalizing vacuum [or at least, diagonalizing in all modes for which inequality (3.8) is satisfied] at the chosen instant is the lowest-energy state in such a sequence. Its "energy deficit" below the Euclidean vacuum is finite in some cases, infinite in others. For example, for nonconformally coupled fields $\sigma_l^{(D)}-\sigma_l^{(E)}$ is negative and of order λ^{-4} for large l at $\eta=\eta_D=0$. Then the difference (4.13) converges for $N=1$ or 2 but diverges to $-\infty$ for $N\geq3$. The existence of states of lower energy, but lower symmetry, than the Euclidean vacuum is an example of a general phenomenon[23] which arises even in Minkowski space.[24]

The choice of a particular vacuum state may be associated with the choice of semiclassical approximation underlying the field theory. In such an approximation the vacuum energy, suitably regularized, is usually taken to provide the "source" appropriate to the given spacetime geometry in the semiclassical Einstein equations. Regularized stress-energy-tensor expectation values for a de Sitter-space scalar field in the Euclidean vacuum have been calculated by, e.g., Dowker and Critchley[25] and Bunch and Davies.[26] The result does not give a consistent semiclassical solution, however: The curvature radius a obtained is less than the Planck length, not within the semiclassical regime. This may be remedied by, for example, treating a very large number (hundreds of thousands or millions) of independent scalar fields,[14] or by supplementing the regularization procedure with an *ad hoc* contribution to the cosmological constant.[15] Either of these effectively decouples the vacuum energy of the particular field considered from the source required by the Einstein equations. But if this is done consistently with the field in the Euclidean vacuum, then no other de Sitter-invariant vacuum state can be accommodated within the resulting semiclassical approximation: The effects of the infinitely greater energy, and, hence, the energy density, of any such state cannot be neglected; the approximation must break down if the field gets into any such state. The regularization procedure could be altered to construct a different semiclassical approximation, consistent with a different vacuum state (de Sitter invariance ensures that the stress-energy expectation values are of suitable form). Within this approximation vacua infinitely different in energy from that chosen, including the Euclidean vacuum, could not be accommodated. Thus a semiclassical approximation corresponds to each vacuum choice, within which only states sufficiently close in energy to that vacuum can be treated consistently. "Sufficiently close" means that the gravitational effects of the energy-density difference between these states and the chosen vacuum must be negligible; the energy difference must, at the very least, be finite. Of course, the vacuum choice affects more than the vacuum energy and the semiclassical Einstein equations; in particular it determines the properties of the theory's two-point functions. The Euclidean-vacuum choice implies, e.g., analyticity of the Feynman function on the Euclidean continuation of de Sitter space,[27] regularity of the symmetric and Feynman functions at antipodal points, and short-distance behavior of Hadamard form. Other vacuum choices alter these properties, the time-ordered behavior of the Feynman function, etc.[8] Hence, the semiclassical approximations associated with other vacuum choices may not be as suitable as that of the Euclidean as bases for the field theory.

V. EXCITED STATES—"PARTICLE" ENERGIES

A mode-by-mode complete set of excited states $\Psi_{\{n_L\}}$ is arrayed above any choice of vacuum state. The excitation energy of a state, the difference between its Hamiltonian expectation value (3.5a) and that of the associated vacuum (again, a quantity independent of any regularization or renormalization[20,21]) is $\mathcal{E}_{\{n_L\}}=\sum_L n_L\omega_L(1+\sigma_L)$. Naturally, this corresponds to a collection of independent field quanta or "particles," n_L in each mode L.

The energy of a single "particle" exhibits gravitational effects on the dynamics of the field. This energy is $E_L=\omega_L(1+\sigma_L)$. The σ_L contribution makes both the value and the time dependence of E_L differ from the classical mode frequency ω_L. For example, E_L can increase as t increases from zero, though classical energies "redshift." Most striking is the behavior of E_L at late and early times. This depends *qualitatively* on the mass and curvature coupling of the field and the vacuum choice. For fields with $\mu > N/(2a)$ the late-time behavior of E_L is

$$E_L \sim \frac{\mu_0^2}{q}\cosh\alpha_L + \frac{N}{2a}\sinh\alpha_L\left[\frac{\mu^2a^2(1-4\xi)-N\xi}{\mu q a^2}\cos(2qt-\beta_L) + \frac{2\xi}{\mu a}\sin(2qt-\beta_L)\right], \tag{5.1a}$$

with errors of order $e^{-2t/a}$, for $t/a \gg 1$ and $(\mu a\sec\eta)^2 \gg L^2$. The constants α_L and β_L are

$$\alpha_L = 2\,\text{arccosh}|\kappa_L^{(+)}| = 2\,\text{arcsinh}|\kappa_L^{(-)}| \tag{5.1b}$$

and

$$\beta_L = \arg\left[\kappa_L^{(+)}\kappa_L^{(-)*}\left[\frac{N}{2}+iqa\right]\right] - 2\arg\Gamma(1+iqa)\ , \tag{5.1c}$$

with $\kappa_L^{(\pm)}$ the coefficients in Eq. (2.10a), for χ_L as fixed by Eq. (3.4). The Euclidean-vacuum choice yields

$$\alpha_L^{(E)} = \text{arcsinh}[\,\text{csch}(\pi qa)] \tag{5.2a}$$

and

$$\beta_L^{(E)} = 2qa\ln 2 + 2\arg\frac{\Gamma(-iqa)}{\Gamma\left[\dfrac{\frac{1}{2}+\lambda-iqa}{2}\right]\Gamma\left[\dfrac{\frac{1}{2}-\lambda-iqa}{2}\right]} + \arcsin(q/\mu) + \vartheta\ , \tag{5.2b}$$

with

$$\vartheta = \begin{cases} 2(-1)^\lambda\arctan[\tanh(\pi qa/2)] & \text{for odd } N,\ \text{i.e., integral } \lambda\ , \\ \pi & \text{for even } N,\ \text{with } \lambda = 2j+\frac{1}{2}\ , \\ 0 & \text{for even } N,\ \text{with } \lambda = 2j+\frac{3}{2}\ , \end{cases} \tag{5.2c}$$

where j is any non-negative integer. The "out" vacuum defined by Eq. (4.11) or (4.12b) is the unique choice yielding $\alpha_L = 0$. Hence, the energy of a "particle," as defined by almost any vacuum choice including the Euclidean, approaches no limit at late times, but oscillates about a fixed value with constant amplitude and frequency. Only the energy of an "out particle" approaches a constant value. The early-time ($t \to -\infty$) behavior of E_L is similar; in that limit only E_L corresponding to the "in"-vacuum choice approaches a constant value. These features are illustrated in Fig. 1, which shows E_L values (from the exact formulas) for a minimally coupled field with $q = 0.5a^{-1}$ in four dimensions, using the Euclidean- and "out"-vacuum choices.

In some cases E_L oscillates between positive and negative values. Its value at $t = 0$ is positive for any field with real μ_0 and μ. But form (5.1a) implies E_L becomes negative periodically at late times if

$$\mu_0^2 a^2 + N(N-1)\xi \left[1 - \frac{4N\xi}{N-1}\right] + \frac{\operatorname{csch}^2 \alpha_L}{q^2 a^2} \mu_0^4 a^4 < 0 \tag{5.3}$$

is satisfied. This occurs for no field with real μ_0 and with ξ between the minimal and conformal values inclusively. Nonetheless, with the Euclidean value (5.2a) for α_L, for example, parameters μ_0 and ξ (hence q) can be found satisfying this inequality. In that case it is satisfied for some range of q values near zero if and only if

$$\mu_0^2 a^2 + N(N-1)\xi \left[1 - \frac{4N\xi}{N-1}\right] < -\pi^2 \mu_0^4 a^4 \tag{5.4}$$

obtains. Hence, ξ must satisfy

$$\xi > \frac{N-1}{8N} + \frac{1}{2N} \left[\frac{(N-1)^2}{16} + \mu_0^2 a^2 + \pi^2 \mu_0^4 a^4\right]^{1/2} \tag{5.5a}$$

and

$$\xi > \frac{N}{4(N+1)} - \frac{\mu_0^2 a^2}{N(N+1)}\ , \tag{5.5b}$$

the latter from the assumption $\mu > N/(2a)$. That is, for a given μ_0, if ξ is greater than these two bounds and the corresponding q is small, then the oscillating E_L crosses into negative values in each cycle. The bounds imply that only fields with curvature coupling stronger than conformal have this property. Negative excitation energies suggest the possibility of a runaway instability of the field, but it must be remembered that the evolution of field states is governed by the canonical Hamiltonian, and the canonical energy (the formal $\xi \to 0$ limit of E_L, at fixed μ or q) is always positive. Moreover for the free fields considered here the time evolution of the wave functionals (3.2a), in which the n_L are conserved, is exact. Perhaps interactions coupled to the gravitational energy would show the effects of negative E_L. These negative energy values also do not alter the ordering of vacuum energies described above, as that depends on modes in the limit of large λ—in particular, modes with $\lambda \gg \mu a \cosh(t/a)$, for which the late-time limit of form (5.1a) is yet to be attained.

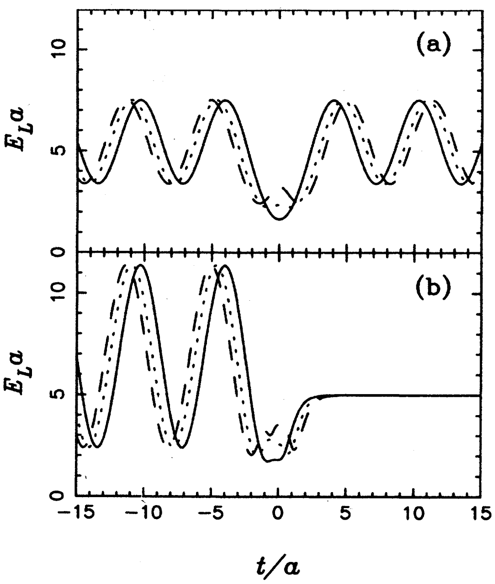

FIG. 1. Single-excitation energies for a massive, minimally coupled field in (3+1)-dimensional de Sitter space, with $q = 0.5a^{-1}$, i.e., $\mu \simeq 1.581a^{-1}$. Shown are the energies of "particles" defined via the (a) Euclidean- and (b) "out"-vacuum choices, with angular momenta $l = 0$ (solid curve), $l = 1$ (dotted curve), and $l = 2$ (dashed curve).

The above results even can be applied to tachyonic fields, with $\mu_0^2 < 0$. As long as ξ is large enough to yield $\mu > N/(2a)$, all the preceding formulas hold without change. [This implies, via the bound (5.5b), a curvature coupling stronger than conformal.] For such fields E_L oscillates at late times about a negative value. If inequality (5.3) holds it oscillates between negative and positive values; if not, it varies only over negative values in the late-time limit.

Excitation energies for fields with $\mu < N/(2a)$ exhibit exponential behavior at late times, rather than the oscillations of form (5.1a) and Fig. 1. With $q \equiv i\zeta/a$, and $\zeta > 0$ to be definite, the late-time behavior of E_L is

$$E_L \sim \frac{N}{4\zeta a} \frac{\Gamma(1+\zeta)}{\Gamma(1-\zeta)} |\kappa_L^{(+)}|^2 [(N-2\zeta)(1-4\xi) - 4\xi] e^{2\zeta t/a} + \left[\frac{\mu_0^2 a}{\zeta} (\kappa_L^{(+)} \kappa_L^{(-)*} + \kappa_L^{(+)*} \kappa_L^{(-)}) + \frac{N}{4\zeta a} \frac{\Gamma(1-\zeta)}{\Gamma(1+\zeta)} |\kappa_L^{(-)}|^2 [(N+2\zeta)(1-4\xi) - 4\xi] e^{-2\zeta t/a}\right]. \tag{5.6}$$

The first (constant) term in the large parentheses is valid only for $\zeta < 1$, the second ($e^{-2\zeta t/a}$) term only for $\zeta < \frac{1}{2}$, since terms of relative order $e^{-2t/a}$ are neglected in deriving this expression.

Excitation energies of "conformally invariant" fields, i.e., those with $\mu^2=(N^2-1)/(4a^2)$, show both classical and nonclassical features. The exact E_L are

$$E_L=\left[\frac{\lambda}{a\cosh(t/a)}+\frac{N-1-4N\xi}{4\lambda a}\left[N\cosh(t/a)-\frac{N-1}{\cosh(t/a)}\right]\right]\cosh A_L$$
$$+\frac{N-1-4N\xi}{2a}\left[\frac{1}{2\lambda}\left[N\cosh(t/a)-\frac{N-1}{\cosh(t/a)}\right]\cos(2\lambda\eta-B_L)+\tanh(t/a)\sin(2\lambda\eta-B_L)\right]\sinh A_L\ , \quad (5.7a)$$

with

$$A_L=2\,\mathrm{arccosh}|k_L^{(+)}|=2\,\mathrm{arcsinh}|k_L^{(-)}| \quad (5.7b)$$

and

$$B_L=\arg(k_L^{(+)}k_L^{(-)*})\ . \quad (5.7c)$$

The coefficients $k_L^{(\pm)}$ are those of Eq. (2.15). The Euclidean-vacuum choice corresponds to $A_L\equiv 0$; for de Sitter-invariant choices condition (4.1) is equivalent to $A_L=A_l$ and $B_L=B_l$, and condition (4.6) is equivalent to $A_{l+1}=A_l$ and $B_{l+1}=B_l+\pi$. In the *strictly* conformally invariant, i.e., massless and conformally coupled case, E_L reduces to $\lambda\cosh A_L/[a\cosh(t/a)]$ and, hence, behaves exactly as the classical energy of a relativistic particle (though unlike ω_L). But for massive fields with a correspondingly smaller curvature coupling E_L also contains, for non-Euclidean vacuum choices, components which approach a finite limit at early and late times, and, in general, components which grow as $\cosh(t/a)$, i.e., as the radius of the space. This is an example of "superadiabatic amplification,"[28] arising from the breaking of strict conformal invariance.

These conformally invariant fields are the only ones for which exact thermal-equilibrium states can be defined in de Sitter space. Such states are naturally associated with the Euclidean-vacuum choice, that state being their zero-temperature limit. The excitation energies of these thermal states show the same behavior as seen in Eq. (5.7a) with the Euclidean-vacuum choice.[29]

VI. EXCITED-STATE INTERACTIONS: "PARTICLE DETECTORS"

The dynamics of the field is manifest in its interactions with other systems. The response of a point oscillator coupled linearly to the field, an Unruh-DeWitt "monopole detector," is a simple example of such an interaction.[30,31]

The response of the detector to the field in a state $\Psi_{\{n_L\}}$ is given by the function[32]

$$\mathcal{F}_{\{n_L\}}(E)=\int_{-\infty}^{+\infty}d\tau\int_{-\infty}^{+\infty}d\tau' e^{-iE(\tau-\tau')}\times\langle\varphi[x(\tau)]\varphi[x(\tau')]\rangle_{\{n_L\}}\ , \quad (6.1)$$

with E the detector transition energy and $x(\tau)$ and $x(\tau')$ the coordinates at proper times τ and τ' on the detector trajectory. Up to a factor depending only on the internal structure of the detector, this is the total probability, over the entire history of the detector, for it to make a transition of energy E in interaction with the field in state $\Psi_{\{n_L\}}$.

This response function can be evaluated explicitly for a detector comoving in the coordinates used here. The field expectation value can be calculated using the wave functional (3.2a), transforming the field variables between times τ and τ' via the propagator (2.23); equivalently, the expansion (2.5), with χ_L as specified by Eq. (3.4), can be used. The result is

$$\mathcal{F}_{\{n_L\}}(E)=\mathcal{F}_{\{0\}}(E)+a^{-N}\sum_L n_L[|X_L(E)|^2+|X_L(-E)|^2]\mathcal{Y}_L^2(\Omega)\ , \quad (6.2a)$$

where $\mathcal{F}_{\{0\}}$ is the response function in the corresponding vacuum state, the angular coordinates denoted Ω are the (fixed) values on the detector trajectory, and X_L is the Fourier transform of χ_L, viz.,

$$X_L(E)\equiv\int_{-\infty}^{+\infty}e^{-iEt}\chi_L(t)dt\ . \quad (6.2b)$$

With χ_L given by Eq. (2.10a), this integral yields

$$X_L(E)=\frac{2^{(N-2)/2}a}{(2q)^{1/2}\Gamma(N/2)}\times[\kappa_L^{(+)}\Xi(+q,\lambda,E)+\kappa_L^{(-)}\Xi(-q,\lambda,E)]\ , \quad (6.3a)$$

with

$$\Xi(q,\lambda,E)=\left[\frac{\Gamma(1-iqa)}{\Gamma(1+iqa)}\right]^{1/2}\Gamma\left[\frac{N}{4}+\frac{i(E+q)a}{2}\right]\Gamma\left[\frac{N}{4}-\frac{i(E+q)a}{2}\right]$$
$$\times{}_3F_2\left[\frac{1}{2}-\lambda,\frac{1}{2}+\lambda,\frac{N}{4}+\frac{i(E+q)a}{2};1+iqa,\frac{N}{2};1\right] \quad (6.3b)$$

and ${}_3F_2$ a generalized hypergeometric function. For conformally invariant fields Eq. (2.15) implies the alternate form

$$X_L(E)=2^{(3-N)/2}\left[\frac{a^3}{2\lambda}\right]^{1/2}[k_L^{(+)}Z_l(E)+k_L^{(-)}Z_l(-E)]\ , \tag{6.4a}$$

with

$$Z_l(E)=\Gamma\left[\lambda-\frac{N-3}{2}\right]e^{-\pi Ea/2}\left[e^{i\pi l/2}\frac{\Gamma((N-1)/2-iEa)}{\Gamma(\lambda+1-iEa)}F\left[-\frac{N-3}{2}-iEa,\lambda-\frac{N-3}{2};1+\lambda-iEa;-1\right]+\text{c.c.}\right] \tag{6.4b}$$

and F an ordinary hypergeometric function. In all cases the detector response is the same for any fields with a given μ or q value.

Although the first term on the right-hand side of Eq. (6.2a) is usually associated with "the particle content of the vacuum," it is the second term which represents the detection of "particles" or excitations, in the sense of characterizing or distinguishing different field states.[33] In this sense the vacuum term represents a detector "dark current," identical for all states $\Psi_{\{n_L\}}$ with a given vacuum choice. This vacuum response has been studied by several authors;[5,7,27] here attention is focused on the excitation-dependent response.

The response associated with an excitation, e.g., the difference between $\mathcal{F}_{\{n_L\}}$ with one n_L one and the rest zero, and $\mathcal{F}_{\{0\}}$, exhibits effects of the dynamic spacetime geometry. In Minkowski space, with χ proportional to $e^{-i\omega t}$, this response is proportional to the square of a delta function in energy. One delta-function factor enforces energy conservation in the response; the other represents the total time interval, by which the response probability is divided to give a response rate.[32] But in de Sitter space, with χ as in Eq. (2.10a) or (2.15), the transform X as per Eqs. (6.3) or (6.4) is finite for all E values (for any field with $\mu>0$). Thus the single-excitation response *probability,* over the entire detector history, is finite. It also has finite width in energy. These features are illustrated in Fig. 2, which shows single-excitation responses for a field with $q=0.5a^{-1}$ in four dimensions, using the Euclidean-vacuum choice. The same response functions for a conformally invariant field (with $q=0.5a^{-1}i$) are very similar, as shown in Fig. 3, despite the differences in form between Eqs. (6.3) and (6.4).

VII. CONCLUSIONS

A detailed examination of the spectrum of quantum states of a scalar field in de Sitter space has revealed gravitational effects on the dynamics of the field, over and above the well-known effects associated with vacuum states. The states are described via Schrödinger-picture wave functionals, complementing the familiar Fock-state description. Field-energy expectation values and monopole-detector responses also characterize the states, and exhibit these gravitational effects.

The Fock spaces of states available to a linear, real scalar field in $(N+1)$-dimensional de Sitter space are

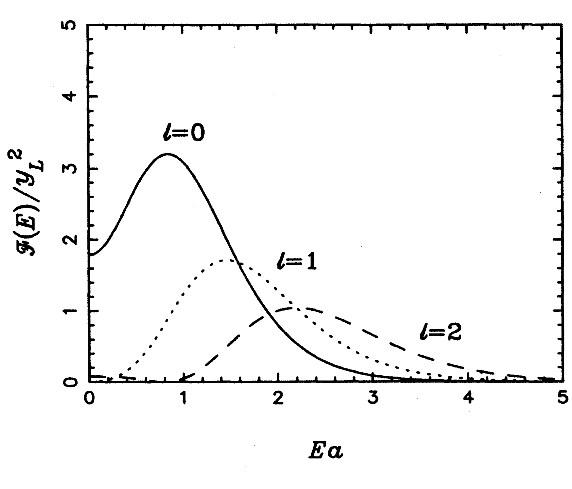

FIG. 2. Single-excitation detector-response functions, with angular dependence divided out, for the field with $q=0.5a^{-1}$, i.e., $\mu\simeq 1.581a^{-1}$, in (3+1)-dimensional de Sitter space. Shown are responses to "particles" defined via the Euclidean-vacuum choice, with angular momenta $l=0$ (solid curve), $l=1$ (dotted curve), and $l=2$ (dashed curve).

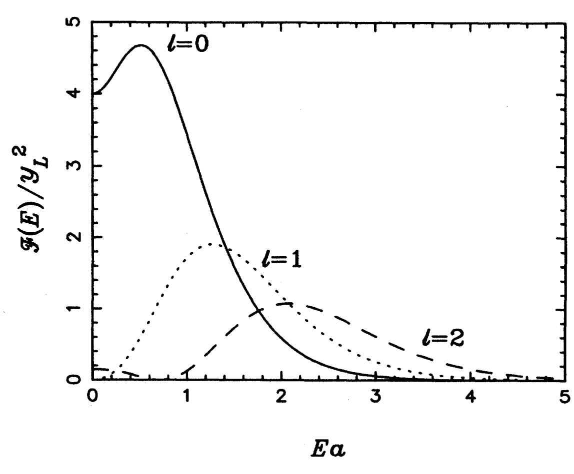

FIG. 3. Single-excitation detector-response functions, with angular dependence divided out, for conformally invariant fields, i.e., fields with $\mu\simeq 1.414a^{-1}$, in (3+1)-dimensional de Sitter space. Shown are responses to "particles" defined via the Euclidean-vacuum choice, with angular momenta $l=0$ (solid curve), $l=1$ (dotted curve), and $l=2$ (dashed curve), as in Fig. 2.

spanned by states with wave functionals consisting of products, over the field normal modes, of Hermite polynomials times Gaussian functions of the normal-mode amplitudes.[34] The functional Schrödinger equation determines the time dependence of the coefficients multiplying the normal-mode amplitudes in these functions, given by Eqs. (3.2). A mode-by-mode complete set of states is specified by the values of these coefficients, one complex number for each mode, at $t=0$, the instant of minimum expansion of the space (in spatially closed coordinates). A one-to-one correspondence exists between the sets of such values and the choices of positive-frequency modes and, hence, creation and annihilation operators, in the canonical Fock-space description of the theory. This is explicitly given by relation (3.4). The orders of the Hermite polynomials in these wave functionals, excitation or "particle" numbers in the Fock-space description, are constants of the states evolution. Hence, for a fixed positive-frequency-mode choice, no "particle" production takes place. What is usually termed "particle production" in time-dependent spacetimes arises from a change in mode choice, i.e., vacuum-state choice. In this alternative picture "particle production" entails quantum transitions between field states, as it does in flat-spacetime theory. Such transitions do not occur for a field interacting only with a fixed spacetime geometry.[35]

States with Gaussian wave functionals, then, are vacuum states.[5,7,15,16] These constitute an infinite-complex-parameter family (one parameter per mode). Vacua invariant under the connected de Sitter group form a one-complex-parameter subfamily; states invariant under the full group, including time reversal, are a one-real-parameter subset of these. These are the vacua described in the literature.[6–8,16] The Euclidean vacuum[6,8] is fully de Sitter invariant. The "in" and "out" vacua[18] are invariant under the connected group, but are time-reversal invariant (and identical) only in an odd number of spacetime dimensions; otherwise they transform into each other under time reversal. Vacuum states obtained via diagonalization[17] of the canonical or gravitational Hamiltonians used here are not de Sitter invariant, except the $H^{(g)}$-diagonalizing vacuum for a massless, conformally coupled field, which coincides with the Euclidean vacuum. The Euclidean vacuum is the lowest-energy de Sitter-invariant state, in the sense that the difference between the Hamiltonian expectation value in any other such state and that in the Euclidean vacuum (a renormalization-independent quantity[20,21]) is a divergent sum of positive terms. The terms correspond to a uniform excitation in all modes. For any massive or nonconformally coupled field, however, noninvariant vacua of lower energy at any instant can be constructed by Hamiltonian diagonalization in some or all modes. This suggests the possibility of the instability of the Euclidean vacuum or "spontaneous de Sitter-symmetry breaking," in the presence of an interaction capable of mediating a transition between the states.

States with nonzero excitation numbers exhibit nonvacuum gravitational effects. The field energy of one "particle," i.e., the difference between the Hamiltonian expectation values of two states differing by one excitation, behaves in general quite unlike the associated classical mode frequency. The dissimilarity is most apparent at early or late times: While the classical frequency asymptotically approaches the field mass in these limits, the quantum energy can oscillate or grow exponentially. For fields with a mass–curvature-coupling sum μ larger than $N/(2a)$, with a the minimum radius of the space, the energy oscillates with fixed amplitude and frequency. For some fields with curvature coupling stronger than conformal it can even oscillate between positive and negative values. "Particles" defined via almost any vacuum choice, including the Euclidean, exhibit this behavior; only with the "in" and "out" vacuum choices do "particle" energies approach fixed limits at early and late times, respectively. For fields with $\mu<N/(2a)$, including conformally invariant fields, these energies exhibit a growing exponential rather than oscillatory behavior. Only for a massless, conformally coupled field do they obey the redshift law for classical relativistic-particle energies. The response of a monopole detector[30,31] to a "particle"—again, the difference between the responses in two states differing by one excitation—shows effects of the spacetime geometry as well. As a function of (detector-transition) energy, the response is a peak of finite height and width, in contrast with the delta-function response obtained in flat spacetime.

Naturally, these gravitational effects are most pronounced for fields with a Compton wavelength μ^{-1} comparable to the curvature radius a of the space. The behavior of the field states rapidly approaches that in flat spacetime with increasing μ or q.

Gravitational effects similar to those seen here are to be expected in more general contexts as well. The effects here all originate, mathematically, in the fact that the normal-mode time-dependence functions χ, given by Eqs. (2.10a), (2.11), and (2.14), or (2.15), and the associated time dependence of the wave functionals, are not of the simple complex-exponential form found in flat spacetime. Thus the effects might be characterized as consequences of a "modulation" of the field normal modes by the time dependence of the metric, entailing both the classical redshift-blueshift reflected, e.g., in the $\operatorname{sech}(t/a)$ factors in χ, and further modulation represented in the associated-Legendre-function factors. A general quantized field in any dynamic spacetime will experience similar modulation, and exhibit similar gravitational-dynamical effects. Moreover, when interactions, i.e., time-dependent perturbations, and the consequent quantum-state transitions are considered, the complex-exponential time dependence of wave functions in flat spacetime gives rise to energy conservation. The nonexponential time dependence of wave functionals found here, and in general dynamic spacetimes, suggests that gravitational effects will alter the simple intuitive picture of (local) energy conservation in quantum transitions.

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