Nonstationary general-relativistic "strings"

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We examine cylindrically symmetric solutions to the Einstein equations under the assumption of functional separability of the metric coefficients. Under the further assumption that the energy-momentum tensor is of the form $T^t_t = T^z_z = -\sigma(r,t)$, all such solutions, with their vacuum exteriors, are found. It is shown that the vacua cannot be matched smoothly onto a Robertson-Walker background.

I. INTRODUCTION

In recent years it has been proposed that the spontaneous breaking of symmetries in grand unified theories during phase transitions in the early Universe could lead to the formation of such structures as domain walls, cosmic strings, and monopoles.[1-3] In this paper we concern ourselves only with "strings."[4] It has been suggested that strings could act as gravitational lenses, giving multiple images of distant objects.[5,6] They may also be responsible for creating the density perturbations in the early Universe necessary for galaxy formation.[7,8] It is also thought that as they move through the Universe they might tend to focus the matter into large-scale structures such as filaments and sheets.[9,10]

Most of the analytical work concerning the gravitational effects of strings has been based on a general relativistic, static model of a straight string surrounded by a vacuum.[5,6] Strings need not be straight and may exist either as infinitely long kinked strings or as closed arbitrarily shaped loops (but see Unruh *et al.*[11] for the relationship between the string's shape and the background curvature). However, anything other than an infinitely long straight string or possibly a circular closed loop would be virtually impossible to treat analytically in general relativity. Here we will be concerned with infinitely long straight strings as this imposes cylindrical symmetry on the spacetime.

The standard string model is described by an energy-momentum tensor of the form $T^t_t = T^z_z = -\sigma = p_z$ (all other components are zero) where σ is the energy density and p_z is the pressure along the axis of the string. This form of the energy-momentum tensor was derived under the assumption that any lateral stresses inside the string would be negligible and that the string is Lorentz invariant along its axis.[5] This allows the energy-momentum tensor, as derived from a Lagrangian describing the string, to be averaged over the cross section of the string, resulting in the form given above. The energy density in this case is a function only of the radial distance r from the axis. As Garfinkle[12] has shown, this string model is somewhat idealized since, in general, the field equations as derived from a Lagrangian lead to nonzero lateral stresses. In particular, in the case of a static, flat background metric, the other diagonal components of the energy-momentum tensor can be as large as $\frac{1}{5}$ of the energy density. In spite of this inconsistency the idealized model is useful in that it can give us information about the general properties of strings (e.g., the bending of light).

Linet[13] has shown that under these conditions the most general solution of the Einstein equations depends on one arbitrary function of radius, $\omega(r)$, which is subject to some regularity conditions. This arbitrary function determines the radial profile of the energy density. If $\omega(r)$ is linear in r, the energy density is zero and the spacetime represents vacuum. This allows the string's interior to be embedded in an exterior vacuum simply by extending $\omega(r)$ linearly outside some finite radius. All that is required is that $\omega(r)$ and its first derivative be continuous at the surface of the string. The external spacetime in this case is conical vacuum in the sense that the metric can be transformed to that of Minkowski space in which the azimuthal coordinate ϕ has an angular deficit given by $\Delta\phi = 8\pi u$ where u is the linear energy density of the string. That is, ϕ ranges from 0 to $2\pi - \Delta\phi$. This result also applies if the string is taken to be a line source.[13]

This model of cosmic strings is, however, somewhat incomplete in the sense that the strings are embedded in a vacuum cannot be smoothly patched onto a FRW background.[14] [The analysis in Ref. 14 is flawed; their metric (9) is incorrect, but the result is correct as will be shown in Sec. IV]. Here we will search for possible models of strings which are nonstationary. We will assume an idealized energy-momentum tensor of the same form as in the static model but with σ depending on time as well ground.[14] (The analysis in Ref. 14 is flawed; their metric (9) is incorrect, but the result is correct as will be shown in Sec. IV). Here we will search for possible models of strings which are nonstationary. We will assume an idealized energy-momentum tensor of the same form as in the static model but with σ depending on time as well as radius. The Einstein equations and their solutions will be presented in Sec. II. Solutions will be found under the further assumption that the metric coefficients are separable functions of their arguments. This is done for simplicity's sake since it reduces the equations to a set of ordinary, rather than partial, differential equations.

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While there is no *a priori* reason to assume that a cosmological string model should be separable, the assumption does lead to very definite conclusions. Whether or not the assumption of separability significantly detracts from the step from static to nonstatic models can only be judged when nonstatic nonseparable models are available. In Sec. III we will confirm that the models are nonstationary and in Sec. IV we will determine whether or not they can be embedded in a FRW background.

While the particular form of the energy-momentum tensor used here does not hold in general (e.g., lateral pressures may be important at some time[12]), models based on it may be valid at some time during their evolution. Since we are concerned with embedding strings in a FRW background, we are looking at strings relatively late in their evolution and by this time the lateral stresses may indeed have dissipated. Thus a string model which is based on the energy-momentum tensor used here and is embedded in a FRW background may represent a fairly consistent model for the late stages of a string's evolution.

II. FIELD EQUATIONS

Since we are concerned with infinitely long straight strings, we begin with the general cylindrically symmetric metric (see, e.g., Kramer *et al.*[15])

$$ds^2=-e^{2(K-U)}(dt^2-dr^2)+e^{-2U}W^2d\phi^2+e^{2U}dz^2 , \quad (1)$$

where U, K, and W are functions of r and t, ϕ is the azimuthal coordinate, and z is the axial coordinate. We define the axis of cylindrical symmetry to be the value of r for which $g_{\phi\phi}$ vanishes (i.e., there is no azimuthal component to the metric on the axis). We use the notation $(t,r,\phi,z)=(x^0,x^1,x^2,x^3)$. If we assume an energy-momentum tensor of the form $T^0_0=T^3_3=-\sigma(r,t)$, the Einstein equations ($G^\alpha_\beta=8\pi T^\alpha_\beta$) are

$$e^{2(U-K)}\left[\dot K\frac{\dot W}{W}-\frac{W''}{W}+K'\frac{W'}{W}-\dot U^2-U'^2\right]=8\pi\sigma , \quad (2)$$

$$\frac{\ddot W}{W}-\dot K\frac{\dot W}{W}-K'\frac{W'}{W}+\dot U^2+U'^2=0 , \quad (3)$$

$$\dot U^2-U'^2+\ddot K-K''=0 , \quad (4)$$

$$e^{2(U-K)}\left[\frac{\ddot W}{W}-2\dot U\frac{\dot W}{W}-2\ddot U+\dot U^2+\ddot K -\frac{W''}{W}+2U'\frac{W'}{W}+2U''-U'^2-K''\right]=8\pi\sigma , \quad (5)$$

and

$$K'\frac{\dot W}{W}-\frac{\dot W'}{W}+\frac{\dot K W'}{W}-2\dot U U'=0 , \quad (6)$$

where an overdot denotes $\partial/\partial t$ and a prime denotes $\partial/\partial r$.

Using Eqs. (2) and (3) we find

$$8\pi\sigma=e^{2(U-K)}\left[\frac{\ddot W}{W}-\frac{W''}{W}\right] \quad (7)$$

and Eqs. (2)–(5) give

$$\dot U\frac{\dot W}{W}+\ddot U=U'\frac{W'}{W}+U'' . \quad (8)$$

Henceforth we shall use (7) and (8) in place of (2) and (5) since (2) and (5) can be recovered using (3), (4), (7), and (8).

The conservation equations ($T^\alpha_{\beta;\alpha}=0$) give the relations

$$\dot\sigma+\sigma\left[\dot K+\frac{\dot W}{W}-2\dot U\right]=0 \quad (9)$$

and

$$K'\sigma=0 . \quad (10)$$

We see immediately from (10) that, in order for the energy density to be nonzero, K' must be zero, i.e., $K=K(t)$. Equation (9) can be integrated to give

$$\sigma=C(r)\frac{e^{2U-K}}{W} , \quad (11)$$

where $C(r)$ is an arbitrary function of r.

We now make the simplifying assumption that the metric coefficients are separable functions of their arguments, i.e., $U=u(t)+\mu(r)$ and $W=w(t)\omega(r)$. Equation (11) then shows that σ will also be a separable function of its arguments. Stein-Schabes[16] attempted to find solutions to the field equations using separable metric coefficients and assuming that the energy density is separable. As we have seen, the separability of the energy density follows from that of the metric coefficients. Stein-Schabes found three classes of solutions. However, one class of solutions is invalid [his Eqs. (31) and (39) are incorrect] and Stein-Schabes did not note that another class of solutions is equivalent to the usual static solutions [his class (iii)]. In what follows we will find all possible classes of solutions.

Equations (7), (3), (4), and (8) can now be written as

$$8\pi\sigma=e^{2(u+\mu-K)}\left[\frac{\ddot w}{w}-\frac{\omega''}{\omega}\right] , \quad (12)$$

$$\frac{\ddot w}{w}-\dot K\frac{\dot w}{w}+\dot u^2=\alpha=-\mu'^2 , \quad (13)$$

$$\dot u^2+\ddot K=\beta=\mu'^2 , \quad (14)$$

$$\dot u\frac{\dot w}{w}+\ddot u=\gamma=\mu'\frac{\omega'}{\omega}+\mu'' , \quad (15)$$

where α, β, and γ are constants. Equation (6) now becomes

$$\frac{-\omega'\dot w}{\omega w}+\dot K\frac{\omega'}{\omega}-2\mu'\dot u=0 . \quad (16)$$

Thus we have $\alpha=-\beta\le 0$ and write Eq. (16) as

$$\frac{\omega'}{\omega}\left[\dot K-\frac{\dot w}{w}\right]=2\sqrt{\beta}\dot u . \quad (17)$$

We will now consider the possible classes of solutions.

A. $\beta=0$

In this case we have $\mu'=0$ and (15) shows that $\gamma=0$. Equation (16) then demands that either $\omega'=0$ or $\dot{K}=\dot{w}/w$. If $\omega'=0$ we see that $g_{\phi\phi}$ is independent of r so that there is no axis and we do not consider this to be a string. Thus we consider only $\dot{K}=\dot{w}/w$, i.e., $e^K=c_1 w$ (in what follows, c_n represents an arbitrary constant) and we see that $\omega(r)$ is arbitrary. In this case Eqs. (13) and (14) are equivalent and, if $\dot{u}\neq 0$, Eq. (15) becomes

$$\frac{\ddot{u}}{\dot{u}}=-\frac{\dot{w}}{w}$$

which can be integrated to give

$$\dot{u}=\frac{c_2}{w}\ . \tag{18}$$

The case $\dot{u}=0$ can therefore be included simply by setting $c_2=0$. Equation (13) now becomes

$$\dot{w}^2-w\ddot{w}=c_2^2 \tag{19}$$

which allows different classes of solution depending on whether or not $c_2=0$.

1. $c_2=0$

Equation (19) now demands that either $\dot{w}=0$ or $w=c_3 e^{c_4 t}$ while (18) gives $e^u=c_5$. If $\dot{w}=0$ we have $w=c_6$ and the metric can be written as

$$ds^2=-dt^2+dr^2+\omega^2(r)d\phi^2+dz^2\ , \tag{20}$$

where the constants have been set to unity by scaling r, t, and z or absorbed by $\omega(r)$ [since the field equations define $\omega(r)$ only up to an arbitrary multiplying factor]. This is simply the metric used to describe static strings.[13] With $w=c_3 e^{c_4 t}$, the metric can be written as

$$ds^2=e^{2t}[-dt^2+dr^2+\omega^2(r)d\phi^2]+dz^2\ , \tag{21}$$

where, again, the arbitrary constants c_3, c_4, and c_5 have been eliminated by simple coordinate transformations. The energy density in this case is given by

$$8\pi\sigma=e^{-2t}\left[1-\frac{\omega''}{\omega}\right]\ . \tag{22}$$

Because of the arbitrary nature of $\omega(r)$, the string described by (21) and (22) can easily be embedded in an external vacuum. From (22) we see that this solution will represent vacuum if $\omega''=\omega$, i.e., $\omega=c_7\sinh r+c_8\cosh r$. It will be shown below that this vacuum is the usual static vacuum. If we choose a particular form of $\omega(r)$ to represent the interior of the string, we simply require that at some boundary value of r, say r_0, the values of the internal form of $\omega(r)$ and its first derivative match those of the external vacuum form given above. That is, we assume that the surface of the string satisfies the Lichnerowicz boundary conditions;[17] that the (t,r,ϕ,z) coordinate system is admissible. In order to ensure that the energy density is positive inside the string and finite on the axis, we must impose the conditions $\omega''<\omega$ and $\lim_{\omega\to 0}(\omega''/\omega)=\text{const.}$

As an example let us choose as the interior solution $\omega_{int}=c_9 r$. The two conditions on the energy density are then easily satisfied since $\omega''=0$. From (22) we see that this represents a homogeneous interior (since σ is independent of r) and that the spacetime is anisotropic since the pressure is anisotropic. Matching the interior and exterior forms of $\omega(r)$ at r_0 then gives the two relations

$$c_9 r_0=c_7\sinh r_0+c_8\cosh r_0$$

and

$$c_9=c_7\cosh r_0+c_8\cosh r_0$$

from which we find

$$r_0^2=1-\frac{c_7^2-c_8^2}{c_9^2}$$

so that we must have $c_7^2<c_8^2+c_9^2$.

2. $c_2\neq 0$.

If we make the substitution $\dot{w}^2=f(w)$ in Eq. (19) we find

$$\frac{-1}{c_2^2-f}\frac{df}{dw}=\frac{2}{w}\ ,$$

which, upon integration, gives

$$\dot{w}^2=c_2^2(c_{10}w^2+1)\ . \tag{23}$$

Equation (23) suggests that we now transform to w as a coordinate using

$$dt=\frac{dw}{c_2(c_{10}w^2+1)^{1/2}}$$

so that

$$\frac{du}{dw}=\frac{\dot{u}}{\dot{w}}=\frac{\pm 1}{w(1+c_{10}w^2)^{1/2}}\ . \tag{24}$$

The solutions in this case depend on whether or not $c_{10}=0$.

(a) $c_{10}=0$. Equation (24) gives $e^u=c_{11}w^{\pm 1}$. With $e^u=c_{11}w^{+1}$, the metric can be written as

$$ds^2=-dw^2+dr^2+\omega^2(r)d\phi^2+w^2dz^2\ , \tag{25}$$

where arbitrary constants have been scaled to unity. This is simply the static metric (20) in a different coordinate system as can be seen from the transformations $w^2=T^2-Z^2$ and $\tanh z=Z/T$.

If $e^u=c_{11}w^{-1}$ the metric is

$$ds^2=w^4[-dw^2+dr^2+\omega^2(r)d\phi^2]+w^{-2}dz^2 \tag{26}$$

and the energy density is given by

$$8\pi\sigma=-w^{-4}\frac{\omega''}{\omega}\ . \tag{27}$$

The string described by (26) and (27) can also be easily embedded in vacuum. According to (27), the vacuum region will have $\omega''=0$, i.e., $\omega_{\text{vac}}=c_{12}r+c_{13}$. The interior form of $\omega(r)$ will have to satisfy the conditions $\omega''<0$ and $\lim_{\omega\to 0}\omega''/\omega=\text{const}$ so that the energy density is positive as well as finite on the axis. As an example let us choose the interior solution to be $\omega_{\text{int}}=\sin(c_{14}r)$. Matching the interior and exterior at r_0 requires

$$\sin(c_{14}r_0)=c_{12}r_0+c_{13} \; ,$$

$$c_{14}\cos(c_{14}r_0)=c_{12} \; ,$$

which gives

$$r_0=\frac{(c_{14}^2-c_{12}^2)^{1/2}}{c_{14}c_{12}}-\frac{c_{13}}{c_{12}} \; .$$

(b) $c_{10}\neq 0$. In this case, Eq. (24) can be integrated to give

$$u=\mp\ln\left[\frac{1+(1+c_{10}w^2)^{1/2}}{w}\right]+c_{15} \tag{28}$$

in which case there are two possible classes of solution depending on whether we use the + or − sign.

Using the − sign in (28) the metric can be written as

$$ds^2=[1+(1+c_{10}w^2)^{1/2}]^2 \times\left[\frac{-dw^2}{c_2^2(1+c_{10}w^2)}+dr^2+\omega^2(r)d\phi^2\right] +\frac{w^2dz^2}{[1+(1+c_{10}w^2)^{1/2}]^2} \tag{29}$$

and σ is given by

$$8\pi\sigma=\frac{1}{[1+(1+c_{10}w^2)^{1/2}]^2}\left[c_{10}c_2^2-\frac{\omega''}{\omega}\right] . \tag{30}$$

A solution equivalent to the $c_{10}>0$ case has been given by Stein-Schabes [case (i) of Ref. 16]. The $c_{10}<0$ case is, as far as we know, new.

The solutions given by (29) and (30) can be embedded in vacuum. If $c_{10}>0$, the embedding is done exactly as it was for the metric (21). If $c_{10}<0$ the external vacuum has $\omega_{\text{vac}}=c_{16}\sin(\sqrt{-c_{10}}c_2r+c_{17})$. In the interior region we require that ω''/ω be finite on the axis. We also require that $\omega''/\omega<c_{10}c_2^2$ for the energy density to be positive. As an example, let us choose the interior solution to be $\omega_{int}=\sin(\sqrt{-c_{10}}c_{18}r)$ where $|c_{18}|>|c_2|$. Matching the interior and exterior at r_0 then gives

$$\sin(\sqrt{-c_{10}}c_{18}r_0)=c_{16}\sin(\sqrt{-c_{10}}c_2r_0+c_{17})$$

and

$$c_{18}\cos(\sqrt{-c_{10}}c_{18}r_0)=c_{16}c_2\cos(\sqrt{-c_{10}}c_2r_0+c_{17}) \; .$$

The interior of the string is homogeneous for this form of ω_{int}.

If $c_{10}<0$ in the metric (29), w is limited to the range $0<w^2<|c_{10}|^{-1}$. The proper radius of the string for a given w, ϕ, and z is

$$R=\int_0^{r_0}\sqrt{g_{rr}}dr=[1+(1+c_{10}w^2)^{1/2}]r_0 \tag{31}$$

and the proper length between two values of z for a given w, r, and ϕ is

$$L=\int_{z_1}^{z_2}\sqrt{g_{zz}}dz=\frac{w}{1+(1+c_{10}w^2)^{1/2}}(z_2-z_1) \; . \tag{32}$$

We see that as $w\to 0$ the proper radius is $2r_0$ and the proper length between two values of z goes to zero. As $w\to\pm|c_{10}|^{-1/2}$ the proper radius goes to zero and the proper length between z_2 and z_1 goes to a maximum value of $|c_{10}|^{-1/2}(z_2-z_1)$. Thus we see that the string expands in the z direction and contracts radially as w goes from 0 to $|c_{10}|^{-1/2}$ and does the reverse for $-|c_{10}|^{-1/2}<w<0$. The energy density in this case goes from a minimum at $w=0$ of

$$8\pi\sigma=\frac{1}{4}\left[c_{10}c_2^2-\frac{\omega''}{\omega}\right] \tag{33}$$

to a maximum at $w=\pm|c_{10}|^{-1/2}$ of

$$8\pi\sigma=c_{10}c_2^2-\frac{\omega''}{\omega} \; .$$

If $c_{10}>0$ in (29), w ranges from 0 to ∞ (the metric is unchanged under $w\to -w$ so we ignore negative values of w). In this case the proper radius as $w\to 0$ is again $2r_0$ and the proper length is again zero. As $w\to\infty$, however, the proper radius goes to wr_0 and the proper length between z_1 and z_2 is just z_2-z_1. We see that the string begins expanding in both the r and z directions but eventually the radial expansion dominates as the axial expansion slows down. The energy density decreases from a maximum at $w=0$ given by Eq. (33), to a minimum of zero as $w\to\infty$.

If we use the + sign in (28) the metric becomes

$$ds^2=\frac{w^4}{[1+(1+c_{10}w^2)^{1/2}]^2} \times\left[\frac{-dw^2}{c_2^2(1+c_{10}w^2)}+dr^2+\omega^2(r)d\phi^2\right] +\frac{[1+(1+c_{10}w^2)^{1/2}]^2}{w^2}dz^2 \tag{34}$$

and the energy density is given by

$$8\pi\sigma=\frac{[1+(1+c_{10}w^2)^{1/2}]^2}{w^4}\left[c_{10}c_2^2-\frac{\omega''}{\omega}\right] . \tag{35}$$

This solution can be embedded in vacuum in the same way as the metric (29).

The proper radius of the string and the proper separation between two points on the axis are given in this case by

$$R=\frac{w^2}{1+(1+c_{10}w^2)^{1/2}}r_0$$

and

$$L=\frac{1+(1+c_{10}w^2)^{1/2}}{w}(z_2-z_1)\ ,$$

respectively.

If $c_{10}<0$ we see that $R\to 0$ and $L\to\infty$ as $w\to 0$ and as $w\to\pm|c_{10}|^{-1/2}$ we have $R\to|c_{10}|^{-1}r_0$ and $L\to|c_{10}|^{1/2}(z_2-z_1)$. In this case the string expands radially from zero proper radius at $w=0$ to a maximum value as $w\to\pm|c_{10}|^{-1/2}$. It also contracts along the axis and the contraction stops as $w\to\pm|c_{10}|^{-1/2}$. The energy density is infinite at $w=0$ and decreases to a value of

$$8\pi\sigma=\frac{1}{c_{10}^2}\left[c_{10}c_2^2-\frac{\omega''}{\omega}\right]$$

as $w\to\pm|c_{10}|^{-1/2}$.

If $c_{10}>0$, the proper radius and separation along the axis are again 0 and ∞, respectively, at $w=0$. As $w\to\infty$ the proper radius is $R=wr_0/\sqrt{c}\ 10$ and the proper separation is $\sqrt{c_{10}}(z_2-z_1)$. Again the string expands radially and contracts along the axis. However, the radial expansion does not stop as in the previous case. The energy density decreases from ∞ at $w=0$ to zero as $w\to\infty$.

B. $\beta\neq 0$

In this case Eq. (14) gives $\mu'=\sqrt{\beta}$ and from (15) we find $\omega'/\omega=\gamma/\sqrt{\beta}$. The possible solutions depend on whether or not $\gamma=0$.

1. $\gamma=0$

With $\gamma=0$ we have $\omega'=0$ and Eq. (16) shows that $\dot{u}=0$. Equation (15) is then satisfied while (14) gives

$$K=\beta\frac{t^2}{2}+c_{18}t+c_{19}\ .$$

Equation (13) now becomes

$$\ddot{w}-(\beta t+c_{18})\dot{w}+\beta w=0\ .$$

The general solution (up to an arbitrary multiplying factor) is

$$w(t)=\beta t+c_{18}+c_{20}\left[-e^{\beta(t^2/2)+c_{18}t}+(\beta t+c_{18})\int e^{\beta(t^2/2)+c_{18}t}dt\right]\ .$$

The metric in this case can be written as

$$ds^2=-e^{\beta t^2+2c_{18}t}e^{-2\sqrt{\beta}r}(dt^2-dr^2)+e^{-2\sqrt{\beta}r}w^2(t)d\phi^2+e^{2\sqrt{\beta}r}dz^2 \tag{36}$$

and the energy density is

$$8\pi\sigma=\frac{e^{2\sqrt{\beta}r}}{e^{(\beta t^2/2)+c_{18}t}}\left[\frac{c_{20}\beta}{w(t)}\right]\ .$$

The constant c_{19} has been eliminated by scaling r and t. We see that the axis is defined by $r=\infty$ and that the energy density diverges on the axis. The Kretschmann scalar ($K=R_{\alpha\beta\delta\gamma}R^{\alpha\beta\delta\gamma}$) also diverges on the axis so the spacetime is singular there. This divergence may be due to the particular form of the energy-momentum tensor. The inclusion of other components of $T_{\mu\nu}$ might eliminate the divergence but then the metric would not be given by (36). Thus we do not consider this (and other singular solutions) to be an acceptable string model.

2. $\gamma\neq 0$

If $\gamma\neq 0$, we have

$$\omega=e^{(\gamma/\sqrt{\beta})r}\ . \tag{37}$$

Equation (16) is now

$$\left[\dot{K}-\frac{\dot{w}}{w}\right]\frac{\gamma}{\sqrt{\beta}}=2\sqrt{\beta}\dot{u}$$

which gives

$$K=\frac{2\beta}{\gamma}u+\ln w+c_{21}\ .$$

Using this, Eqs. (13) and (14) can be reduced to

$$\dot{u}^2+2\frac{\beta}{\gamma}\ddot{u}+\left[\frac{\dot{w}}{w}\right]^{\cdot}=\beta \tag{38}$$

and

$$\frac{\dot{u}\dot{w}}{w}+\ddot{u}=\gamma\ . \tag{39}$$

Equation (39) is the same as Eq. (15) so we see that (15) is redundant in this case. Equations (38) and (39) admit a vacuum solution with $(\dot{w}/w)^2=\gamma^2/\beta$ and $u=(\beta\ln w)/\gamma$. However, they also admit a class of nonvacuum solutions as we now show.

If we let $\dot{u}=\sqrt{\beta}X$, $\dot{w}/w=\gamma Y/\sqrt{\beta}$, and $t=\tilde{t}\sqrt{\beta}/\gamma$ we obtain, from (38) and (39),

$$X^*=1-XY \tag{40}$$

and

$$Y^*=\frac{\beta^2}{\gamma^2}(2XY-1-X^2)\ , \tag{41}$$

where an asterisk denotes $\partial/\partial\tilde{t}$. In order for the energy density to remain positive, it is necessary that $\ddot{w}/w\geq\gamma^2/\beta$. In terms of the variables X, Y, and $\tilde{t}$ this translates into the condition

$$Y^2+Y^*\geq 1$$

or, equivalently,

$$\left[Y+\frac{\beta^2}{\gamma^2}X\right]^2\geq\left[1+\frac{\beta^2}{\gamma^2}\right]\left[1+\frac{\beta^2}{\gamma^2}X^2\right]\ . \tag{42}$$

If a particular solution of Eqs. (40) and (41) satisfies (42) everywhere then it will be a valid solution. The functions U and W may then be obtained from X and Y by doing the appropriate integrations. Instead of integrating Eqs. (40) and (41) for $X(\tilde{t})$ and $Y(\tilde{t})$ we can integrate them for $Y(X)$ and determine whether there are any solutions in the (X,Y) plane which satisfy (42).

The results of integrating these equations are shown in Fig. 1 for $(\beta/\gamma)^2=0.5$. It can be seen that there do exist solutions which lie entirely within the region bounded by Eq. (42) (in this figure, the region above the dotted line). For large values of X, the solutions approach the form

$$Y=\left\{\mp\left[\frac{\beta}{\gamma}\right]\left[1+\left[\frac{\beta}{\gamma}\right]^2\right]^{1/2}-\left[\frac{\beta}{\gamma}\right]^2\right\}X+\delta \qquad (\delta=\text{const})$$

whereas the boundary is given by

$$Y=\left\{\mp\left[\frac{\beta}{\gamma}\right]\left[1+\left[\frac{\beta}{\gamma}\right]^2\right]^{1/2}-\left[\frac{\beta}{\gamma}\right]^2\right\}X\ .$$

We see that the solutions are parallel to the boundary so that if they are within the boundary for large values of X, they will remain inside.

The energy density in this case is given by

$$8\pi\sigma=\frac{e^{2(1-2\beta/\gamma)u}}{w^2}\omega^{2\beta/\gamma}\left[\frac{\ddot{w}}{w}-\frac{\gamma^2}{\beta}\right] \qquad (43)$$

and the coefficient of $d\phi^2$ in the metric is

$$g_{\phi\phi}=\omega^{2(1-\beta/\gamma)}w^2e^{-2u}\ .$$

Since the axis of symmetry lies where $g_{\phi\phi}=0$, we see that if $\beta=\gamma$, $g_{\phi\phi}$ is independent of r and there is no axis. Thus we do not consider the case $\beta=\gamma$ to represent a realistic string. If $\beta>\gamma>0$, the axis occurs where $\omega\rightarrow\infty$ and from (37) we see that this corresponds to $r\rightarrow\infty$. However, from (43) we see that the energy density diverges as $r\rightarrow\infty$ and it can also be shown that the Kretschmann scalar diverges as well. If $\gamma<0$ (recall that $\beta>0$) the axis is at $\omega=0$, i.e., at $r=-\infty$ and again the energy density and Kretschmann scalar are infinite on the axis.

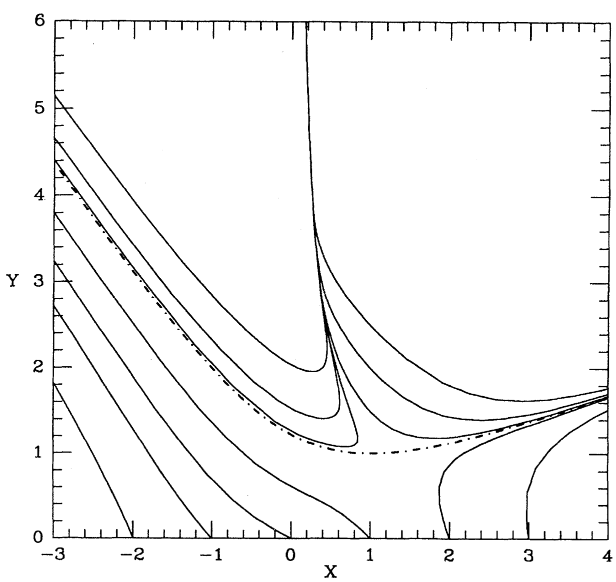

FIG. 1. Solutions of Eqs. (40) and (41) in the (X,Y) plane for $(\beta/\gamma)^2=0.5$. The dashed line indicates the vacuum solution and the region above the dashed line contains solutions with positive energy density.

The axis also occurs at $\omega=0$ if $\gamma>\beta$ and according to (37) this corresponds to $r=-\infty$. The energy density goes to zero on the axis in this case but diverges as $r\rightarrow\infty$. Thus a realistic string in this class should have $\gamma>\beta$ and the solution must be terminated at some finite radius (which may be a function of t) and patched onto a nonsingular exterior. If we wish the exterior region to be vacuum, this task is complicated by the fact that there is no unique cylindrically symmetric vacuum so that we do not know *a priori* which vacuum solution should be patched onto the string. It is further complicated by the fact that this solution can only be given numerically so that, in order to satisfy the boundary conditions, the entire space of possible numerical solutions would have to be explored. For these reasons we will not consider this class of solutions any further.

In this section we have demonstrated the existence of several solutions. The solutions which are nonsingular on the axis and can be easily embedded in vacuum are given by the metrics (21), (26), (29), and (34) [as well as the static solutions given by (20) or (25)]. In what follows we will determine if these models are nonstationary. As noted above we will not consider the solution with $\beta\neq0$ and $\gamma\neq0$.

III. NONSTATIONARITY

As we have seen, the static metric (20) can be written in a form which, at first glance, appears to be nonstationary [metric (25)]. It is therefore necessary that we demonstrate explicitly whether or not the remaining solutions presented above are indeed nonstationary. This can be accomplished by showing that the spacetimes do not admit any timelike Killing vectors. Killing vectors are vectors ξ that satisfy the Killing equation

$$\xi_{\beta;\alpha}+\xi_{\alpha;\beta}=0\ .$$

For the general form of the metric used in this paper, any Killing vector with component $\xi_t=0$ is necessarily spacelike. Thus, if any of the metrics that we are considering as models of strings [i.e., metrics (21), (26), (29), and (34)] admit only Killing vectors with $\xi_t=0$, then those metrics are nonstationary since they admit no timelike Killing vectors. We note that these models consist of two parts; the string itself and the surrounding vacuum. Each must be examined separately.

The metric (21) can be seen to be equivalent to

$$ds^2=-dt^2+t^2dr^2+t^2\omega^2(r)d\phi^2+dz^2 \qquad (44)$$

by using the transformation $t=e^{\tilde{t}}$ where the tilde refers to the coordinate in (21). In Appendix A it is shown that in the interior of the string, this metric admits no timelike Killing vectors so that the string's interior is nonstationary. In the vacuum region where $\omega(r)$ is given by $\omega=c_7\sinh r+c_8\cosh r$, the metric can be seen to be equivalent to the static vacuum metric

$$ds^2=-dT^2+dR^2+\gamma^2R^2d\phi^2+dz^2 \quad (45)$$

from the transformations

$$T=\frac{t}{\gamma}(c_7\cosh r+c_8\sinh r)$$

and

$$R=\frac{t}{\gamma}(c_7\sinh r+c_8\cosh r)\ ,$$

where $\gamma^2=c_7^2-c_8^2$. Thus we have a nonstationary string surrounded by a static vacuum.

Since the boundary of the string is defined by $r=r_0=\text{const}$, we see that, in the coordinates used in (45), the boundary is given by $R=T(c_7\sinh r_0+c_8\cosh r_0)/(c_7\cosh r_0+c_8\sinh r_0)$. This model represents a nonstationary string that has zero radius at time $T=0$ and which expands radially into a static vacuum [when considered in the (R,T) coordinates].

The second metric of interest is (26). The Killing vectors for this metric are calculated in Appendix B (after letting $w\to t$). We find that, in the vacuum region as well as inside a homogeneous string, the spacetime admits the following four Killing vectors (in contravariant form):

$$\begin{aligned}\xi_A^\alpha&=(0,0,0,1)\ ,\\ \xi_B^\alpha&=(0,0,1,0)\ ,\\ \xi_C^\alpha&=\left[0,\sin,\phi,\frac{\omega'}{\omega}\cos\phi,0\right]\ ,\\ \xi_D^\alpha&=\left[0,\cos\phi,\frac{-\omega'}{\omega}\sin\phi,0\right]\ ,\end{aligned} \quad (46)$$

where $\omega(r)$ takes the form appropriate to either vacuum or a homogeneous string. The interior of an inhomogeneous string admits only the two Killing vectors ξ_A^α and ξ_B^α. As we can see, these Killing vectors are all spacelike so the spacetime is nonstationary in both the string and vacuum regions.

The Killing vectors for the metric (29) depend on whether $c_{10}>0$ or $c_{10}<0$. If $c_{10}>0$, the transformation $w=(c_{10})^{-1/2}\sinh t$ allows the metric to be written as

$$ds^2=(1+\cosh t)^2\left[-\frac{dt^2}{c_{10}c_2^2}+dr^2+\omega^2(r)d\phi^2\right]+\frac{\cosh t-1}{\cosh t+1}dz^2\ . \quad (47)$$

The Killing vectors for this metric are found in Appendix C. Again we find that the vacuum and homogeneous string spacetimes admit the four Killing vectors given by (46) whereas an inhomogeneous string admits only ξ_A^α and ξ_B^α. Therefore the metric (47) is also nonstationary.

With $c_{10}<0$, the transformation $w=|c_{10}|^{-1/2}\sin t$ allows us to write (29) as

$$ds^2=(1+\cos t)^2\left[-\frac{dt^2}{|c_{10}|c_2^2}+dr^2+\omega^2(r)d\phi^2\right]+\frac{1-\cos t}{1+\cos t}dz^2\ . \quad (48)$$

The Killing vectors for this metric are also given in Appendix C. The results are exactly the same as for the metrics (26) and (47).

The metric (34) can also be written as

$$ds^2=(\cosh t-1)^2\left[-\frac{dt^2}{c_{10}c_2^2}+dr^2+\omega^2(r)d\phi^2\right]+\frac{\cosh t+1}{\cosh t-1}dz^2 \quad (49)$$

if $c_{10}>0$, and

$$ds^2=(1-\cos t)^2\left[\frac{-dt^2}{|c_{10}|c_2^2}+dr^2+\omega^2(r)d\phi^2\right]+\frac{1+\cos t}{1-\cos t}dz^2 \quad (50)$$

if $c_{10}<0$ by using the same transformations as used to find metrics (47) and (48). The calculation of the Killing vectors in this case is virtually the same as that for the metrics (47) and (48) and the same results hold.

We have shown that the string's interior is nonstationary in the models described by the metrics (44), (26), and (47)–(50). In the case of (44) the exterior vacuum is static while in the rest of the cases it is nonstationary. In the following section we will determine whether or not the vacuum regions surrounding the strings can be joined onto a Robertson-Walker background.

IV. ROBERTSON-WALKER BACKGROUND

Before considering the matching of the string models onto a Robertson-Walker (RW) exterior, we make the following observations and considerations.

When matching two cylindrically symmetric spacetimes across a cylindrically symmetric surface, we may take the azimuthal coordinate ϕ to be continuous through the boundary since it is the only coordinate in both spacetimes that is identified only on the finite range $[0,2\pi)$. However, we cannot generally take the axial coordinate to be continuous as can be seen from the following. The two metrics

$$ds^2=-dt^2+dr^2+r^2d\phi^2+dz^2 \quad (51)$$

and

$$ds^2=-dT^2+dR^2+R^2d\phi^2+T^2d\eta^2 \quad (52)$$

are equivalent to each other (as shown in Sec. II). The coordinates are related by the transformations $r=R$, $t=T\cosh\eta$, and $z=T\sinh\eta$. The surfaces $r=r(t)$ and $R=R(T)$ are both cylindrically symmetric but, if we attempt to match the two metrics (51) and (52) across these surfaces and we assume continuity of the axial coordinate (i.e., $z=\eta$) we see immediately that we cannot have $g_{zz}=g_{\eta\eta}$ and the match cannot be accomplished. The reason for this is that the surface $r=r(t)$ actually corresponds to the surface $R=R(T\cosh\eta)$ or, conversely, the surface $R=R(T)$ corresponds to $r=r(t^2-z^2)$. The results of this are twofold: we cannot necessarily assume continuity of the axial coordinate, and cylindrically symmetric surfaces are not necessarily defined by $r=r(t)$ but may also be defined by $r=r(t,z)$.

Whether or not we can take the axial coordinate to be continuous can be determined if we consider the following. Consider two spacetimes M and N which are to be joined across a surface Σ. Assume that we can generally characterize Σ in terms of the coordinates in one of the spacetimes, say M [e.g., $r=r(t,z)$]. Also assume that each spacetime admits a number of linearly independent Killing vectors, say M admits m of them and N admits n (m and n are not necessarily equal). Since we can characterize Σ in M we can determine the normal to the surface (in M) and we denote this as $n_{M\alpha}$. If we have a general characterization of the surface in N we can also determine the normal in N and denote it as $n_{N\alpha}$.

If we find the product $n_{M\alpha}\xi_M^\alpha$ for each of the m Killing vectors in M, those Killing vectors for which this quantity is zero are orthogonal to the normal and thus lie in the surface Σ (Σ must be either timelike or spacelike). These Killing vectors then represent inherent isometries of Σ. However, since any linear combination of Killing vectors is a Killing vector, this procedure will not give us the number of independent Killing vectors that lie in the surface (i.e., the values of the products $n_{M\alpha}\xi_M^\alpha$ are dependent on which m linearly independent Killing vectors we choose from a potentially infinite number of possibilities for a basis). We can circumvent this problem by taking the product of $n_{M\alpha}$ with a linear combination of all the Killing vectors:

$$n_{M\alpha}\sum_{i=1}^{m} a_i\xi_{M_i}^\alpha \ .$$

If we set this product equal to zero, we will find that some of the coefficients a_i must necessarily be zero, while others will be completely arbitrary and some may be related to the arbitrary ones. The number of independent, arbitrary, nonzero coefficients a_i is then the maximum number of Killing vectors that lie in the surface. These a_i's then allow us to determine which Killing vectors lie in the surface (e.g., if we find that a_1 and a_2 are completely arbitrary, a_3 equals $3a_1$, and all the rest must be zero, then $\xi_1+3\xi_3$ and ξ_2 are the two linearly independent Killing vectors which lie in the surface).

In the region N we can form the product

$$n_{N\alpha}\sum_{i=1}^{n} b_i\xi_{N_i}^\alpha$$

and again, set this equal to zero (the b_i's are constant coefficients). We then require that the number of independent, arbitrary b_i's match the number of a_i's. That is, we require the number of Killing vectors inherent in the surface to be the same whether we are using the coordinates in M or in N. The Killing vectors of N that lie in the surface must then be expressible as coordinate transformations of linear combinations of the Killing vectors of M which lie in the surface. In this way we can determine whether or not a particular coordinate is continuous and if not, how it is related in the spacetimes (if at all). We illustrate this procedure below by attempting to match a RW spacetime onto the static, cylindrically symmetric vacuum and then we consider the nonstationary vacua.

Cocke[18] has shown that the metric for the vacuum exterior to a cylinder of FRW dust can be expressed in the Einstein-Rosen form. While Cocke's analysis is for a cylinder of fluid embedded in an external vacuum, it should also hold for a cylinder of vacuum removed from a FRW background. Unfortunately the particular form of the Einstein-Rosen metric for which this can be accomplished is not known. If it was known, all that would remain would be to determine if any of the vacuua being considered here are coordinate transformations of the Einstein-Rosen metric. However, since it is not, we will consider matching these vacua onto a RW exterior.

The cylindrical form of the RW metric in comoving coordinates is

$$ds^2=-dT^2+S^2(T)\left[\frac{dR^2}{1-kR^2}+R^2d\phi^2 +(1-kR^2)d\eta^2\right] , \tag{53}$$

where $k=0,\pm1$. This can be seen to be equivalent to the usual spherically symmetric form

$$ds^2=-dT^2+S^2(T)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2}+\bar{r}^2(d\theta^2+\sin^2\theta\, d\phi^2)\right]$$

from the transformations

$$R=\bar{r}\sin\theta$$

and

$$\eta=\begin{cases}\arctan\left[\dfrac{\bar{r}}{\sqrt{1-\bar{r}^2}}\cos\theta\right], & k=+1\ ,\\ \bar{r}\cos\theta, \quad k=0 & \\ \text{arctanh}, \left[\dfrac{\bar{r}}{\sqrt{1+\bar{r}^2}}\cos\theta\right], & k=-1\ .\end{cases} \tag{54}$$

Let the surface across which the vacuum and RW exterior are to be joined be denoted by Σ and the coordinates intrinsic to Σ be ζ^i ($i=1,2,3$). As noted above, the interpretation of the coordinate η may be ambiguous so we will assume that Σ is characterized by

$$R=R_b(T,\eta) \tag{55}$$

in the RW region. The normal to Σ is then given by

$$n_\alpha=\frac{1}{C}\left[-\frac{\partial R_b}{\partial T},1,0,-\frac{\partial R_b}{\partial\eta}\right] , \tag{56}$$

where

$$C=\left[-\left[\frac{\partial R_b}{\partial T}\right]^2+\frac{1-kR_b^2}{S^2(T)} +\left[\frac{\partial R_b}{\partial\eta}\right]^2\frac{1}{S^2(T)(1-kR_b^2)}\right]^{1/2} .$$

For the sake of simplicity we will consider only the case

$k=0$ although the results can easily be generalized to $k=\pm 1$.

The RW metric (with $k=0$) admits the six Killing vectors

$$\xi_{\rm I}^{\alpha}=\left[0,\cos\phi,-\frac{\sin\phi}{R},0\right],$$

$$\xi_{\rm II}^{\alpha}=\left[0,\sin\phi,\frac{\cos\phi}{R},0\right],$$

$$\xi_{\rm III}^{\alpha}=(0,0,1,0),$$

$$\xi_{\rm IV}^{\alpha}=(0,0,0,1),$$

$$\xi_{\rm V}^{\alpha}=\left[0,-\eta\sin\phi,-\eta\frac{\cos\phi}{R},R\sin\phi\right],$$

$$\xi_{\rm VI}^{\alpha}=\left[0,\eta\cos\phi,-\eta\frac{\sin\phi}{R},-R\cos\phi\right].$$

Taking the product of n_α with a linear combination of these Killing vectors and setting it equal to zero on the surface we get

$$Cn_\alpha\sum_{i=I}^{\rm VI}a_i\xi_i^\alpha=0=a_{\rm I}\cos\phi+a_{\rm II}\sin\phi-a_{\rm IV}\frac{\partial R_b}{\partial\eta}-a_{\rm V}\left[\eta\sin\phi+R_b\frac{\partial R_b}{\partial\eta}\sin\phi\right]+a_{\rm VI}\left[\eta\cos\phi+R_b\frac{\partial R_b}{\partial\eta}\cos\phi\right]. \quad (57)$$

This is clearly independent of $a_{\rm III}$ so $a_{\rm III}$ is arbitrary and thus $\xi_{\rm III}^\alpha$ is one of the vectors that lies on the surface (this Killing vector simply corresponds to rotation around the axis). For a cylindrical surface we require that there be one (and only one) more Killing vector which lies on the surface. From the coefficients of $\cos\phi$ and $\sin\phi$ we have

$$a_{\rm I}+a_{\rm VI}\left[\eta+R_b\frac{\partial R_b}{\partial\eta}\right]=0 \quad (58)$$

and

$$a_{\rm II}-a_{\rm V}\left[\eta+R_b\frac{\partial R_b}{\partial\eta}\right]=0. \quad (59)$$

Let $X=\eta+R_b(\partial R_b/\partial\eta)$. If $X=0$, then $a_{\rm IV}$ and $a_{\rm V}$ are both arbitrary and we have two more Killing vectors $\xi_{\rm V}^\alpha$ and $\xi_{\rm IV}^\alpha$ that lie on the surface. Since we can allow only one more Killing vector on the surface we must eliminate the case $X=0$ [in this case we have $R_b^2+\eta^2=f(T)$ which can be seen to be the same as the surface $\bar{r}^2=f(t)$ in spherical coordinates, i.e., the surface is actually spherically symmetric]. If $X=$const we can still have $a_{\rm V}$ and $a_{\rm VI}$ arbitrary and $a_{\rm I}$ and $a_{\rm II}$ will be proportional to them. Again, there will be too many Killing vectors lying on the surface since we have two more arbitrary a_i's. Thus X is not constant so we see from (58) and (59) that $a_{\rm V}$ and $a_{\rm VI}$ must therefore be zero. It then follows that $a_{\rm I}$ and $a_{\rm II}$ must also be zero. Equation (57) now becomes

$$0=a_{\rm IV}\frac{\partial R_b}{\partial\eta}.$$

We cannot have $a_{\rm IV}=0$ since then there would be only one Killing vector on Σ ($\xi_{\rm III}^\alpha$). Thus it is necessary that $\partial R_b/\partial\eta=0$ and $a_{\rm IV}$ is arbitrary. This means that the second Killing vector that lies on the surface is $\xi_{\rm IV}^\alpha$ which simply corresponds to translation along the axis. The surface is now given by $R=R_b(T)$ which is what we would expect intuitively.

The Darmois junction conditions[19] for a boundary surface are

$$[g_{ij}]=0 \quad (60)$$

and

$$[K_{ij}]=0, \quad (61)$$

where

$$g_{ij}=\frac{\partial x^\alpha}{\partial\xi^i}\frac{\partial x^\beta}{\partial\xi^j}g_{\alpha\beta},$$

$$K_{ij}=\frac{\partial x^\alpha}{\partial\xi^i}\frac{\partial x^\beta}{\partial\xi^j}n_{\beta;\alpha},$$

and $[X]=X|_\Sigma^+-X|_\Sigma^-$ is the jump in the quantity X when evaluated on the surface Σ in the two different regions denoted by $+$ and $-$ (here $+$ represents the RW exterior and $-$ the vacuum interior). On a boundary surface the following conditions also follow:[20]

$$[T_\alpha^\beta n^\alpha n_\beta]=0 \quad (62)$$

and

$$\left[T_\alpha^\beta\frac{\partial x^\alpha}{\partial\xi^i}n_\beta\right]=0. \quad (63)$$

In the RW region, the energy-momentum tensor is given by

$$T_\alpha^\beta=(\epsilon+p)\tilde{u}^\beta\tilde{u}_\alpha+p\delta_\alpha^\beta,$$

where ϵ is the energy density, p is the isotropic pressure, and $\tilde{u}^\beta$ is the (timelike) four-velocity of the fluid which is taken to be comoving. In the vacuum region $T_\alpha^\beta=0$. Contracting (63) with u^i, the tangent to Σ, we have

$$\left[T_\alpha^\beta u^i\frac{\partial x^\alpha}{\partial\xi^i}n_\beta\right]=0$$

and since the four-velocity of the surface is given by $u^\alpha=u^i\partial x^\alpha/\partial\xi^i$ this becomes

$$[T_\alpha^\beta u^\alpha n_\beta]=0.$$

Using T_α^β as given above we find

$$(\epsilon+p)\tilde{u}_\alpha u^\alpha\tilde{u}^\beta n_\beta=0$$

since $u^\alpha n_\alpha=0$. We do not consider the case $p=-\epsilon$ and $\tilde{u}_\alpha u^\alpha$ cannot be 0 since this would imply that Σ is spacelike so we must have $\tilde{u}^\beta n_\beta=0$. Thus $\partial R_b/\partial T=0$ and we have $R=R_b=$const on the surface. Equation (58) gives

$$(\epsilon+p)(n^\alpha \tilde{u}_\alpha)^2+(n^\alpha n_\alpha)p=0$$

and since $n^\alpha \tilde{u}_\alpha=0$ and $n^\alpha n_\alpha=1$ we have $p=0$. Thus the only way that the RW region can be joined onto a vacuum is across a comoving boundary when the matter content is dust ($p=0$). Conditions (62) and (63) do not allow us to put any conditions on the nature of the boundary as determined in the vacuum region. The intrinsic metric of the surface can now be written as

$$ds^2=-dT^2+T^{4/3}(R_b^2 d\phi^2+d\eta^2) \tag{64}$$

since $S(T)=T^{2/3}$ for dust.

The static, cylindrically symmetric vacuum as given by (20) with $\omega(r)=ar$ (a is a constant) admits the ten Killing vectors

$$\xi_1^\alpha=(1,0,0,0)\ ,$$

$$\xi_2^\alpha=\left[0,\cos a\varphi,-\frac{\sin a\phi}{ar},0\right]\ ,$$

$$\xi_3^\alpha=\left[0,\sin a\phi,\frac{\cos a\phi}{ar},0\right]\ ,$$

$$\xi_4^\alpha=(0,0,0,1)\ ,$$

$$\xi_5^\alpha=(0,0,1,0)\ ,$$

$$\xi_6^\alpha=\left[0,-z\sin a\phi,\frac{-z}{ar}\cos a\phi, r\sin a\phi\right]\ ,$$

$$\xi_7^\alpha=\left[0,z\cos a\phi,\frac{-z}{ar}\sin a\phi,-r\cos a\phi\right]\ ,$$

$$\xi_8^\alpha=\left[r\cos a\phi,-t\cos a\phi,\frac{t}{ar}\sin a\phi,0\right]\ ,$$

$$\xi_9^\alpha=\left[r\sin a\phi,-t\sin a\phi,-\frac{t}{ar}\cos a\phi,0\right]\ ,$$

$$\xi_{10}^\alpha=(z,0,0,t)\ .$$

These are easily obtained from the Killing vectors of Minkowski space since the vacuum is simply Minkowski space with an angular deficit.

Again we assume that the joining of the vacuum onto the RW region is to be done on a surface given by $r=r_b(t,z)$. The normal to this surface is

$$n_\alpha=\frac{1}{D}\left[-\frac{\partial r_b}{\partial t},1,0,-\frac{\partial r_b}{\partial z}\right]\ , \tag{65}$$

where

$$D=\left[-\left[\frac{\partial r_b}{\partial t}\right]^2+1+\left[\frac{\partial r_b}{\partial z}\right]^2\right]^{1/2}\ .$$

Setting the product of n_α with a linear combination of the ξ^α's equal to zero we have

$$\begin{aligned} Dn_\alpha\sum_{i=1}^{10} b_i\xi_i^\alpha=0=&-b_1\frac{\partial r_b}{\partial t}+b_2\cos a\phi+b_3\sin a\phi-b_4\frac{\partial r_b}{\partial z}-b_6\left[z\sin a\phi+r_b\frac{\partial r_b}{\partial z}\sin a\phi\right]\\ &+b_7\left[z\cos a\phi+r_b\frac{\partial r_b}{\partial z}\cos a\phi\right]-b_8\left[r_b\frac{\partial r_b}{\partial t}\cos a\phi+t\cos a\phi\right]-b_9\left[r_b\frac{\partial r_b}{\partial t}\sin a\phi+t\sin a\phi\right]\\ &-b_{10}\left[z\frac{\partial r_b}{\partial t}+t\frac{\partial r_b}{\partial z}\right]\ . \end{aligned} \tag{66}$$

Equation (66) is independent of b_5 so we see that ξ_5^α lies in the surface. ξ_5^α corresponds to rotation about the axis. We see that ξ_5^α and $\xi_{\rm III}^\alpha$ (from the RW spacetime) are the same Killing vector since we have assumed continuity of ϕ. Again, we need one and only one more Killing vector that lies in the surface.

Equation (66) can be written as

$$\begin{aligned} 0=&-\frac{\partial r_b}{\partial t}(b_1+b_{10}z)+\cos a\phi\left[b_2+b_7\left[z+\frac{\partial r_b}{\partial z}r_b\right]-b_8\left[t+\frac{\partial r_b}{\partial t}r_b\right]\right]+\sin a\phi\left[b_3-b_6\left[z+\frac{\partial r_b}{\partial z}r_b\right]\right.\\ &\left.-b_9\left[t+\frac{\partial r_b}{\partial t}r_b\right]\right]-\frac{\partial r_b}{\partial z}(b_4+b_{10}t)\ . \end{aligned} \tag{67}$$

From the $\cos a\phi$ and $\sin a\phi$ terms we have

$$b_2+b_7X-b_8Y=0 \tag{68}$$

and

$$b_3-b_6X-b_9Y=0\ , \tag{69}$$

where $X=z+r_b\partial r_b/\partial z$ and $Y=t+r_b\partial r_b/\partial t$. If $X=0$, then b_7 and b_6 are arbitrary and there are two more Killing vectors on Σ. However, this is one too many so we must have $X\neq 0$. Similarly, we must have $Y\neq 0$. If either X or Y is a constant, (68) and (69) then demand that the other is also a constant. In this case (68) and (69) represent two equations for the six unknown b_i's, i.e., four of them are arbitrary. However, we can allow only one more arbitrary b_i so this case can be dismissed. If X and Y are not constants, then (68) and (69) each demand that they be linearly related, i.e., $X=\alpha Y+\beta$, where α and β are constants. In this case (68) and (69) become

$$b_2+\beta b_7+(\alpha b_7-b_8)Y=0$$

and

$$b_3-\beta b_6+(-\alpha b_6-b_9)Y=0 .$$

Thus we can take b_7 and b_6 to be arbitrary and these equations then give $b_2=-\beta b_7$, $b_8=\alpha b_7$, $b_3=\beta b_6$, and $b_9=-\alpha b_6$. However, since there are two arbitrary b_i's, there will be two more Killing vectors lying in the surface, which is not allowed. Thus, since all possible forms of X and Y for which (68) and (69) can be satisfied lead to more than one more Killing vector lying in Σ, we must conclude that the coefficients of X and Y in (68) and (69) are all zero. The two coefficients b_2 and b_3 must also then be zero.

Equation (66) is then

$$0=\frac{\partial r_b}{\partial t}(b_1+b_{10}z)+\frac{\partial r_b}{\partial z}(b_4+b_{10}t) . \tag{70}$$

If all three of b_1, b_4, and b_{10} are nonzero, since we can only allow one arbitrary b_i, the three of them must be proportional to each other. We can then write $b_1=\alpha b_{10}$ and $b_4=\beta b_{10}$. Equation (70) then becomes

$$\frac{\partial r_b}{\partial t}(\alpha+z)=-\frac{\partial r_b}{\partial z}(\beta+t)$$

which implies $r_b=r_b[(\beta+t)^2-(\alpha+z)^2]$. This reflects the fact noted earlier that $r(t^2-z^2)$ is also a cylindrically symmetric surface. The constants α and β are present since the metric is unchanged under simple translations on the t and z directions. In this case the Killing vector which lies on the surface is $\alpha\xi_1+\beta\xi_4+\xi_{10}=(z+\alpha,0,0,t+\beta)$.

If only two of b_1, b_4, and b_{10} are to be nonzero, then we must have either $b_1=0$, $b_4=0$, or $b_{10}=0$. The cases $b_1=0$ and $b_4=0$ can be included in the previous case simply by setting α or β to zero. If $b_{10}=0$ then we must have $b_1=\gamma b_4$ and Eq. (70) becomes

$$\frac{\gamma\partial r_b}{\partial t}=-\frac{\partial r_b}{\partial z}$$

which implies $r_b=r_b(t-\gamma z)$. This reflects the fact that the metric is invariant under Lorentz boosts in the z direction. The Killing vector which lies on this surface is $\gamma\xi_1+\xi_4=(\gamma,0,0,1)$.

If only one of b_1, b_4, and b_{10} is nonzero, then the other two must be zero. The case $b_1=b_4=0$ can be included in the previous work by setting $\alpha=\beta=0$. The case $b_{10}=b_1=0$ can be included in the preceding case by setting $\gamma=0$. If $b_{10}=b_4=0$, Eq. (70) becomes

$$\frac{\partial r_b}{\partial t}=0 ;$$

i.e., $r_b=r_b(z)$ and the Killing vector lying on the surface is just $\xi_1=(1,0,0,0)$.

To recap, there are three possible classes of surface in the static vacuum that are cylindrically symmetric. Along with the Killing vector that lies on the surface they are

$$\begin{aligned}&\text{(i)}\ \ r=r_b[(\beta+t)^2-(\alpha+z)^2] ,\\ &\xi^\alpha_\Sigma=(z+\alpha,0,0,t+\beta) ,\end{aligned} \tag{71}$$

$$\text{(ii)}\ r=r_b(t-\gamma z),\quad \xi^\alpha_\Sigma=(\gamma,0,0,1) , \tag{72}$$

and

$$\text{(iii)}\ \ r=r_b(z),\quad \xi^\alpha_\Sigma=(1,0,0,0) . \tag{73}$$

Each of these surfaces also admits the Killing vector ξ^α_5 corresponding to rotation about the axis.

In each of these three cases, the Killing vector ξ^α_Σ must be a coordinate transformation of the RW Killing vector $\xi^\alpha_{\text{IV}}=(0,0,0,1)$. Case (iii) is not admissible since ξ^α_Σ in this case is timelike whereas ξ^α_{IV} is spacelike. In case (ii) the constant γ must satisfy $\gamma^2<1$ for ξ^α_Σ to be spacelike.

The covariant forms of the Killing vectors must also be equivalent to each other. Letting $\xi_{\Sigma\alpha}$ and ξ^α_Σ be coordinate transformations of $\xi_{\text{IV}\alpha}$ and ξ^α_{IV}, respectively, we find, for case (i),

$$\frac{\partial T}{\partial t}(z+\alpha)+\frac{\partial T}{\partial z}(t+\beta)=0$$

and

$$-\frac{\partial T}{\partial t}(z+\alpha)+\frac{\partial T}{\partial z}(t+\beta)=0 .$$

We see that $\partial T/\partial z=\partial T/\partial t=0$ so T cannot be expressed in terms of the coordinates t and z and the match cannot be accomplished.

In case (ii) we find the transformation equations

$$\gamma\frac{\partial T}{\partial t}+\frac{\partial T}{\partial z}=0,\quad -\gamma\frac{\partial T}{\partial t}+\frac{\partial T}{\partial z}=0 ,$$

$$\frac{\partial z}{\partial \eta}=1,\quad \frac{\partial z}{\partial \eta}=\frac{1}{S^2(T)} .$$

From the first two equations we have $\partial T/\partial z=0$ and if T is going to be related to t we must have $\gamma=0$. The last two equations show that the RW spacetime must have $S(T)=1$. This is simply Minkowski space and does not represent a cosmological fluid. We see that the static vacuum metric contains no cylindrical surface that can be matched onto a RW exterior which represents a possible cosmological fluid.

We now examine the matching of the nonstationary solutions onto a RW exterior. The nonstationary vacuum metrics (26) and (47)–(50) all admit only the four Killing vectors given by (46). If the normal to the surface is again given by (65) except with

$$D=(g_{\alpha\beta}n^\alpha n^\beta)^{1/2} ,$$

where $g_{\alpha\beta}$ is appropriate to whichever metric we are examining, we can see from inspection that the only way in which there can be another Killing vector in the boundary, in addition to the one corresponding to translation in ϕ, is to have $\partial r_b/\partial z=0$. The second Killing vector lying on the surface is then $\xi_A^\alpha=(0,0,0,1)$. Equating this with the RW Killing vector $\xi_{\rm IV}^\alpha$ we find $\partial z/\partial\eta=1$ and $\partial\eta/dz=1$ so that $z=\eta$ and $T=T(t)$. The axial coordinate can thus be taken to be continuous through Σ.

For the metric (26) the vacuum region has $\omega(r)=c_{12}r+c_{13}$ so that on the surface Σ the metric becomes

$$ds^2=t^4\left[-\left[\frac{\partial t}{\partial T}\right]^2+\left[\frac{\partial r_b}{\partial T}\right]^2\right]dT^2 +t^4(c_{12}r_b+c_{13})^2d\phi^2+\frac{1}{t^2}d\eta^2 \ . \tag{74}$$

Comparing this with (64), the continuity of the intrinsic metric [condition (60)] gives

$$t=T^{-2/3} \ , \tag{75}$$

$$t^2(c_{12}r_b+c_{13})=T^{2/3}R_b \ , \tag{76}$$

and

$$t^4\left[-\left[\frac{\partial t}{\partial T}\right]^2+\left[\frac{\partial r_b}{\partial T}\right]^2\right]=-1 \ . \tag{77}$$

Using Eqs. (75) and (76) we find that Eq. (77) is inconsistent and the match cannot be realized.

The vacuum region for the metric (47) has $\omega(r)=c_7\sinh r+c_8\cosh r$ so that the intrinsic metric of the surface Σ is given by

$$ds^2=(1+\cosh t)^2\left[\frac{-1}{c_{10}c_2^2}\left[\frac{\partial t}{\partial T}\right]^2+\left[\frac{\partial r_B}{\partial T}\right]^2\right]dT^2 +(1+\cosh t)^2(c_7\sinh r_b+c_8\cosh r_b)^2d\phi^2 +\left[\frac{\cosh t-1}{\cosh t+1}\right]^2d\eta^2 \ . \tag{78}$$

The continuity of the intrinsic metric then gives

$$\frac{\cosh t-1}{\cosh t+1}=T^{2/3} \ ,$$

$$(1+\cosh t)(c_7\sinh r_b+c_8\cosh r_b)=T^{2/3}R_b \ ,$$

and

$$(1+\cosh t)^2\left[\frac{-1}{c_{10}c_2^2}\left[\frac{\partial t}{\partial T}\right]^2+\left[\frac{\partial r_b}{\partial T}\right]^2\right]=-1 \ .$$

Again, the third equation is inconsistent with the first two so the two regions cannot be joined smoothly. It is easily verified that the same results hold for the metrics (48)–(50). Thus, none of the nonstationary vacua join smoothly onto the RW exterior.

V. SUMMARY

We have found all separable, cylindrically symmetric solutions to the Einstein equations with $T_t^t=T_z^z=-\sigma(r,t)$. The solutions given by the metrics (21), (26), (29), and (34) represent nonstationary strings that are nonsingular on the axis and which can easily be embeded in an external vacuum. In the case of (21) the exterior vacuum is static while in the other two cases it is nonstationary. The solution given by metric (36) is singular on the axis. We have also demonstrated the existence of another class of solutions (Sec. II B 2); however, these solutions diverge as $r\to\infty$ and cannot easily be embeded in an exterior vacuum due to the numerical nature of the solution.

The vacua surrounding the strings in the cases of metrics (21), (26), (29), and (34) cannot be joined smoothly onto an exterior RW spacetime.

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APPENDIX A

The Killing equations for the metric (44) are

$$\frac{\partial\xi_0}{\partial t}=0 \ , \tag{A1}$$

$$\frac{\partial\xi_0}{\partial r}+\frac{\partial\xi_1}{\partial t}-\frac{2}{t}\xi_1=0 \ , \tag{A2}$$

$$\frac{\partial\xi_0}{\partial\phi}+\frac{\partial\xi_2}{\partial t}-\frac{2}{t}\xi_2=0 \ , \tag{A3}$$

$$\frac{\partial\xi_0}{\partial z}+\frac{\partial\xi_3}{\partial t}=0 \ , \tag{A4}$$

$$\frac{\partial\xi_1}{\partial r}-t\xi_0=0 \ , \tag{A5}$$

$$\frac{\partial\xi_1}{\partial\phi}+\frac{\partial\xi_2}{\partial r}-2\frac{\omega'}{\omega}\xi_2=0 \ , \tag{A6}$$

$$\frac{\partial\xi_1}{\partial z}+\frac{\partial\xi_3}{\partial r}=0 \ , \tag{A7}$$

$$\frac{\partial\xi_2}{\partial\phi}-t\omega^2\xi_0+\omega\omega'\xi_1=0 \ , \tag{A8}$$

$$\frac{\partial\xi_2}{\partial z}+\frac{\partial\xi_3}{\partial\phi}=0 \ , \tag{A9}$$

$$\frac{\partial\xi_3}{\partial z}=0 \ . \tag{A10}$$

From (A1) and (A10) we have $\xi_0=\xi_0(r,\phi,z)$ and $\xi_3=\xi_3(t,r,\phi)$. Equations (A2)–(A4) can be integrated to give

$$\xi_1=t\frac{\partial\xi_0}{\partial r}+t^2f(r,\phi,z) \ , \tag{A11}$$

$$\xi_2=t\frac{\partial\xi_0}{\partial\phi}+t^2g(r,\phi,z) \ , \tag{A12}$$

and

$$\xi_3 = -t\frac{\partial \xi_0}{\partial z} + h(r,\phi) , \quad (A13)$$

where f, g, and h are arbitrary functions of their arguments. Equation (A9) then becomes

$$t^2 \frac{\partial g}{\partial z} + \frac{\partial h}{\partial \phi} = 0 .$$

Thus $g = g(r,\phi)$ and $h = h(r)$. Equation (A7) is then

$$t^2 \frac{\partial f}{\partial z} + \frac{\partial h}{\partial r} = 0$$

so that h =const and $f = f(r,\phi)$. With these results, Eq. (A5) can be written as

$$t\left[\frac{\partial^2 \xi_0}{\partial r^2} - \xi_0\right] + t^2 \frac{\partial f}{\partial r} = 0 .$$

This means that $f = f(\phi)$ as well as

$$\frac{\partial^2 \xi_0}{\partial r^2} = \xi_0 . \quad (A14)$$

Equation (A6) is now

$$2t\left[\frac{\partial^2 \xi_0}{\partial \phi\, \partial r} - \frac{\omega'}{\omega}\frac{\partial \xi_0}{\partial \phi}\right] + t^2\left[\frac{\partial f}{\partial \phi} + \frac{\partial g}{\partial r} - 2\frac{\omega'}{\omega} g\right] = 0$$

so we have

$$\frac{\partial^2 \xi_0}{\partial \phi\, \partial r} - \frac{\omega'}{\omega}\frac{\partial \xi_0}{\partial \phi} = 0 .$$

Differentiating this with respect to r and using (A14) we find

$$\frac{\partial \xi_0}{\partial \phi}\left[1 - \frac{\omega''}{\omega}\right] = 0 .$$

Inside the string we have $\omega'' \neq \omega$ ($\omega'' = \omega$ defines the vacuum in this case) so that $\partial\xi_0/\partial\phi = 0$. Equation (A8) can now be written as

$$t^2\left[\frac{\partial g}{\partial \phi} + \omega\omega' f\right] + t\left[\omega\omega'\frac{\partial \xi_0}{\partial r} - \omega^2 \xi_0\right] = 0 .$$

Thus we have

$$\frac{\partial \xi_0}{\partial r}\frac{\omega'}{\omega} = \xi_0 . \quad (A15)$$

Differentiating this we find

$$\frac{\omega''}{\omega} - \left[\frac{\omega'}{\omega}\right]^2 = 1 - \frac{\xi_0 \partial^2 \xi_0/\partial r^2}{(\partial \xi_0/\partial r)^2}$$

if $\partial\xi_0/\partial r \neq 0$. However, using (A14) and (A15) we find that this implies $\omega'' = \omega$. Since $\omega'' \neq \omega$ inside the string we conclude that $\partial\xi_0/\partial r = 0$ and from (A15) we see that $\xi_0 = 0$. Thus there are no timelike Killing vectors inside the string so it is nonstationary.

APPENDIX B

The Killing equations for the metric (26) are (after letting $w \to t$)

$$\frac{\partial \xi_0}{\partial t} - \frac{2\xi_0}{t} = 0 , \quad (B1)$$

$$\frac{\partial \xi_0}{\partial r} + \frac{\partial \xi_1}{\partial t} - \frac{4}{t}\xi_1 = 0 , \quad (B2)$$

$$\frac{\partial \xi_0}{\partial \phi} + \frac{\partial \xi_2}{\partial t} - \frac{4}{t}\xi_2 = 0 , \quad (B3)$$

$$\frac{\partial \xi_0}{\partial z} + \frac{\partial \xi_3}{\partial t} + \frac{2}{t}\xi_3 = 0 , \quad (B4)$$

$$\frac{\partial \xi_1}{\partial r} - \frac{2}{t}\xi_0 = 0 , \quad (B5)$$

$$\frac{\partial \xi_1}{\partial \phi} + \frac{\partial \xi_2}{\partial r} - 2\frac{\omega'}{\omega}\xi_2 = 0 , \quad (B6)$$

$$\frac{\partial \xi_1}{\partial z} + \frac{\partial \xi_3}{\partial r} = 0 , \quad (B7)$$

$$\frac{\partial \xi_2}{\partial \phi} - 2\frac{\omega^2}{t}\xi_0 + \omega\omega'\xi_1 = 0 , \quad (B8)$$

$$\frac{\partial \xi_2}{\partial z} + \frac{\partial \xi_3}{\partial \phi} = 0 , \quad (B9)$$

$$\frac{\partial \xi_3}{\partial z} + \frac{\xi_0}{t^7} = 0 . \quad (B10)$$

Equation (B1) shows that $\xi_0 = t^2 f(r,\phi,z)$ so Eq. (B4) may be written as

$$t^2 \frac{\partial f}{\partial z} + \frac{1}{t^2}\frac{\partial (t^2 \xi_3)}{\partial t} = 0 .$$

This means that ξ_3 is of the form

$$\xi_3 = -\frac{t^3}{5}\frac{\partial f}{\partial z} + \frac{i(r,\phi,z)}{t^2} .$$

Using this in (B10) we find

$$-\frac{t^3}{5}\frac{\partial^2 f}{\partial z^2} + \frac{1}{t^2}\frac{\partial i}{\partial z} + \frac{1}{t^5} f = 0$$

so that $i = i(r,\phi)$ and $f = 0$ and thus $\xi_0 = 0$. Therefore, this spacetime is nonstationary in both the string and vacuum regions.

Now we have $\xi_3 = i(r,\phi)/t^2$. Equation (B5) shows that $\xi_1 = \xi_1(t,\phi,z)$ and (B2) can be integrated to give $\xi_1 = t^4 j(\phi,z)$. Using these in (B7) we find $\partial i/\partial r = \partial j/\partial z = 0$ so that $i = i(\phi)$ and $j = j(\phi)$. Upon integration (B3) gives $\xi_2 = t^4 l(r,\phi,z)$ so that (B9) is

$$t^4 \frac{\partial l}{\partial z} + \frac{1}{t^2}\frac{\partial i}{\partial \phi} = 0 ,$$

i.e., $l = l(r,\phi)$ and i =const (say α) so we have $\xi_3 = \alpha/t^2$.

With $\xi_1 = t^4 j(\phi)$ and $\xi_2 = t^4 l(r,\phi)$, Eqs. (B6) and (B8) become

$$\frac{\partial j}{\partial \phi} + \frac{\partial l}{\partial r} - 2\frac{\omega'}{\omega} l = 0 \quad (B11)$$

and

$$\frac{\partial l}{\partial \phi}+\omega\omega' j=0 \ . \tag{B12}$$

Differentiating (B11) with respect to ϕ and using (B12) we find

$$\frac{\partial^2 j}{\partial \phi^2}=-(\omega'^2-\omega''\omega)j \ . \tag{B13}$$

This allows two classes of solution depending on whether or not $\omega'^2-\omega''\omega$ is a constant.

If $\omega'^2-\omega''\omega=\epsilon$ (ϵ=const), we have, upon differentiation,

$$\omega'\omega''-\omega\omega'''=0$$

so that, if $\omega''\neq 0$, we have

$$\omega''=c\omega \ ,$$

where c is a constant. The case $\omega''=0$ can be included simply by setting $c=0$. In this case we are dealing with either the vacuum ($c=0$) or a homogeneous string ($c\neq 0$). From Eq. (27) we see that we must require $c\leq 0$ so that the energy density is non-negative. With $c\leq 0$ we see that ϵ must be positive. Since $\omega(r)$ is only defined by the field equations up to an arbitrary multiplicative factor, we can let $\omega\rightarrow\sqrt{\epsilon}\omega$ so that ϵ can be set to 1.

With $\epsilon=1$, Eq. (B13) becomes

$$\frac{\partial^2 j}{\partial \phi^2}=-j$$

so that

$$j=\beta\sin\phi+\delta\cos\phi \ ,$$

where β and δ are constants. Equation (B12) can be integrated to give

$$l=\beta\omega\omega'\cos\phi-\delta\omega\omega'\sin\phi+m(r)$$

and using this in (B11) we find

$$\frac{\partial m}{\partial r}-2\frac{\omega'}{\omega}m=0 \ .$$

Thus we have $m=\gamma\omega^2$. ξ_1 and ξ_2 are now given by

$$\xi_1=t^4(\beta\sin\phi+\delta\cos\phi)$$

and

$$\xi_2=t^4(\beta\omega\omega'\cos\phi-\delta\omega\omega'\sin\phi+\gamma\omega^2) \ .$$

With these results we see that the vacuum and a homogeneous string admit the four Killing vectors

$$\xi_{A\alpha}=(0,0,0t^{-2}) \ ,$$

$$\xi_{B\alpha}=(0,0,\omega^2t^4,0) \ ,$$

$$\xi_{C\alpha}=(0,t^4\sin\phi,t^4\omega\omega'\cos\phi,0) \ ,$$

$$\xi_{D\alpha}=(0,t^4\cos\phi,-t^4\omega\omega'\sin\phi,0) \ ,$$

or, in contravariant form,

$$\xi^\alpha_A=(0,0,0,1) \ ,$$

$$\xi^\alpha_B=(0,0,1,0) \ ,$$

$$\xi^\alpha_C=\left[0,\sin\phi,\frac{\omega'}{\omega}\cos\phi,0\right] \ ,$$

$$\xi^\alpha_D=\left[0,\cos\phi,-\frac{\omega'}{\omega}\sin\phi,0\right] \ .$$

If $\omega'^2-\omega''\omega$ is not constant, i.e., inside an inhomogeneous string, Eq. (B13) demands that $j(\phi)=0$. Equation (B11) then gives $l=\gamma\omega^2$. In this case $\xi_1=0$ and $\xi_2=\gamma\omega^2t^4$ and the spacetime admits only the two Killing vectors ξ^α_A and ξ^α_B.

APPENDIX C

For the metric (47) the Killing equations are

$$\frac{\partial\xi_0}{\partial t}-\frac{\sinh t}{1+\cosh t}\xi_0=0 \ , \tag{C1}$$

$$\frac{\partial\xi_0}{\partial r}+\frac{\partial\xi_1}{\partial t}-\frac{2\sinh t}{1+\cosh t}\xi_1=0 \ , \tag{C2}$$

$$\frac{\partial\xi_0}{\partial \phi}+\frac{\partial\xi_2}{\partial t}-\frac{2\sinh t}{1+\cosh t}\xi_2=0 \ , \tag{C3}$$

$$\frac{\partial\xi_0}{\partial z}+\frac{\partial\xi_3}{\partial t}-\frac{2}{\sinh t}\xi_3=0 \ , \tag{C4}$$

$$\frac{\partial\xi_1}{\partial r}-c_{10}c_2^2\frac{\sinh t}{1+\cosh t}\xi_0=0 \ , \tag{C5}$$

$$\frac{\partial\xi_1}{\partial \phi}+\frac{\partial\xi_2}{\partial r}-2\frac{\omega'}{\omega}\xi_2=0 \ , \tag{C6}$$

$$\frac{\partial\xi_1}{\partial z}+\frac{\partial\xi_3}{\partial r}=0 \ , \tag{C7}$$

$$\frac{\partial\xi_2}{\partial \phi}-c_{10}c_2^2\omega^2\frac{\sinh t}{1+\cosh t}\xi_0+\omega\omega'\xi_1=0 \ , \tag{C8}$$

$$\frac{\partial\xi_2}{\partial z}+\frac{\partial\xi_3}{\partial \phi}=0 \ , \tag{C9}$$

$$\frac{\partial\xi_3}{\partial z}-c_{10}c_2^2\frac{\sinh t\,\xi_0}{(1+\cosh t)^4}=0 \ . \tag{C10}$$

Equation (C1) can be integrated to give $\xi_0=(1+\cosh t)f(r,\phi,z)$. Differentiating (C4) with respect to z we have

$$\frac{\partial^2\xi_0}{\partial z^2}+\frac{\partial^2\xi_3}{\partial z\,\partial t}-\frac{2}{\sinh t}\frac{\partial\xi_3}{\partial z}=0 \ .$$

Using (C10) in this we find

$$(1+\cosh t)\frac{\partial^2 f}{\partial z^2}-\frac{1-2\cosh t}{(1+\cosh t)^3}c_{10}c_2^2f=0$$

so we have $\partial^2 f/\partial z^2=f=0$. Thus $\xi_0=0$ and the Killing vectors are all spacelike so both the string and vacuum regions are nonstationary.

Equations (C2), (C3), and (C4) can now be integrated to give

$$\xi_1 = (1+\cosh t)^2 j(r,\phi,z) ,$$

$$\xi_2 = (1+\cosh t)^2 l(r,\phi,z) ,$$

and

$$\xi_3 = \frac{\cosh t - 1}{\cosh t + 1} i(r,\phi,z) .$$

Equations (C5) and (C10) then show that $i = i(r,\phi)$ and $j = j(\phi,z)$. Using these in (C7) we see that $i = i(\phi)$ and $j = j(\phi)$. Equation (C9) then gives $i = \text{const}$ (say, α) as well as $l = l(r,\phi)$. Equations (C6) and (C8) are then exactly the same as (B11) and (B12). The solutions for $l(r,\phi)$ and $j(\phi)$ are then the same as before so we have

$$\xi_1 = (1+\cosh t)^2(\beta \sin\phi + \delta \cos\phi) ,$$

$$\xi_2 = (1+\cosh t)^2(\beta\omega\omega' \cos\phi - \delta\omega\omega' \sin\phi + \gamma\omega^2) ,$$

and

$$\xi_3 = \frac{\cosh t - 1}{\cosh t + 1} i(r,\phi,z) .$$

In contravariant form the Killing vectors are the same as those given in Appendix B.

The Killing equations for the metric (48) are essentially the same as those for the metric (47) except with $\cos t$ replacing $\cosh t$. The equations can then be solved in exactly the same manner as above and we find

$$\xi_1 = (1+\cos t)^2(\beta \sin\phi + \delta \cos\phi) ,$$

$$\xi_2 = (1+\cos t)^2(\omega\omega'\beta \cos\phi - \omega\omega'\delta \sin\phi + \gamma\omega^2) ,$$

and

$$\xi_3 = \alpha \frac{1-\cos t}{1+\cos t} .$$

Again, the contravariant forms of the Killing vectors are the same as those given in Appendix B.

[1]Ya. B. Zel'dovich, I. Yu. Kobzarev, and L. B. Okun, Zh. Eksp. Teor. Fiz. **67**, 3 (1974) [Sov. Phys. JETP **40**, 1 (1975)].

[2]T. W. Kibble, J. Phys. A **9**, 1387 (1976); Phys. Rep. **67**, 183 (1980).

[3]A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[4]Here we are concerned with nonstationary cylindrically symmetric solutions to the Einstein equations. These do not, in general, have the invariance properties associated with (current-free) trapped regions of false vacuum. Their "stringlike" nature arises out of their symmetry. In the text we use, rather loosely, the term "string" to indicate a nonvacuum cylindrically symmetric solution.

[5]A. Vilenkin, Phys. Rev. D **23**, 852 (1981).

[6]J. R. Gott, Astrophys. J. **288**, 422 (1985).

[7]Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. **192**, 663 (1980).

[8]A. Vilenkin, Phys. Rev. Lett. **46**, 1169 (1981).

[9]W. A. Hiscock and J. B. Lail, Phys. Rev. D **37**, 869 (1988).

[10]J. Silk and A. Vilenkin, Phys. Rev. Lett. **53**, 1700 (1984).

[11]W. G. Unruh, G. Hayward, W. Israel, and D. McManus, Phys. Rev. Lett. **62**, 2897 (1989).

[12]D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).

[13]B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985).

[14]C. C. Dyer, L. M. Oates, and G. D. Starkman, Gen. Relativ Gravit. **20**, 71 (1988).

[15]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[16]J. A. Stein-Schabes, Phys. Rev. D **33**, 3545 (1986).

[17]A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnétisme* (Masson, Paris, 1955), Chap. III.

[18]W. J. Cocke, J. Math. Phys. **7**, 1171 (1966).

[19]G. Darmois, *Mèmorial de Sciences Mathèmatiques, Fascicule XXV* (Gauthier-Villars, Paris, 1927), Chap. V.

[20]W. Israel, Nuovo Cimento **44B**, 1 (1966); see also **48B**, 463 (1967).