Perturbative description of dissipation in nonequilibrium field theory

Ian D. Lawrie
Department of Physics, The University, Leeds LS2 9JT, England
(Received 15 November 1988)

Dissipative effects, such as the relaxation of quasiparticle occupation numbers, arise from absorptive parts of Green's functions, which typically appear first at the second order of perturbation theory. Within the closed-time-path formalism, it is shown, using a generalized renormalization technique, that these absorptive parts may be approximately resummed so as to appear in unperturbed propagators. In this way, it becomes possible to study, in low-order perturbation theory, the evolution in time of a field theory which is driven away from thermal equilibrium by the presence in its Hamiltonian of explicitly time-dependent parameters. Particular attention is given to a scalar field with time-dependent mass, which is relevant to the dynamics of phase transitions in the very early Universe. Under favorable conditions, the analysis leads to a kinetic equation of the Boltzmann type, and an approximate numerical solution of this equation is presented for illustrative purposes.

I. INTRODUCTION

This is one of a series of papers[1–3] in which we develop theoretical tools needed for a systematic study of the dynamics of symmetry-breaking phase transitions in the early Universe. We have in mind scenarios of the "new inflationary" kind[4] which modify the original idea of Guth[5] by supposing that a scalar "inflaton" field undergoes a second-order transition. In these scenarios, inflation lasts while the initial symmetric state of the field slowly evolves into a broken-symmetry state. Our program, explained and motivated in detail in Ref. 2, is to assume the existence of an initial high-temperature, spatially homogeneous, symmetric state, and to follow the evolution of the density matrix, or at least of Green's functions, through the phase transition, while using the thermal expectation value of the stress tensor to calculate the evolution of the Robertson-Walker scale factor. This is a fairly obvious and well-defined, if technically complicated, approach. Of course, the existence of an initial state with the above properties is by no means assured; indeed, it is explicitly denied in scenarios of the "chaotic" type.[6] Nevertheless, a primary attraction of the idea of inflation is that memory of the initial state is effectively lost and we, therefore, hope to obtain reliable information about the final state by studying systematically a problem which is mathematically well posed. A study of this general kind has, indeed, been described by Mazenko.[7,8] He found, in particular, density fluctuations much smaller than those estimated by earlier methods.[9] Since the earlier estimates were greatly in excess of the values required by prevailing theories of galaxy formation, this line of investigation is clearly of practical importance. Mazenko's estimate, on the other hand, is far too low. To some extent, our program may be viewed as an attempt to generalize his analysis which, for example, relies heavily on special properties of the large-N limit of O(N)-symmetric scalar field theory.

In pursuit of this program we have encountered (so far) two major technical obstacles. The first, foreseen from the outset, is that neither the expectation value of the inflaton field ϕ nor local fluctuation regions (in which this expectation value is nonzero although its spatial average vanishes) will naturally emerge from our assumed initial state unless they are inserted by hand—an arbitrary procedure we wish strenuously to avoid. This problem was solved in principle in Refs. 1 and 2, where we showed that all relevant properties of the broken-symmetry state can be recovered if we recast the theory in terms of ϕ^2, avoiding all reference to $\langle\phi\rangle$. The second problem, to which this paper is addressed, is the following. The presence in the field-theory Hamiltonian of the time-dependent scale factor naturally drives the theory away from thermal equilibrium. Feynman rules for perturbative calculations in the nonequilibrium theory were derived some time ago by Semenoff and Weiss,[10] and these appear to be just what we require. Unfortunately, these rules describe, in traditional fashion, perturbations about a noninteracting theory. Now, the occupation numbers of single-particle or quasiparticle modes evolve with time only via scattering processes. In elementary transport theory, for example, this evolution is described by a Boltzmann equation.[11] Therefore, the occupation numbers in the unperturbed propagators are fixed by the initial distribution and do not adequately reflect the changing state of the system.

Evolution of the occupation numbers is a dissipative relaxation effect, which may be expected to arise from absorptive parts of higher-order contributions to the full Green's functions. Since very few of these can be calculated in practice, it is highly desirable to effect some form of resummation which would incorporate at least the leading effects into unperturbed propagators. In standard renormalization theory, such resummations can be accomplished by adding a suitable counterterm to the free-field part of the Lagrangian, subtracting it from the interaction part, and determining its value self-consistently by minimizing the net effect of the modified interaction.

We follow the same route here, constructing what will be referred to as a "dissipative counterterm" in a manner consistent with the general structure of the propagators.

In this paper we consider only the example of a single self-interacting scalar field with a time-dependent effective mass, since this already gives rise to considerable technical complexity. However, the method we use to construct the dissipative counterterm is a systematic one and should be readily adaptable to more general theories. Indeed, the problem we address pervades the whole of nonequilibrium statistical mechanics, and we would expect the techniques proposed here to be very widely applicable. Even for theories with time-independent Hamiltonians, it is desirable to include in unperturbed propagators as much information as possible about the state of the system. When this state changes with time, the method developed here is, we believe, the appropriate one to use.

We follow Semenoff and Weiss in adopting the closed-time-path formalism[12,13] to obtain real-time Green's functions. The version appropriate to thermal equilibrium has been elegantly derived in terms of path integrals by Niemi and Semenoff,[14] who point out that it resolves ambiguities encountered, for example, by Dolan and Jackiw[15] in their real-time formulation of thermal field theory. (The imaginary-time formalism given by the latter authors, is, of course, perfectly sound, but it cannot be extended to encompass nonequilibrium states.) Further developments have been presented by several authors[16] and a closely related method for calculating time-dependent expectation values is described by Jordan.[17] This formalism is briefly described in Sec. II, where we exhibit the structure of the 2×2 matrix of real-time propagators. In Sec. III we introduce the dissipative counterterm, while in Sec. IV we obtain a formal solution for the modified unperturbed dissipative propagator matrix to which it gives rise. In an interacting theory, single quasiparticle modes are not necessarily well defined. Section V discusses the extent to which the propagators can be interpreted in terms of time-dependent occupation numbers and exhibits an approximation in which a kinetic equation for these numbers can be derived. The counterterm is evaluated at the lowest nontrivial order of perturbation theory in Sec. VI, and it is shown that, under sufficiently favorable circumstances, the kinetic equation reduces to a Boltzmann equation of the standard type. An illustrative numerical solution of this equation is described in Sec. VII. Finally, in Sec. VIII our findings are summarized and compared with other approaches to nonequilibrium field theory. An appendix reviews the relevance of our model to scalar field theory in Robertson-Walker universes.

II. CLOSED-TIME-PATH FORMALISM AND THE STRUCTURE OF PROPAGATORS

We consider a Hermitian scalar field, denoted in the Heisenberg picture by $\phi(\mathbf{x},t)$, whose time evolution is governed by the Lagrangian density

$$\mathcal{L}(\phi)=\tfrac{1}{2}(\partial_t\phi)^2-\tfrac{1}{2}\nabla\phi\cdot\nabla\phi-\tfrac{1}{2}m^2(t)\phi^2-\frac{\lambda}{4!}\phi^4 \tag{2.1}$$

or, equivalently, by the Hamiltonian density

$$\mathcal{H}(\phi)=\tfrac{1}{2}\pi^2+\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+\tfrac{1}{2}m^2(t)\phi^2+\frac{\lambda}{4!}\phi^4\ , \tag{2.2}$$

where $\pi(\mathbf{x},t)=\dot{\phi}(\mathbf{x},t)$ is the conjugate momentum. The mass carries an explicit time dependence which, for the purposes of this paper, we take to be externally prescribed. The Lagrangian density of a scalar field in a spatially homogeneous Robertson-Walker universe can be expressed in this form, as discussed in the Appendix. If, at time t_0, the state of the field is described by the normalized density matrix ρ, then, taking the Schrödinger and Heisenberg pictures to coincide at t_0, the generating functional for time-ordered Green's functions at later times is

$$Z(j)=\mathrm{Tr}\left[\rho T\exp\left[i\int_{t_0}^{\infty}dt\int d^3x\, j(\mathbf{x},t)\phi(\mathbf{x},t)\right]\right]\ , \tag{2.3}$$

where T is the time-ordering operator. The expectation value of any function of ϕ at times later than t_0 may be expressed in terms of functional derivatives of $Z(j)$.

The time-ordered exponential in (2.3) may be rewritten in terms of Schrödinger-picture operators as

$$\left[\bar{T}\exp\left[i\int_{t_0}^{\infty}dt\int d^3x\,\mathcal{H}_S(t)\right]\right]\times\left[T\exp\left[-i\int_{t_0}^{\infty}dt\int d^3x[\mathcal{H}_S(t)-j(\mathbf{x},t)\phi_S(\mathbf{x})]\right]\right], \tag{2.4}$$

where $\bar{T}$ is the anti-time-ordering operator and $\mathcal{H}_S(t)$ is the Hamiltonian density obtained by inserting Schrödinger-picture operators $\phi_S(\mathbf{x})$ and $\pi_S(\mathbf{x})$ in (2.2). It is now useful to generalize (2.3) and (2.4) by writing

$$Z(j_1,j_2)=\mathrm{Tr}\left\{\rho\left[\bar{T}\exp\left[i\int d^4x(\mathcal{H}_S+j_2\phi_S)\right]\right]\left[T\exp\left[-i\int d^4x(\mathcal{H}_S-j_1\phi_S)\right]\right]\right\}. \tag{2.5}$$

In particular, the propagators

$$G_{ab}(\mathbf{x},t;\mathbf{x}',t')=-\frac{\delta}{\delta j_a(\mathbf{x},t)}\frac{\delta}{\delta j_b(\mathbf{x}',t')}\times Z(j_1,j_2)\Big|_{j_1=j_2=0} \tag{2.6}$$

have the following interpretation in terms of the Heisenberg-picture field, with $x\equiv(\mathbf{x},t)$:

$$\begin{bmatrix}G_{11}(x;x') & G_{12}(x;x')\\ G_{21}(x;x') & G_{22}(x;x')\end{bmatrix}=\mathrm{Tr}\left[\rho\begin{bmatrix}T[\phi(x)\phi(x')] & [\phi(x')\phi(x)]\\ [\phi(x)\phi(x')] & \bar{T}[\phi(x)\phi(x')]\end{bmatrix}\right]. \tag{2.7}$$

For free particles in thermal equilibrium, or in the Minkowski vacuum, the expectation value $\langle \phi(x)\phi(x') \rangle$ is analytic at $t=t'$. In the presence of dissipation, however, we may expect it to decay at large time separations, perhaps as $\exp(-\text{const}\times|t-t'|)$. While we make no special assumption about the nature of this decay, we do wish to allow for the possibility that the expectation value assumes different forms for t greater or less than t'. If we write

$$\mathrm{Tr}[\rho\phi(x)\phi(x')]=\theta(t-t')H(x;x') +\theta(t'-t)K(x;x') , \tag{2.8}$$

then the Hermiticity of ϕ and ρ implies that $K(x;x')=H^*(x';x)$. One easily finds that (2.7) has the structure

$$\begin{bmatrix} G_{11}(x;x') & G_{12}(x;x') \\ G_{21}(x;x') & G_{22}(x;x') \end{bmatrix} =\theta(t-t') \begin{bmatrix} H(x;x') & H^*(x;x') \\ H(x;x') & H^*(x;x') \end{bmatrix} +\theta(t'-t) \begin{bmatrix} H(x';x) & H(x';x) \\ H^*(x';x) & H^*(x';x) \end{bmatrix} . \tag{2.9}$$

Using standard techniques,[10,14–16] we may derive a path-integral representation of (2.5) in the form

$$Z(j_1,j_2)=Z_0\int [d\bar{\phi}_1(\mathbf{x})d\bar{\phi}_2(\mathbf{x})]\langle \bar{\phi}_1|\rho|\bar{\phi}_2\rangle \bar{Z}(j_1,j_2) , \tag{2.10}$$

$$\bar{Z}(j_1,j_2)=\int [d\phi_1(\mathbf{x},t)d\phi_2(\mathbf{x},t)] \times\exp\left[i\int_{t_0}^{\infty}dt\int d^3x[\mathcal{L}(\phi_1)-\mathcal{L}(\phi_2) +j_1\phi_1+j_2\phi_2]\right] , \tag{2.11}$$

where the functional integral in (2.11) is subject to the boundary condition $\phi_a(\mathbf{x},t_0)=\bar{\phi}_a(\mathbf{x})$, and Z_0 is a normalizing constant ensuring that $Z(0,0)=1$. The c-number fields ϕ_1 and ϕ_2 may be envisaged as residing on segments C_1 and C_2 of a contour in the complex time plane, C_1 running along the real axis from t_0 to ∞ and C_2 running from ∞ to t_0. Causal boundary conditions are equivalent to giving each segment an infinitesimal downward slope. In the case that ρ is the canonical density operator $\exp[-\beta_0 H(t_0)]/\mathrm{Tr}\exp[-\beta_0 H(t_0)]$, the matrix element in (2.1) may be represented by a Euclidean path integral, the corresponding field ϕ_3 residing on a third segment C_3 which descends from t_0 to $t_0-i\beta_0$. Periodic boundary conditions then apply at the end points t_0 and $t_0-i\beta_0$ of the whole contour. At present, however, we refrain from taking this step.

III. THE DISSIPATIVE COUNTERTERM

The Feynman rules for perturbative evaluation of the Green's functions generated by (2.10) include a matrix of propagators $g_{ab}(x;x')$, namely, the lowest-order approximation to (2.9), which invert the quadratic part of the effective action in (2.11). These propagators satisfy equations of the form

$$\mathcal{D}_{ac}(x)g_{cb}(x;x')=g_{ac}(x;x')\overleftarrow{\mathcal{D}}_{cb}(x') =-i\delta_{ab}\delta(x-x') , \tag{3.1}$$

where $\mathcal{D}$ is a second-order differential operator. Boundary conditions on this equation arise from the manner in which (2.11) is embedded in (2.10) and will be discussed later. In practice, it will be possible to carry out only very-low-order calculations, and it is essential to define $g(x;x')$ so as to mimic as closely as possible the properties of $G(x;x')$. In particular, finite lifetimes of quasiparticle excitations, which in turn permit the evolution of occupation numbers in response to a changing environment, are possible only in the presence of interactions, and are absent from $g(x;x')$ if perturbation theory is organized in the most obvious manner. The strategy we propose is to modify $g(x;x')$ by adding to the quadratic part of the composite Lagrangian $\mathcal{L}(\phi_1)-\mathcal{L}(\phi_2)$ a local counterterm $\frac{1}{2}\phi_a(x)\mathcal{M}_{ab}\phi_b(x)$ and subtracting it from the interaction part. Initially arbitrary functions in $\mathcal{M}$ can then be used to optimize the resulting $g(x;x')$ as an approximation to $G(x;x')$. The notation is intended to emphasize an obvious analogy with mass renormalization which, in zero-temperature Minkowski-space field theory permits the use of an unperturbed propagator having the same mass shell as the full propagator. Indeed, a mass renormalization counterterm is included in the diagonal part of $\mathcal{M}_{ab}$. We shall see, however, that the decay width for quasiparticle excitations must be represented by off-diagonal terms. These make sense only within the closed-time-path representation and have no straightforward interpretation in terms of the original field $\phi(x)$ and its Lagrangian (2.1) alone. For example, the addition and subtraction of an anti-Hermitian counterterm in the Hamiltonian (2.2) to simulate decay widths would not work.

Clearly, there are limits to what can be achieved in this way. Consider the self-energy matrix $\Sigma(x;x')$ defined by

$$G(x;x')=g(x;x')+i\int d^4x''d^4x'''g(x;x'')\Sigma(x'';x''') \times G(x''';x') . \tag{3.2}$$

We shall want to choose $\mathcal{M}(x)$ so as to minimize the net effect of the second term in this equation. However, the contribution of this local counterterm to $\Sigma(x;x')$ is of the form $\mathcal{M}(x)\delta(x-x')$. When the state is homogeneous in space and time, Σ depends only on $(x-x')$. The standard renormalization technique expands its Fourier transform about a selected reference momentum, in effect expressing Σ as a series of derivatives of $\delta(x-x')$, and the two leading terms yield mass and wave-function renormalizations in a more or less unambiguous manner. Here we consider states which are homogeneous in space but not in time. Evidently, the temporal inhomogeneity cannot be represented correctly by a local counterterm, and, although we shall offer a prescription for extracting a suitable counterterm, this is by no means unique. There is some hope that a local counterterm does adequately

simulate the behavior of the propagators and their first few derivatives evaluated at equal times, which is sufficient for many low-order calculations. Conceivably, the technique described here might be generalized using a nonlocal counterterm, but we have not pursued this possibility.

Given that a local counterterm is to be employed, its structure must be consistent with (2.9). That is, the differential operator $\mathcal{D}(x)$ must be such that (3.1) admits a solution of the form (2.9), in which we denote by $h(x;x')$ the lowest-order approximation to $H(x;x')$. We restrict attention to spatially homogeneous states, and take Fourier transforms with respect to the argument $(\mathbf{x}-\mathbf{x}')$. We need to construct an operator $\mathcal{D}_k(t,\partial/\partial t)$ such that the equations

$$\mathcal{D}_k(t,\partial/\partial t)g_k(t,t')=g_k(t,t')\overleftarrow{\mathcal{D}}_k(t',-\overleftarrow{\partial}/\partial t') =-i\delta(t-t') \tag{3.3}$$

have a solution of the form (2.9) with $H(x;x')$ replaced by $h_k(t,t')$. To accomplish this, we first observe that the operator appropriate to a temporally homogeneous state must be included as a special case. For this case, let

$$h_k(t,t')=u_k(t-t')+iv_k(t-t') , \tag{3.4}$$

with u and v real. For the Fourier transform of the propagator matrix $g_k(t-t')=g_k(t,t')$, we find (cf. Ref. 13)

$$\hat{g}_k(\omega)\equiv\int_{-\infty}^{\infty}dt\, e^{-i\omega t}g_k(t) =\begin{bmatrix} A_k(\omega)+iB_k(\omega) & A_k(\omega)+C_k(\omega) \\ A_k(\omega)-C_k(\omega) & A_k(\omega)-iB_k(\omega)\end{bmatrix} , \tag{3.5}$$

with

$$\begin{aligned} A_k(\omega)&=2\int_0^{\infty}dt\, u_k(t)\cos\omega t , \\ B_k(\omega)&=2\int_0^{\infty}dt\, v_k(t)\cos\omega t , \\ C_k(\omega)&=-2\int_0^{\infty}dt\, v_k(t)\sin\omega t . \end{aligned} \tag{3.6}$$

Equation (3.3) becomes an algebraic equation $\hat{\mathcal{D}}_k(\omega)\hat{g}_k(\omega)=\hat{g}_k(\omega)\hat{\mathcal{D}}_k(\omega)=-i$, whose solution (suppressing momentum and frequency arguments) is

$$\hat{\mathcal{D}}=\frac{1}{A^2+C^2}\begin{bmatrix} -(B+iA) & i(A+C) \\ i(A-C) & (B-iA)\end{bmatrix} . \tag{3.7}$$

To construct the corresponding differential operator, we naturally replace ω by $i\partial/\partial t$. We demand that $h_k(t,t')$ be a smooth function, so that the δ function in (3.3) arises from differentiating step functions. Then $\mathcal{D}$ must be second order, with second derivatives appearing only in its diagonal elements. We fix the coefficients of these to be ±1, in accordance with (2.1) and (2.11). Taking into account that A, B, and C are real and that A and B are even in ω while C is odd, we discover that $\mathcal{D}_k(\partial/\partial t)$ has the form

$$\mathcal{D}_k\left[\frac{\partial}{\partial t}\right]=\begin{bmatrix} \left[\frac{\partial^2}{\partial t^2}+\beta_k-i\alpha_k\right] & \left[\gamma_k\frac{\partial}{\partial t}+i\alpha_k\right] \\ \left[-\gamma_k\frac{\partial}{\partial t}+i\alpha_k\right] & \left[-\frac{\partial^2}{\partial t^2}-\beta_k-i\alpha_k\right]\end{bmatrix} , \tag{3.8}$$

where the real constants α_k, β_k, and γ_k arise from expanding A, B, and C in powers of ω.

A straightforward division of the Lagrangian into free-particle and interaction terms obviously leads to the identification $\beta_k=m^2+k^2$ and $\alpha_k=\gamma_k=0$, with m^2 independent of time to allow a stationary state. The quantities α_k, $\beta_k-m^2-k^2$, and γ_k must each be at least of the order of the coupling constant λ, and these, of course, constitute the counterterm $\mathcal{M}$. To clarify their physical significance, consider the propagator which solves (3.3) with $\mathcal{D}_k$ given by (3.8). The function $h_k(t,t')$ from which it is constructed may be written as

$$\begin{aligned} h_k(t,t')=&\tfrac{1}{4}e^{-i[\omega_k-(1/2)i\gamma_k](t-t')} \\ &\times\left[\frac{1}{\omega_k}+\frac{2n_k+1}{\omega_k-\frac{1}{2}i\gamma_k}\right] \\ &+\tfrac{1}{4}e^{i[\omega_k+(1/2)i\gamma_k](t-t')} \\ &\times\left[-\frac{1}{\omega_k}+\frac{2n_k+1}{\omega_k+\frac{1}{2}i\gamma_k}\right] \end{aligned} \tag{3.9}$$

with

$$\omega_k^2=\beta_k-\tfrac{1}{4}\gamma_k^2 , \tag{3.10}$$

$$2n_k+1=\alpha_k/\omega_k\gamma_k . \tag{3.11}$$

Obviously, γ_k can be interpreted as the decay width for quasiparticle excitations of momentum k. Provided that α_k and γ_k are of the same order in λ (and we shall discover that each is of order λ^2), the free particle limit $\lambda\to0$ may be taken, leaving a finite value for n_k. In this limit, with $\gamma_k=0$, (3.9) reproduces the standard unperturbed thermal propagator, with occupation numbers n_k for single-particle modes. In thermal equilibrium, these should be given by the Bose-Einstein distribution

$$n_k=(e^{\beta\omega_k-\alpha}-1)^{-1} \tag{3.12}$$

and we expect this distribution to arise from the boundary conditions on (3.3) together with an appropriate prescription for determining $\mathcal{M}$.

To accommodate time-dependent states, we allow α_k, β_k, and γ_k in (3.8) to vary with time. Since the time derivative must act either forwards or backwards in (3.3), we make the replacement

$$\gamma_k\frac{\partial}{\partial t}\to\gamma_k^{1/2}(t)\frac{\partial}{\partial t}\gamma_k^{1/2}(t)=\gamma_k(t)\frac{\partial}{\partial t}+\tfrac{1}{2}\dot{\gamma}_k(t) . \tag{3.13}$$

Thus, the differential operator to be inserted in (3.3) is

$$\begin{aligned} \mathcal{D}_k\left[t,\frac{\partial}{\partial t}\right]=&\begin{bmatrix}1&0\\0&-1\end{bmatrix}\left[\frac{\partial^2}{\partial t^2}+\beta_k(t)\right] \\ &+\begin{bmatrix}0&1\\-1&0\end{bmatrix}\left[\gamma_k(t)\frac{\partial}{\partial t}+\tfrac{1}{2}\dot{\gamma}_k(t)\right] \\ &+\begin{bmatrix}1&-1\\-1&1\end{bmatrix}[-i\alpha_k(t)] \end{aligned} \tag{3.14}$$

and the counterterm to be added to the self-interaction in

(2.11) is, after spatial Fourier transformation, $\frac{1}{2}\phi_{b,-k}(t)\mathcal{M}_{ab,k}(t)\phi_{b,-k}(t)$ with

$$\mathcal{M}_k\left[t,\frac{\partial}{\partial t}\right]=\begin{bmatrix}1&0\\0&-1\end{bmatrix}[\beta_k(t)-k^2-m^2(t)]+\begin{bmatrix}0&1\\-1&0\end{bmatrix}\left[\gamma_k(t)\frac{\partial}{\partial t}+\tfrac{1}{2}\dot\gamma_k(t)\right]+\begin{bmatrix}1&-1\\-1&1\end{bmatrix}[-i\alpha_k(t)]\ . \tag{3.15}$$

The term involving $\dot\gamma_k(t)$ in (3.15) appears to make no net contribution to the counterterm, but we retain it for future convenience [see the discussion following (6.3) below].

IV. THE DISSIPATIVE PROPAGATOR

In this section we obtain a formal solution of (3.3) with the differential operator (3.14). We need to find the function $h_k(t,t')$ which generalizes (3.9) when α_k, β_k, and γ_k are time dependent. Given the structure (2.9) with h_k replacing H, it is straightforward to show that both the forward and backward versions of (3.3) are satisfied if $h_k(t,t')$ obeys the following conditions:

$$\frac{\partial}{\partial t}[h_k(t,t')-h_k(t',t)]\bigg|_{t'=t}=-i\ , \tag{4.1}$$

$$\frac{\partial}{\partial t}[h_k^*(t,t')-h_k(t',t)]\bigg|_{t'=t}=0\ , \tag{4.2}$$

$$\left[\frac{\partial^2}{\partial t^2}+\gamma_k(t)\frac{\partial}{\partial t}+\beta_k(t)+\tfrac{1}{2}\dot\gamma_k(t)\right]h_k(t,t')=0\ , \tag{4.3}$$

$$\left[\frac{\partial^2}{\partial t^2}+\beta_k(t)-i\alpha_k(t)\right]h_k(t',t)+\left[\gamma_k(t)\frac{\partial}{\partial t}+\tfrac{1}{2}\dot\gamma_k(t)+i\alpha_k(t)\right]h_k^*(t',t)=0\ . \tag{4.4}$$

Guided by (3.9), we write $h_k(t,t')$ in terms of three auxiliary functions $\Omega_k(t)$, $C_k(t,t')$, and $N_k(t,t')$ as

$$h_k(t,t')=\tfrac{1}{4}\exp\left[-i\int_{t'}^{t}[\Omega_k(t'')-\tfrac{1}{2}i\gamma_k(t'')]dt''\right]\times C_k(t,t')[1+N_k(t,t')]+\tfrac{1}{4}\exp\left[i\int_{t'}^{t}[\Omega_k(t'')+\tfrac{1}{2}i\gamma_k(t'')]dt''\right]\times C_k(t,t')[-1+N_k^*(t,t')]\ . \tag{4.5}$$

We take Ω_k and C_k to be real, but make no further assumption about C_k or N_k, so the representation is quite general.

First of all, we discover that (4.1) and (4.2) are satisfied if

$$C_k(t,t)=\frac{1}{\Omega_k(t)} \tag{4.6}$$

and

$$\frac{\partial}{\partial t}\mathrm{Re}[N_k(t,t')-N_k(t',t)]\bigg|_{t'=t}=\gamma_k(t)\mathrm{Re}N_k(t,t)-2\Omega_k(t)\mathrm{Im}N_k(t,t)\ . \tag{4.7}$$

Next, the positive- and negative-frequency parts of (4.5) satisfy (4.3) provided that

$$C_k(t,t')=[\Omega_k(t)\Omega_k(t')]^{-1/2} \tag{4.8}$$

and that $\Omega_k(t)$ and $N_k(t,t')$ satisfy

$$\frac{1}{2}\frac{\ddot\Omega_k(t)}{\Omega_k(t)}-\frac{3}{4}\frac{\dot\Omega_k^2(t)}{\Omega_k^2(t)}+\Omega_k^2(t)=\beta_k(t)-\tfrac{1}{4}\gamma_k(t)\ , \tag{4.9}$$

$$\left[\frac{\partial}{\partial t}-\frac{\dot\Omega_k(t)}{\Omega_k(t)}-2i\Omega_k(t)\right]\frac{\partial}{\partial t}N_k(t,t')=0\ . \tag{4.10}$$

Finally, to satisfy (4.4), we require, in addition,

$$\left[\frac{\partial}{\partial t}+\gamma_k(t)+2i\Omega_k(t)-\frac{\dot\Omega_k(t)}{\Omega_k(t)}\right]\left[\frac{\partial}{\partial t}+\gamma_k(t)\right]\times N_k(t',t)=2i\alpha_k(t)\ . \tag{4.11}$$

To write down the solution of (4.10) and (4.11) we suppress the subscripts k and introduce the abbreviation

$$\gamma(t_1,t_2)=\int_{t_1}^{t_2}\gamma(t)dt$$

with $\Omega(t_1,t_2)$ defined similarly. The general solution is then given by

$$N(t,t')=e^{-\gamma(t_0,t')}(N_1+N_2e^{-2i\Omega(t_0,t')}+N_3e^{-2i\Omega(t_0,t)}+N_4e^{2i\Omega(t',t)})+\int_{t_0}^{t'}dt''e^{-\gamma(t'',t')}(1-e^{-2i\Omega(t'',t')})\frac{\alpha(t'')}{\Omega(t'')}\ , \tag{4.12}$$

where N_1, N_2, N_3, and N_4 are complex constants of integration, with $\mathrm{Im}N_1=\mathrm{Im}N_4$ to satisfy (4.7). On substituting this solution into (4.5), we find that only the combinations $(N_1+N_4^*)$ and $(N_2+N_3^*)$ appear independently, so it is sufficient to set $N_3=N_4=0$ and make N_1 real. Thus, $N_k(t,t')$ depends only on t'.

In summary, our propagator matrix is constructed via the spatial Fourier transform of (2.9) from the function

$$h_k(t,t')=\tfrac{1}{4}[\Omega_k(t)\Omega_k(t')]^{-1/2}\exp\left[-\tfrac{1}{2}\int_{t'}^{t}\gamma_k(t'')dt''\right]\left[[1+N_k(t')]\exp\left[-i\int_{t'}^{t}\Omega_k(t'')dt''\right]+[-1+N_k^*(t')]\exp\left[i\int_{t'}^{t}\Omega_k(t'')dt''\right]\right]\ , \tag{4.13}$$

where $\Omega_k(t)$ is a solution of (4.9) and $N_k(t)$ satisfies

$$\left[\frac{\partial}{\partial t}+\gamma_k(t)+2i\Omega_k(t)-\frac{\dot{\Omega}_k(t)}{\Omega_k(t)}\right]\times\left[\frac{\partial}{\partial t}+\gamma_k(t)\right]N_k(t)=2i\alpha_k(t)\ . \quad (4.14)$$

At this point, a few comments are in order. First, we observe that when the dissipative counterterm is removed, so that $\alpha_k(t)=\gamma_k(t)=\beta_k(t)-k^2-m^2(t)=0$, our propagator matrix is consistent with that derived by Semenoff and Weiss.[10] These authors treat a more general theory, allowing time-dependent coefficients $\gamma_0(t)$ and $\gamma_1(t)$ for the time and space derivative terms in the Lagrangian, so in our case $\gamma_0(t)=\gamma_1(t)=1$. Their mode functions are given in our notation by

$$f_1(k,t)=[2\Omega_k(t)]^{-1/2}\exp\left[\mp i\int_{t_0}^{t}\Omega_k(t')dt'\right]\ . \quad (4.15)$$
2

They determine the full 3×3 matrix of propagators which arises from using the canonical density matrix in (2.10) and, in effect, evaluate the constants N_1 and N_2 in (4.12) by applying continuity and periodicity conditions to the whole contour. Here, the initial conditions on (4.14) are determined by the embedding of (2.11) in (2.10), which amounts to the same thing. In particular, without the dissipative counterterm, N_1 is given simply by the initial Bose-Einstein occupation numbers, and these are not seen to evolve with time.

Up to a point, we are now able to remedy this situation by interpreting $N_k(t)$ in terms of a set of time-dependent occupation numbers. To see this in outline, suppose that α_k, γ_k, and Ω_k are all slowly varying, and consider the evolution of $N_k(t)$ on a time scale sufficiently long for oscillations of frequency Ω_k to average to zero. Using (4.12), we obtain the estimate

$$N_k(t)\approx\left[N_1-\frac{\alpha_k(t_0)}{\gamma_k(t_0)\Omega_k(t_0)}\right]e^{-\gamma_k(t-t_0)}+\frac{\alpha_k(t)}{\gamma_k(t)\Omega_k(t)}\ . \quad (4.16)$$

Although we have yet to supply a precise means of determining α_k, γ_k, and Ω_k, comparison with (3.11) for thermal equilibrium indicates that memory of the initial state decays, with a mode-dependent relaxation time $1/\gamma_k$, while occupation numbers appropriate to the changing state of the system are updated through $\alpha_k(t)$. It is worth noting, however, that the decay of correlations is not necessarily exponential. If $\int\gamma_k dt$ should turn out to depend logarithmically on time during some stage of the evolution, we should obtain long-time tails with power-law decay as is found in some situations.[11]

A final comment concerns the fact that in (4.13) we have been able to decompose the propagator in terms of functions with a single time argument. While this is convenient from the point of view of practical calculations, it is possible only because we assumed a local counterterm. We cannot expect a similar decomposition for the true propagators. Although the propagator itself involves both $h_k(t,t')$ and $h_k(t',t)$, we see from (2.9) that the argument of N_k is always the earlier of the two times. To the extent that $N_k(t)$ can legitimately be interpreted in terms of time-dependent occupation numbers, it would seem, therefore, that the evolution of these over the time interval spanned by the propagator is not properly accounted for. As stated earlier, we anticipate that $g_k(t,t')$ will at best be a good approximation to $G_k(t,t')$ only when $t\approx t'$.

V. AN APPROXIMATE KINETIC EQUATION

In the next section, we shall see that the functions $\alpha_k(t)$ and $\gamma_k(t)$ are to be determined self-consistently at the lowest nontrivial order by matching the counterterm (3.15) against a two-loop diagram whose propagators are themselves constructed from (4.13). It may be imagined that some drastic approximation will be required to reduce this task to manageable proportions. Before proceeding, we wish in this section to pursue the interpretation of the function $N_k(t)$ in terms of a time-dependent distribution of occupation numbers by exhibiting an approximation to (4.14) which will subsequently be shown to reduce to a kinetic equation of the Boltzmann type. As well as making contact with established ideas of kinetic theory, this will serve to suggest a possible means of rendering explicit computations tractable.

Comparing (4.13) with (3.9), we would like, roughly, to identify occupation numbers $n_k(t)$ through

$$N_k(t)\approx\frac{2n_k(t)+1}{1-i\gamma_k(t)/2\Omega_k(t)}\ . \quad (5.1)$$

To this end, we formally rewrite (4.14) as

$$\left[\frac{\partial}{\partial t}+\gamma_k(t)\right]N_k(t)=\Omega_k(t)\left[1+[\gamma_k(t)+2i\Omega_k(t)]^{-1}\frac{\partial}{\partial t}\right]^{-1}\times\left[\frac{\alpha_k(t)}{\Omega_k(t)[\Omega_k(t)-\frac{1}{2}i\gamma_k(t)]}\right]\ . \quad (5.2)$$

Under suitable conditions, it may be permissible to use a time-derivative expansion on the right-hand side, and the lowest term of this expansion yields

$$\left[\frac{\partial}{\partial t}+\gamma_k(t)\right]N_k(t)\approx[\alpha_k(t)/\Omega_k(t)]\times[1-i\gamma_k(t)/2\Omega_k(t)]^{-1}\ . \quad (5.3)$$

The validity of such an approximation is very difficult to assess reliably. As a rough estimate we observe, assuming $\gamma_k\ll\Omega_k$, that the first correction will be negligible if $|d(\alpha_k/\Omega_k^2)/dt|\ll|\alpha_k/\Omega_k|$. If we further assume that the denominator in (5.1) is slowly varying, which will be true if $\gamma_k\ll\Omega_k$, then (5.3) may be written as

$$\frac{d}{dt}n_k(t)\approx\tfrac{1}{2}\left[\frac{\alpha_k(t)}{\Omega_k(t)}-\gamma_k(t)\right]-\gamma_k(t)n_k(t)\ . \quad (5.4)$$

This is the approximation we wished to derive. When, in the following section, we have derived concrete expressions for α_k and γ_k, it will transpire that some further approximations serve to cast the right-hand side in the form of a scattering integral, which admits the Bose-Einstein distribution as a stationary solution. In view of (3.11), indeed, we may already anticipate that the right-hand side of (5.4) vanishes in thermal equilibrium.

VI. EVALUATION OF THE DISSIPATIVE COUNTERTERM

Our vaguely stated criterion for determining the arbitrary functions in the dissipative counterterm (3.15) was to optimize the unperturbed propagator as an approximation to the full propagator by minimizing the second term in (3.2). We now attempt to make this more precise, working in momentum space, but suppressing momentum arguments and integrations for clarity of notation. First, we extract from $\Sigma(t,t')$ the contribution of $\mathcal{M}$ and express the remainder in terms of the average and difference of its time arguments:

$$\Sigma(t,t')=\mathcal{M}(t,\partial/\partial t)\delta(t-t')+\tilde{\Sigma}(t-t',\tfrac{1}{2}(t+t')) . \tag{6.1}$$

Now Fourier transform the first argument of $\tilde{\Sigma}$, separating components which are odd and even in the frequency:

$$\tilde{\Sigma}(\tau,\bar{t})=\int \frac{d\omega}{2\pi} e^{i\omega\tau}[\Sigma^{(1)}(\omega^2,\bar{t})\omega+\Sigma^{(2)}(\omega^2,\bar{t})] . \tag{6.2}$$

The quantity $G_{11}(t,t')-g_{11}(t,t')$ contains, among other terms,

$$i\int dt''dt'''g_{11}(t,t'')\Sigma_{12}(t'',t''')G_{21}(t''',t')$$
$$=i\int dt''dt'''g_{11}(t,t'')\int\frac{d\omega}{2\pi}\left[\gamma(\bar{t})\frac{\partial}{\partial t''}+i\alpha(\bar{t})-i\Sigma_{12}^{(1)}(\omega^2,\bar{t})\frac{\partial}{\partial t''}+\Sigma_{12}^{(2)}(\omega^2,\bar{t})\right]e^{i\omega(t''-t''')}G_{21}(t''',t') , \tag{6.3}$$

where $\bar{t}=\frac{1}{2}(t''+t''')$. We have been slightly cavalier in our use of the Fourier representation of $\delta(t''-t''')$ and its derivative. To justify the appearance of $\gamma(\bar{t})$ where $\mathcal{M}$ contained $\gamma(t'')$, consider that, for any differentiable function $f(\tau)$,

$$\int d\tau f(\tau)\left[\gamma(\bar{t}+\tfrac{1}{2}\tau)\frac{\partial}{\partial t}+\tfrac{1}{2}\dot{\gamma}(\bar{t}+\tfrac{1}{2}\tau)\right]\delta(\tau)$$
$$=\int d\tau f(\tau)\frac{\partial}{\partial\tau}\left[\gamma(\bar{t}+\tfrac{1}{2}\tau)\delta(\tau)\right]$$
$$=-f'(0)\gamma(\bar{t})$$
$$=\int d\tau f(\tau)\gamma(\bar{t})\frac{\partial}{\partial t}\delta(\tau) .$$

Our course is clear from (6.3). The expression in large parentheses cannot be made to vanish for all ω. As in other renormalization methods, we require it to vanish at some reference frequency $\bar{\omega}$, which may depend on k and t:

$$\gamma_k(t)=i\Sigma_{12}^{(1)}(k,\bar{\omega}^2,t) , \tag{6.4}$$

$$\alpha_k(t)=i\Sigma_{12}^{(2)}(k,\bar{\omega}^2,t) . \tag{6.5}$$

An obvious choice for $\bar{\omega}$ is $\Omega_k(t)$ which loosely represents an instantaneous mass shell. In the same way, we shall identify

$$\beta_k(t)=k^2+m^2(t)-\mathrm{Re}\Sigma_{11}^{(2)}(k,\bar{\omega},t) . \tag{6.6}$$

The construction of the counterterm in Sec. III ensures that the structure of $\mathcal{M}$ matches that of Σ, so we could have considered any components of Σ with identical results. There is obviously some measure of arbitrariness in the procedure described here, although we have not actually succeeded in devising any satisfactory alternatives for comparison.

The first few terms in the expansion of the full propagator are given by the familiar set of diagrams drawn in Fig. 1. The propagator lines represent the 2×2 propagator matrix. Each vertex contributes a factor of $i\lambda$ if all four propagators arrive at it with index 1, $-i\lambda$ if they all arrive with index 2, and zero otherwise. Diagram (d) contains the counterterm and is constructed by multiplying matrices in the obvious way. We have not exhibited any mechanism for taking account of the initial density matrix, which may well involve additional diagrams. We leave these aside for now, returning to discuss their effect at the end of this section.

We consider only the lowest nontrivial contribution to (6.4)–(6.6). The first contribution to $\Sigma_{11}(t,t')$ comes from Fig. 1(b). It is real, and given by

$$\Sigma_{11}(k,t,t')=-\tfrac{1}{2}\lambda\int\frac{d^3k}{(2\pi)^3}g_{11,k}(t,t)\delta(t-t')$$
$$=-\tfrac{1}{2}\lambda\int\frac{d^3k}{(2\pi)^3}h_k(t,t)\delta(t-t') . \tag{6.7}$$

Because of the δ function, the right-hand side of (6.6) is independent of $\bar{\omega}$ and we obtain simply

$$\beta_k(t)=k^2+m^2(t)+\tfrac{1}{2}\lambda\int\frac{d^3k}{(2\pi)^3}h_k(t,t) . \tag{6.8}$$

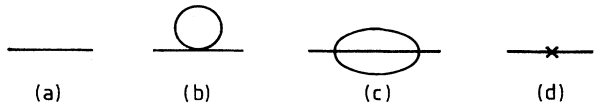

FIG. 1. Low-order contributions to the propagator matrix.

The first contribution to $\Sigma_{12}(t,t')$ comes from Fig. 1(c):

$$\Sigma_{12}(k,t,t')=\tfrac{1}{6}\lambda^2\int\frac{d^3k_1}{(2\pi)^3}\times\int\frac{d^3k_2}{(2\pi)^3}g_{12,k_1}(t,t')g_{12,k_2}(t,t')\times g_{12,k-k_1-k_2}(t,t')\ . \tag{6.9}$$

It is scarcely possible to achieve concrete results from (6.9) as it stands, since the functions $\alpha_k(t)$, $\gamma_k(t)$, and $\Omega_k(t)$ in the integrand are unknown. We now introduce a sequence of further approximations which, when the resulting expressions for $\alpha_k(t)$ and $\gamma_k(t)$ are substituted into (5.4), will lead to the promised Boltzmann equation. We are not, in any case, bound to adhere rigidly to the prescription (6.4)–(6.6). If, by whatever means, we can produce functions $\alpha_k(t)$, $\beta_k(t)$, and $\gamma_k(t)$ which in some measure reflect the properties of the self-energy matrix Σ, then these can be used to construct the dissipative counterterm, and we may expect an improved perturbation theory to result.

According to (6.1), the counterterm should reflect most accurately the properties of $\tilde{\Sigma}$ in the neighborhood of $t=t'$ and we expect (or, more accurately, hope) that this region supplies the most important contribution to the $\Sigma^{(1)}$ and $\Sigma^{(2)}$. On this assumption, denoting $h_k(t_1,t_2)$ by $h_k(\tau,t)$ with $\tau=t_1-t_2$ and $t=\frac{1}{2}(t_1+t_2)$, we approximate (4.13) by

$$h_k(\tau,t)\approx[4\Omega_k(t)]^{-1}e^{-\gamma_k(t)\tau}\times\{[1+N_k(t)]e^{-i\Omega_k(t)\tau}+[-1+N_k^*(t)]e^{i\Omega_k(t)\tau}\}\ . \tag{6.10}$$

For the propagator in (6.9), we then obtain

$$g_{12,k}(\tau,t)=h_k^*(\tau,t)\theta(\tau)+h_k(-\tau,t)\theta(-\tau)$$

$$\approx\int\frac{d\omega}{2\pi}e^{i\omega\tau}\left[\frac{\omega\gamma}{(\omega^2-\Omega^2-\frac{1}{4}\gamma^2)^2+\omega^2\gamma^2}+\frac{\frac{1}{2}i(1-\gamma/2\Omega)N}{\omega^2-(\Omega-i\gamma/2)^2}-\frac{\frac{1}{2}i(1+i\gamma/2\Omega)N^*}{\omega^2-(\Omega+i\gamma/2)^2}\right], \tag{6.11}$$

where the t and k arguments have been suppressed. We now define $n_k(t)$ through (5.1) and assume that it is real. This is not generally true, but it will be a good approximation when, for example, the assumptions leading to (5.4) are valid. We then have

$$g_{12,k}(\tau,t)\approx\int\frac{d\omega}{2\pi}e^{i\omega\tau}\left[\frac{\omega\gamma}{(\omega^2-\Omega^2-\frac{1}{4}\gamma^2)+\omega^2\gamma^2}+\frac{(2n+1)\Omega\gamma}{(\omega^2-\Omega^2+\frac{1}{4}\gamma^2)^2+\Omega^2\gamma^2}\right]. \tag{6.12}$$

This represents a considerable simplification, and may perhaps serve as a starting point for numerical calculations. For illustrative purposes, however, we go a step further and observe that, since (6.4) is already of order λ^2, the limit $\gamma\to0$ may reasonably be taken in (6.12) for the purpose of evaluating Σ_{12} at our low order of approximation. This finally yields the result

$$g_{12,k}(\tau,t)\approx\int d\omega\, e^{i\omega\tau}[\theta(\omega)+n_k(t)]\times\delta(\omega^2-\Omega_k^2(t))\ , \tag{6.13}$$

familiar from the equilibrium theory.[13–16] In the limit of zero decay width, we naturally encounter a well-defined mass shell.

It is now straightforward to obtain the Fourier transform $\tilde{\Sigma}_{12}(k,\omega,t)$ of the relevant component of (6.2):

$$\tilde{\Sigma}_{12}(k,\omega,t)=\frac{-i\lambda^2}{6(2\pi)^5}\int d^3k_1d^3k_2d^3k_3\int d\omega_1d\omega_2d\omega_3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k})\delta(\omega_1+\omega_2+\omega_3-\omega)\times\prod_{i=1}^{3}\delta(\omega_i^2-\Omega_{k_i}^2(t))[\theta(\omega_i)+n_{k_i}(t)]\ . \tag{6.14}$$

On selecting $\Omega_k(t)$ as the renormalization point, we identify

$$\alpha_k(t)=\tfrac{1}{2}i[\tilde{\Sigma}_{12}(k,\Omega_k(t),t)+\tilde{\Sigma}_{12}(k,-\Omega_k(t),t)]\ , \tag{6.15}$$

$$\gamma_k(t)=[i/2\Omega_k(t)][\tilde{\Sigma}_{12}(k,\Omega_k(t),t)-\tilde{\Sigma}_{12}(k,-\Omega_k(t),t)]\ . \tag{6.16}$$

The energy-momentum constraints in (6.14) are those of two-body scattering. If the dispersion relation for quasiparticles were $\Omega_k^2(t)=k^2+m^2(t)$, we should conclude that, of the four quantities $\omega_1,\omega_2,\omega_3,-\omega$, two should be positive, representing incoming particles, and two negative, corresponding to outgoing particles. We assume this to be true, although we have not been able to determine the exact conditions on Ω_k [a solution of (4.9)] which make it so.

With all these approximations, we can write the kinetic equation (5.4) as

$$\frac{d}{dt}n_k(t)=\frac{\lambda^2}{32(2\pi)^5}\int d^3k_1 d^3k_2 d^3k_3 \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}) \times\frac{\delta(\Omega_1+\Omega_2-\Omega_3-\Omega_k)}{\Omega_1\Omega_2\Omega_3\Omega_k} \times[n_1n_2(1+n_3)(1+n_k) -(1+n_1)(1+n_2)n_3n_k]\,, \tag{6.17}$$

where, for example, Ω_1 stands for $\Omega_{k_1}(t)$. This is indeed a Boltzmann equation which admits a stationary solution of the Bose-Einstein form

$$n_k=\{\exp[\beta(\Omega_k-\mu)]-1\}^{-1} \tag{6.18}$$

with inverse temperature β [not to be confused with $\beta_k(t)$] and chemical potential μ.

The one-loop contribution in (6.8) is much easier to compute:

$$\beta_k(t)=k^2+m^2(t)+\tfrac{1}{2}\lambda\int\frac{d^3k}{(2\pi)^3}[\tfrac{1}{2}+n_k(t)] \times\frac{\Omega_k(t)}{\Omega_k^2(t)+\frac{1}{2}\gamma_k^2(t)}\,. \tag{6.19}$$

It would be consistent with the foregoing approximations to set $\gamma_k(t)=0$ in this integral also.

Equations (4.9) and (4.11) or its approximation (6.17) now constitute a closed set which in principle can be solved numerically to determine the evolution of the state of the system, to the extent that this state is described by the propagator. Initial conditions on this evolution are set by the initial density matrix and we must now consider how this enters the formalism. We are able to consider seriously only the case that ρ is the canonical density operator mentioned in Sec. II. Its presence in (2.10) is taken fully into account when, as described in Sec. II, the 2×2 matrix of real-time propagators is augmented to a 3×3 matrix and interaction vertices arising from the initial Hamiltonian are added. In the absence of the dissipative counterterm, the Feynman rules are those given by Semenoff and Weiss.[10] The effect on the real-time propagators is twofold. First, as stated earlier, continuity and periodicity conditions determine, in effect, the initial conditions on (4.9) and (4.11). Second, there are additional diagrams which propagate correlations in the initial state to later times. At the lowest order of perturbation theory, the latter effect is trivial, and may be described as follows. The initial conditions supply occupation numbers of the form (3.12), in which ω_k is derived from the quadratic part of the initial Hamiltonian appearing in ρ. The additional diagrams serve, in lowest order, merely to renormalize these occupation numbers in accordance with interactions in the initial Hamiltonian. Higher-order effects of initial correlations are not so simple. To describe them correctly, it would be necessary to extend the considerations of the present paper to construct a 3×3 dissipative counterterm. While this can no doubt be done, it seems to us that the technical difficulty of actually computing higher-order corrections would make it a largely academic exercise. At the order of approximation considered here, it is sufficient simply to insert the required initial conditions by hand.

VII. A NUMERICAL ILLUSTRATION

By way of illustration, we describe an attempt to follow numerically the evolution described by (6.17) and (6.19) with $\gamma_k(t)=0$, for a specific choice of $m^2(t)$. We simplify (4.9) to the form $\Omega_k^2(t)=\beta_k(t)$, assume that $\Omega_k^2(t)$ has the form

$$\Omega_k^2(t)=k^2+M^2(t) \tag{7.1}$$

and determine $M^2(t)$ self-consistently from (6.19). The equations we obtain after all these approximations owe relatively little to our original formulation, in the sense that they might well have been guessed at by elementary informal arguments. We offer the following account first to provide a little qualitative insight and second in the hope of convincing the reader that our formalism really is capable of producing hard numerical information. On the other hand, we happily concede that more sophisticated approximations than those we have so far devised may be required to make this information reliable.

When $\Omega_k(t)$ has the above form, (6.17) can be reduced to

$$\frac{d}{dt}n_k(t)=\frac{\pi}{16}\left[\frac{\lambda}{4\pi^2}\right]^2\frac{n_k(t)}{k\Omega_k(t)} \times\int_D d\Omega_1 d\Omega_2 dp\; n_1n_2n_3 \times[(1+n_3^{-1})(1+n_k^{-1}) -(1+n_1^{-1})(1+n_2^{-1})]\,. \tag{7.2}$$

Here, p is the magnitude of the total three-momentum in the scattering process. It does not appear in the integrand, but the domain of integration D is over all positive Ω_1 and Ω_2 consistent with the double triangle relation summarized by Fig. 2. In various regions of the (Ω_1,Ω_2) plane, $\int dp$ is the equal to one or another of the k's. As discussed in the Appendix, a conformally coupled scalar field is described by (2.1) with $m^2(t)=a^2(t)m_0^2$,

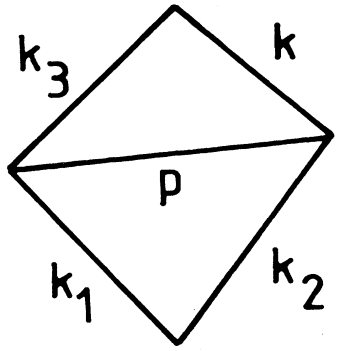

FIG. 2. Double-triangle relation between magnitudes of three-momenta in the scattering integral of Eq. (7.2).

where $a(t)$ is the Robertson-Walker scale factor and m_0 is the bare mass. A straightforward mass renormalization removes the divergent part of the integral in (6.19), and we choose units of momentum such that $a^2(t_0)m_R^2=1$. Then (6.19) takes the form

$$M^2(t)=a^2(t)+\left[\frac{\lambda}{4\pi^2}\right]\int_0^\infty dk\frac{k^2 n_k(t)}{\Omega_k(t)}\ , \tag{7.3}$$

where the scale factor is now normalized to $a(t_0)=1$ and the frequency is given by $\Omega_k^2(t)=k^2+M^2(t)$. We chose an initial distribution of the form (3.12) with $\omega_k=\Omega_k(t_0)$ and $\alpha=0$ and a scale factor of the form

$$a^2(t)=1+A[\tanh(t/\tau)-\tanh(t_0/\tau)]\ . \tag{7.4}$$

If $A\gg 1$, and $t_0\ll -\tau$, then $a^2(t)$ remains approximately constant at early times, expands by a factor of approximately $2A$ with a time scale of τ near $t=0$, and remains approximately constant thereafter. We, therefore, hope to see a transition from an initial equilibrium state to a final one, relaxation towards the final state occurring on a time scale which is implicit in (7.2), but not easily determinable in advance. The function (7.4) was chosen to illustrate such behavior and has no deeper significance.

Even when reduced to the skeleton form of (7.2) and (7.3), the numerical problem is not an easy one. We adopted the following strategy. The integral in (7.3), and similar integrals in the one-loop approximation to any expectation value we may wish to calculate, can be estimated using a Gauss-Legendre quadrature formula, via the mapping $k=(1+x)/(1-x)$ from the interval $-1<x<1$ where the formula is defined. We chose a 20-point formula, and solved (7.2) for the twenty n_k's associated with the quadrature points. These values are not sufficient to estimate the double integral in (7.2), where k_1, k_2, and k_3 are constrained by momentum conservation. To evaluate this integral, we adopted the expedient of fitting the calculated n_k's to a function

$$n_k^{\rm eff}(t)=(\exp\{\beta(t)[\bar{M}^2(t)+k^2]^{1/2}-\alpha(t)\}-1)^{-1}\ . \tag{7.5}$$

This has the Bose-Einstein form, but it coincides with the stationary solution of (7.2) only when $\bar{M}=M$. The fit parameters $\beta(t)$, $\bar{M}(t)$, and $\alpha(t)$ also provide a convenient parametrization of the results. We solved (7.2) using a fourth-order Adams-Bashforth forward integration method, using the Euler point-shape formula to start the solution and vary the time step. At each step, (7.3) was solved for $M(t)$ by the secant method.

Figure 3 shows our solution for parameter values $(\lambda/4\pi^2)=\frac{1}{2}$, $\beta(t_0)=1$, $A=5$, and $t_0=-6\tau$. At first glance, these results are encouraging, and may be understood as follows. Interpreted in terms of an expanding universe, our Hamiltonian generates translations in conformal time, so $\beta(t)$ is the inverse of a "conformal temperature." Roughly speaking, it is related to the physical temperature by $\beta(t)=\beta_{\rm phys}(t)/a(t)$. For a massless free field, this quantity remains constant during the expansion, the physical temperature being redshifted as $1/a(t)$. In Fig. 3 we see that for our massive interacting field also,

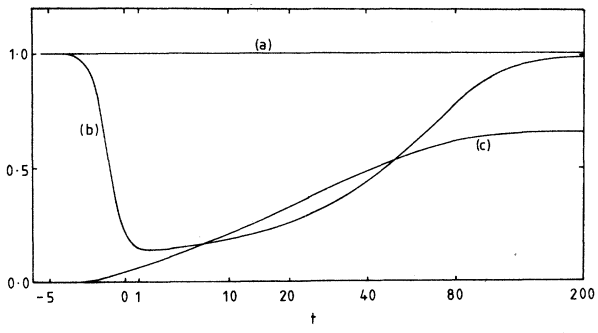

FIG. 3. Evolution with time of (a) $\beta(t)$, (b) $\bar{M}^2(t)/M^2(t)$, and (c) $\alpha(t)$. Time is measured in units of τ on a scale which is linear for negative times and logarithmic for positive times.

$\beta(t)$ is almost constant, increasing by less than 1% during the whole run. The thermal correction to the effective quasiparticle mass in (7.3) is, for our chosen parameter values, fairly small—about 30% at the higher initial physical temperature and about 10% at the final temperature. Thus, $M(t)$ increases roughly in proportion to $a(t)$, but $\bar{M}(t)$ in (7.5) remains close to its initial value $\bar{M}(t_0)=M(t_0)$, so long as the occupation numbers have had no chance to relax. Clearly, the relaxation time is much greater than τ, so the ratio $\bar{M}^2(t)/M^2(t)$ falls sharply during the expansion and then slowly recovers to the value 1 characteristic of the final equilibrium state.

The quantity

$$N_{\rm qp}=\int_0^\infty k^2 n_k(t)dk \tag{7.6}$$

is proportional to the total number of quasiparticles per unit comoving volume and is conserved by (7.2). The major contribution to the integral is from small-k modes, whose occupation numbers we find to remain essentially constant. Therefore, as $\bar{M}(t)$ in (7.5) increases, the effective reduced chemical potential $\alpha(t)$ increases from its initial value of zero to maintain the quasiparticle numbers, as seen in Fig. 3.

What is less encouraging is that (7.5) is not a particularly good fit to the numerically generated distribution, so the apparent approach to a final equilibrium state in Fig. 3 is to some extent illusory. In particular, $n_k(t)$ appears to develop an oscillatory modulation as a function of k. We know of no good reason for this, and our numerical method is far too crude to allow a detailed investigation. From time to time, we evaluated the integral (7.6) and found its value to be preserved within 1% up to times of about 40τ, which suggests that (7.2) may be fairly insensitive to inadequacies of the analytic representation (7.5). By the end of the run, however, $N_{\rm qp}$ was about 50% too high, presumably owing to the accumulation of numerical errors. Changing the detailed procedure for fitting (7.5) to the data had only a minor effect, leaving the qualitative form of the curves unchanged.

One highly desirable feature of this investigation is perhaps worth emphasizing. If our system does indeed relax towards equilibrium, then it does so of its own accord. We have not coupled it to an external heat bath which would prescribe the final ensemble in advance.

Admittedly, though, in our actual calculation, the initial equilibrium state and the use of (7.5) may well have had a propitious influence. Since (7.1)–(7.3) are an extreme simplification of the ones we actually derived, this calculation is essentially only of qualitative interest, and we have, therefore, not striven for great numerical accuracy. It does, however, give reassurance that there is at least one starting point from which we may expect to develop more reliable methods of calculation.

VIII. SUMMARY AND DISCUSSION

Using the closed-time-path approach to nonequilibrium field theory, we have shown how dissipative effects may be incorporated in unperturbed propagators. Because of the finite relaxation times thus introduced, the propagators evolve with time in accordance with the changing state of the quantum fields, and so, therefore, do the expectation values of physical quantities when evaluated using low-order perturbation theory. We stress that this evolution is *not* adequately described by standard perturbation theory. Our method is to add and subtract a suitably constructed counterterm in the composite Lagrangian, which can be used to effect an approximate resummation of the absorptive parts of higher-order contributions to the full propagators. Because we used a local counterterm, we expect this resummation to be most useful when the time arguments of Green's functions span an interval considerably smaller than the relevant relaxation times. This method makes sense only within the closed-time-path formalism, since the counterterm mixes fields defined on different segments of the path. It seems that the more naive idea of adding and subtracting a non-Hermitian part to the canonical Hamiltonian to obtain finite quasiparticle decay rates could not be implemented in a consistent manner. Such a procedure would, of course, involve expanding about a theory with explicitly nonunitary time evolution. In a rough sense, the elements of the counterterm matrix which describe dissipative effects may be understood as assigning suitable finite values to the $i\epsilon$ terms which, for time-independent states, define contours of integration for the various propagators in the complex frequency plane [see, for example, Eq. (4.3) of Ref. 14 where, however, a different time path is used].

We exhibited an approximation in which time-dependent occupation numbers for quasiparticle modes can be identified, and further approximations which lead to a Boltzmann equation for their time evolution. Finally, we presented a numerical solution of this equation which demonstrates, at a qualitative level, that our formalism is indeed capable of describing the time evolution in concrete terms. The approximations essentially assume that time evolution at a momentum scale k is slow compared with the inverse of elementary excitation energies Ω_k, and are especially vulnerable in the neighborhood of a critical point, where infrared singularities must be handled with care.

We are aware of two other approaches to nonequilibrium field theory which have features in common with that described here. The closed-time-path formalism has been used by Calzetta and Hu[18] to study scalar field theory with a time-independent Lagrangian, but with a general Gaussian initial density matrix. By truncating the hierarchy of Dyson-Schwinger equations, and by imagining space-time to be divided into cells such that the state of the field is approximately uniform within each cell, they derive a Boltzmann equation which is essentially equivalent to that exhibited here. Their considerations can probably be generalized to the time-dependent case without undue difficulty. Nevertheless, we believe our formal development to be advantageous, inasmuch as it provides a systematic means both of describing dissipation within low-order perturbation theory and of improving upon the severe approximations needed to obtain the Boltzmann equation. Calzetta, Habib, and Hu[19] have considered noninteracting theories in a general curved spacetime, and derived a kinetic equation of the Vlasov type within an adiabatic expansion.

The thermo field dynamics (TFD) of Umezawa and his co-workers has recently been generalized to deal with nonequilibrium states,[20,21] and this generalization introduces a "semifree" Hamiltonian in a similar spirit to our dissipative counterterm. This theory is based on an extension of canonical field theory, but for equilibrium states is equivalent to the closed-time-path formulation of quantum statistical mechanics.[22] We doubt whether the same is true for nonequilibrium states. One reason for this is that TFD contains a free parameter (denoted by α in Ref. 20) which, in equilibrium, mirrors the freedom to choose different time paths.[22] For nonequilibrium states, only the path described in Sec. II is allowed, either because the initial density matrix does not have the canonical form or because the Hamiltonians at two different times do not commute. However, the corresponding freedom does not disappear from TFD and, indeed, the authors of Refs. 20 and 21 have been able to complete their analyses only for the value $\alpha=\frac{1}{2}$, which corresponds to a disallowed time path. The propagators obtained in Ref. 21 are rather different from ours, but we have not succeeded in establishing whether or not some deeper equivalence between the two formalisms exists. Our own view is that TFD is useful only to the extent that it can be shown to be equivalent to quantum statistical mechanics, but some authors would no doubt wish to contest this view.

Finally, we recall that several quite different approaches to the nonequilibrium evolution of quantum fields in the early Universe have been proposed. Hosoya and Sakagami[23] have derived an equation of motion on the basis of both elementary kinetic theory and a phenomenological nonequilibrium statistical operator. Whether this yields an adequate approximation to the true statistical mechanics problem is not clear to us at present. Guth and Pi[24] have used a Schrödinger picture approach to calculate the time evolution of field modes and wave functionals for an effective free field model, and an extension of this line of investigation to study the evolution of more general Gaussian density matrices has been described recently by Éboli, Jackiw, and Pi.[25] This approach is appealing in its directness, and provides a characterization of the state of the field which cannot

readily be obtained from Green's functions. At present, however, a systematic means of dealing with interacting theories and non-Gaussian density matrices is lacking. Yet another approach, calling upon Fokker-Planck-type equations for Wigner distributions or their associated Langevin equations, has been pursued by several authors.[26]

ACKNOWLEDGMENTS

The work was undertaken in part during a period of study leave spent at the University of British Columbia. The author would like to thank the Physics Department of the University of British Columbia for its hospitality and the Anglo-Canadian Scientific Exchange Scheme for generous financial support. It is a pleasure to acknowledge helpful discussions with T. Arimitsu, E. Calzetta, G. Semenoff, H. Umezawa, W. Unruh, and Y. Yamanaka.

APPENDIX

In a Robertson-Walker universe with line element

$$ds^2=dt^2-a^2(t)d\mathbf{x}^2 , \tag{A1}$$

the covariant action for a self-interacting scalar field may be written as

$$S_{\mathrm{RW}}=\int dt\, d^3x\, a^3 \left[\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}a^{-2}\nabla\phi\cdot\nabla\phi-\tfrac{1}{2}m_0^2\phi^2 -\tfrac{1}{2}\xi R\phi^2-\frac{1}{4!}\lambda\phi^4 \right] , \tag{A2}$$

where ξ measures the strength of the unique renormalizable coupling to the Ricci scalar curvature R. The coordinate t in (A1) is physical time, in the sense that it represents the time measured by an observer with fixed comoving spatial coordinates $\mathbf{x}$. The transformation

$$\phi=a^{-1}\phi_c , \tag{A3}$$

$$\frac{dt_c}{dt}=a^{-1} , \tag{A4}$$

which introduces the conformal time t_c leads to a new action

$$S_c=\int dt_c d^3x \left[\tfrac{1}{2}\dot{\phi}_c^2-\tfrac{1}{2}\nabla\phi_c\cdot\nabla\phi_c -\tfrac{1}{2}a^2[m_0^2+(\xi-\tfrac{1}{6})R]\phi_c^2-\frac{1}{4!}\lambda\phi_c^4 \right] , \tag{A5}$$

where now the overdot means $\partial/\partial t_c$. The case of conformal coupling $\xi=\frac{1}{6}$ was used in Sec. VII.

The two actions (A2) and (A5) differ by the space-time integral of a total derivative, and describe the same dynamics. The Hamiltonians obtained from them are related by

$$H_{\mathrm{RW}}=a^{-1}H_c+\tfrac{1}{2}a^{-1}\int d^3x[(a^{-1}\dot{a}^2+\tfrac{1}{6}a^2R)\phi_c^2 -a^{-1}\dot{a}\{\pi_c,\phi_c\}] , \tag{A6}$$

where π_c is the momentum conjugate to ϕ_c. Although the two Hamiltonians describe the same dynamics, density matrices of the form $\exp[-\beta_{\mathrm{RW}}H_{\mathrm{RW}}(t_0)]$ and $\exp[-\beta_c H_c(t_0)]$ do not describe equivalent statistical ensembles, on account of the additional term in (A6). Indeed, it is well known that field theories in curved spacetimes do not possess unique vacua,[27] and the zero-temperature limits of the above ensembles correspond to two inequivalent candidate vacua. Naively, it seems to us that, at sufficiently early times, $a^2m_0^2$ is very small, leading to an almost time-independent Hamiltonian in the conformal description, and that this description is, therefore, the best means of setting up an initial equilibrium state. This consideration, devoid though it is of all rigor, serves to motivate the particular version of the problem studied in this paper.

[1] I. D. Lawrie, in *Advances on Phase Transitions and Disorder Phenomena,* edited by G. Busiello, L. de Cesare, F. Mancini, and M. Marinaro (World Scientific, Singapore, 1987), p. 548.
[2] I. D. Lawrie, Nucl. Phys. **B301**, 685 (1988).
[3] I. D. Lawrie, J. Phys. A **21**, L823 (1988).
[4] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982); A. Linde, Phys. Lett. **108B**, 389 (1982).
[5] A. Guth, Phys. Rev. D **23**, 347 (1981).
[6] A. Linde, Phys. Lett. **129B**, 177 (1983).
[7] G. F. Mazenko, Phys. Rev. Lett. **54**, 2163 (1985).
[8] G. F. Mazenko, Phys. Rev. D **34**, 2223 (1986).
[9] A. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. Hawking, Phys. Lett. **115B**, 295 (1982); A. Starobinsky, *ibid.* **117B**, 175 (1982); J. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[10] G. W. Semenoff and N. Weiss, Phys. Rev. D **31**, 689 (1985); **31**, 699 (1985).
[11] See, e.g., L. E. Reichl, *A Modern Course in Statistical Physics* (Edward Arnold, London, 1980).
[12] J. Schwinger, J. Math. Phys. **2**, 407 (1961); L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1615 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].
[13] For a review, see K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).
[14] A. J. Niemi and G. W. Semenoff, Ann. Phys. (N.Y.) **152**, 105 (1984).
[15] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[16] A. J. Niemi and G. W. Semenoff, Nucl. Phys. **B230**, 18 (1984); R. L. Kobes, G. W. Semenoff, and N. Weiss, Z. Phys. C **29**, 371 (1985); R. L. Kobes and K. L. Kowalski, Phys. Rev. D **34**, 513 (1986).
[17] R. D. Jordan, Phys. Rev. D **33**, 444 (1986).
[18] E. Calzetta and B. L. Hu, Phys. Rev. D **37**, 2878 (1988).
[19] E. Calzetta, S. Habib, and B. L. Hu, Phys. Rev. D **37**, 2901 (1988).
[20] T. Arimitsu, M. Guida, and H. Umezawa, Physica (Utrecht) **148A**, 1 (1988); T. Arimitsu, H. Umezawa, Y. Yamanaka, and N. J. Papastamatiou, *ibid.* **148A**, 27 (1988).

[21]H. Umezawa and Y. Yamanaka, University of Alberta report, 1987 (unpublished).
[22]H. Matsumoto, Y. Nakano, and H. Umezawa, J. Math. Phys. **25**, 3076 (1984).
[23]A. Hosoya and M. Sakagami, Phys. Rev. D **29**, 2228 (1984).
[24]A. Guth and S.-Y. Pi, Phys. Rev. D **32**, 1899 (1985).
[25]O. Éboli, R. Jackiw, and S.-Y. Pi, Phys. Rev. D **37**, 3557 (1988).
[26]A. D. Linde, Phys. Lett. B **175**, 395 (1986); S.-J. Rey, Nucl. Phys. **B284**, 706 (1987); F. R. Graziani, Phys. Rev. D **38**, 1122 (1988); **38**, 1131 (1988); **38**, 1802 (1988).
[27]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).