Proof of the closed-universe-recollapse conjecture for diagonal Bianchi type-IX cosmologies

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It is proven that there do not exist any diagonal Bianchi type-IX universes which expand for an infinite time, provided only that the matter satisfies the dominant energy condition and has non-negative average principal pressures.

I. INTRODUCTION

For Robertson-Walker cosmological models, there exists an interesting connection between the topology of space and the future evolution of the Universe. Under only the assumption that the energy density of matter is non-negative, it is easy to prove that any "open" (i.e., $k=0,-1$) expanding Robertson-Walker model must continue to expand forever. On the other hand, under the assumption that the average pressure of matter is non-negative (and under further assumptions on the matter that ensure that no singularities occur during the expanding phase), every three-sphere ($k=+1$) expanding Robertson-Walker universe must recollapse within a finite amount of time.[1] It has been conjectured by Barrow, Galloway, and Tipler[1] that this recollapse behavior of the $k=+1$ Robertson-Walker models holds quite generally for all universes with spatial topology S^3 and $S^1 \times S^2$. In this paper, we shall give a proof that such behavior does indeed hold for the case of diagonal Bianchi type-IX universes with appropriate matter content.[2]

To begin, we must define more precisely what we mean here by "expansion" and "contraction" of the Universe. Recall that a *slice* of spacetime is a closed, achronal, edgeless set; a spacelike slice thus can be taken to represent an "instant of time." The Universe may be said to be expanding (contracting) at the instant of time represented by the smooth, spacelike slice Σ if the trace of the extrinsic curvature of Σ is everywhere positive (negative). However, in general, this property may be more reflective of how the slice is chosen than of the dynamics of the spacetime. Indeed, it is not difficult to find *contracting* slices in open Robertson-Walker models which "expand forever." (These contracting slices terminate in the initial singularity.) The situation is improved somewhat if the spacetime is globally hyperbolic, since then one can impose the additional demand that the slice be a Cauchy surface. However, the fact that Minkowski spacetime possesses expanding and contracting Cauchy surfaces makes manifest that a statement such as "the Universe is expanding at 'time' Σ" must be interpreted with some care.

In the case of spatially homogeneous cosmologies, as will be considered in this paper, there is a natural choice of slicing of spacetime: namely, that defined by the surfaces of homogeneity. Thus, when we use the term "expanding" or "contracting" we refer to the trace of the extrinsic curvature of these slices. The issue we shall investigate here is whether an initially expanding diagonal Bianchi type-IX universe with appropriate matter sources must, within a finite time, reach a maximum of expansion and then begin to contract, i.e., "recollapse." Note that if recollapse begins, then it follows from the Raychauduri equation (applied to the congruence of geodesics normal to the homogeneous hypersurfaces) that the contraction rate will diverge to infinity a finite time later (see, e.g., lemma 9.2.1 of Ref. 4). For noncompact spatial slices, this divergence could merely represent a singularity in the slicing rather than in the spacetime structure. However, for the case of compact slices, it is known from the singularity theorems that a spacetime singularity must occur in the sense that not all timelike geodesics can be future complete (see theorem 9.5.2 of Ref. 4). Thus, for a Bianchi type-IX universe, recollapse implies the existence of a "final" singularity. If it is further assumed that the spacetime is globally hyperbolic and that the surfaces of homogeneity are Cauchy surfaces, then this final singularity must be "all encompassing"; i.e., all timelike curves have finite length (see theorem 9.5.1 of Ref. 4).

However, difficulties arise when attempting to prove that recollapse occurs because of the possibility that a singularity could develop during the expanding phase, thereby effectively halting the evolution prior to recollapse. In the vacuum case, it is possible to show (as we shall at the beginning of Sec. II below) that no singularity can occur during the expanding phase of a Bianchi type-IX solution. However, such singularities could arise if matter is present and if no "equation of state" or other suitable conditions are imposed on the matter. In order to avoid imposing such additional conditions, we shall reformulate our basic question as follows: Do there exist any diagonal, Bianchi type-IX solutions with appropriate matter sources which expand for an infinite amount of proper time? If no singularities can occur during the expanding phase, then this question is equivalent to the original question of whether all such universes must recollapse within a finite time.

By definition, the general Bianchi type-IX spacetime has topology $\mathbb{R}\times S^3$, with a simply transitive action of the isometry group SU(2) on the S^3 spatial slices. The metric of a general Bianchi type-IX model can be put in the form

$$ds^2=-d\tau^2+e^{2\alpha}\sum_{i,j=1}^{3}(e^{2\beta})_{ij}\sigma^i\sigma^j \tag{1.1}$$

(see, e.g., Sec. 7.2 of Ref. 4). Here σ^1, σ^2, and σ^3 are isometry invariant one-forms on the three-sphere satisfying $\pounds_\tau \sigma^i=0$ (where τ^a is the unit normal to the homogeneous hypersurfaces), α is a scalar, and $\boldsymbol{\beta}$ is a traceless 3×3 matrix. (Both α and $\boldsymbol{\beta}$ are functions of the proper time τ only.) The diagonal Bianchi type-IX spacetimes are those for which the σ^i can be chosen so that $\boldsymbol{\beta}$ is a diagonal matrix for all time. This requirement is equivalent to demanding that on each homogeneous slice the eigenvectors of the extrinsic curvature tensor $K^a{}_b$ coincide with the eigenvectors of the three-dimensional Ricci tensor ${}^{(3)}R^a{}_b$. For vacuum solutions, this condition is implied by the field equations (see Ref. 5), so the diagonal case encompasses all vacuum Bianchi type-IX spacetimes.

For a diagonal spacetime, let β_1, β_2, and β_3 denote the diagonal elements of the matrix $\boldsymbol{\beta}$. Only two of these quantities are independent since $\beta_1+\beta_2+\beta_3=0$ on account of the tracelessness of $\boldsymbol{\beta}$. We choose the independent variables to be

$$\beta_+=-\tfrac{1}{2}\beta_3 , \tag{1.2}$$

$$\beta_-=\frac{1}{2\sqrt{3}}(\beta_1-\beta_2) . \tag{1.3}$$

In the diagonal case, it is not difficult to show that the Einstein tensor $G^a{}_b$ must be diagonalizable and its eigenvectors must coincide with the vectors $(d\tau)^a$, $(\sigma^1)^a$, $(\sigma^2)^a$, and $(\sigma^3)^a$. Thus, a diagonal Bianchi type-IX universe can admit only matter distributions with a similarly diagonal stress-energy tensor $T^a{}_b$. Let $-\rho$, P_1, P_2, and P_3 denote the eigenvalues of $T^a{}_b$ corresponding, respectively, to the eigenvectors $(d\tau)^a$, $(\sigma^1)^a$, $(\sigma^2)^a$, and $(\sigma^3)^a$. Then Einstein's equation $G_{ab}=8\pi T_{ab}$ for a diagonal, Bianchi type-IX universe, takes the form[5]

$$\left[\frac{d\alpha}{d\tau}\right]^2-\left[\frac{d\beta_+}{d\tau}\right]^2-\left[\frac{d\beta_-}{d\tau}\right]^2+\tfrac{1}{4}e^{-2\alpha}(1-V)=\frac{8\pi}{3}\rho , \tag{1.4}$$

$$\frac{d^2\alpha}{d\tau^2}+\left[\frac{d\alpha}{d\tau}\right]^2+2\left[\left[\frac{d\beta_+}{d\tau}\right]^2+\left[\frac{d\beta_-}{d\tau}\right]^2\right]=-\frac{4\pi}{3}(\rho+P_1+P_2+P_3) , \tag{1.5}$$

$$\frac{d^2\beta_+}{d\tau^2}+3\frac{d\alpha}{d\tau}\frac{d\beta_+}{d\tau}+\frac{1}{8}e^{-2\alpha}\frac{\partial V}{\partial\beta_+}=-\frac{4\pi}{3}(2P_3-P_1-P_2) , \tag{1.6}$$

$$\frac{d^2\beta_-}{d\tau^2}+3\frac{d\alpha}{d\tau}\frac{d\beta_-}{d\tau}+\frac{1}{8}e^{-2\alpha}\frac{\partial V}{\partial\beta_-}=\frac{4\pi\sqrt{3}}{3}(P_1-P_2) . \tag{1.7}$$

Here $V(\beta_+,\beta_-)\equiv 1-\frac{2}{3}{}^{(3)}Re^{2\alpha}$ is given by

$$V(\beta_+,\beta_-)=1-\tfrac{4}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)+\tfrac{1}{3}e^{-8\beta_+}+\tfrac{2}{3}e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-)-1] . \tag{1.8}$$

Note that $V\geq 0$. A plot of the level surfaces of V is given in Fig. 1. In the vacuum case $T_{ab}=0$ there is redundancy in the above system: By use of the Bianchi identity $\nabla^a G_{ab}=0$, Eq. (1.4) and any two of Eqs. (1.5)–(1.7) imply the remaining equation.

The trace of the extrinsic curvature of an arbitrary homogeneous slice is

$$K=3\frac{d\alpha}{d\tau} . \tag{1.9}$$

Hence, the Universe is expanding if and only if $d\alpha/d\tau$ is positive. Thus, we seek to determine whether there exist any solutions of the system (1.4)–(1.7) of ordinary differential equations (with appropriate restrictions imposed upon ρ, P_1, P_2, and P_3) such that $d\alpha/d\tau>0$ over a half-infinite interval of τ time.

It will be convenient to introduce a new time variable t, monotonically related to τ by

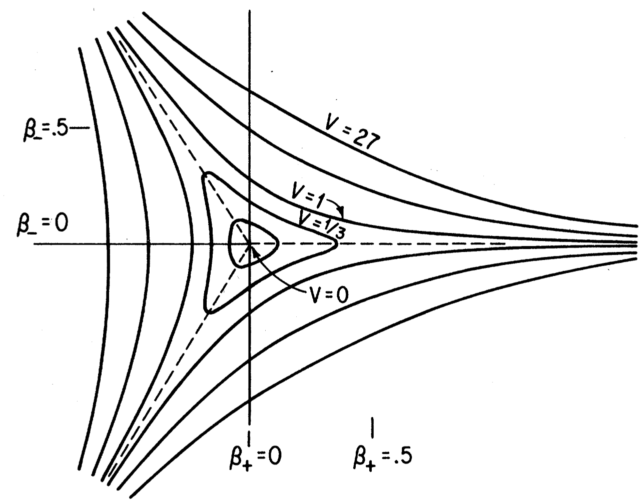

FIG. 1. A plot of the level surfaces of the "potential" $V(\beta_+,\beta_-)$ appearing in the field equations. (This plot is adapted from Ref. 3, with appropriate changes in conventions.) The potential has a symmetry under 120° rotations in the β_+-β_- plane, corresponding to cyclic permutations of β_1,β_2,β_3. The minimum value of $V=0$ is achieved at origin. The contours with $V<1$ are closed, whereas the contours with $V\geq 1$ are asymptotic to the dotted lines $\beta_-=0$ and $\sqrt{3}\beta_+\pm\beta_-=0$ shown in the figure. Thus, for any $C>1$, the contour $V=C$ defines three "channels" along which $\beta_+^2+\beta_-^2$ can diverge to infinity, keeping $V\leq C$.

$$\frac{dt}{d\tau}=e^{-\alpha} . \tag{1.10}$$

In terms of this variable, Einstein's equations (1.4)–(1.7) become

$$\dot\alpha^2-\dot\beta_+^2-\dot\beta_-^2+\tfrac{1}{4}(1-V)=\frac{8\pi}{3}e^{2\alpha}\rho , \tag{1.11}$$

$$\ddot\alpha+2(\dot\beta_+^2+\dot\beta_-^2)=-\frac{4\pi}{3}e^{2\alpha}(\rho+P_1+P_2+P_3) , \tag{1.12}$$

$$\ddot\beta_++2\dot\alpha\dot\beta_++\frac{1}{8}\frac{\partial V}{\partial\beta_+}=-\frac{4\pi}{3}e^{2\alpha}(2P_3-P_1-P_2) , \tag{1.13}$$

$$\ddot\beta_-+2\dot\alpha\dot\beta_-+\frac{1}{8}\frac{\partial V}{\partial\beta_-}=\frac{4\pi\sqrt{3}}{3}e^{2\alpha}(P_1-P_2) , \tag{1.14}$$

where the overdots denote derivatives with respect to t.

From Eq. (1.12), we obtain the very important conclusion that for matter satisfying $\rho+P_1+P_2+P_3\geq 0$ (as we shall assume below), we have

$$\ddot\alpha\leq 0. \tag{1.15}$$

Thus, $\dot\alpha$ decreases with time, and, hence,

$$\alpha(t)\leq\dot\alpha_0(t-t_0)+\alpha_0 , \tag{1.16}$$

where the subscript 0 denotes the initial value at time $t=t_0$. Consequently by integrating Eq. (1.10) we obtain

$$\tau\leq\frac{1}{\dot\alpha_0}e^{\dot\alpha_0(t-t_0)+\alpha_0} . \tag{1.17}$$

From Eq. (1.17), it follows that as $\tau\to\infty$, we have $t\to\infty$. Conversely, if $\dot\alpha\geq 0$ holds for all t, then $\alpha\geq\alpha_0$, so by integrating Eq. (1.10) again, now using this inequality, we find that $\tau\geq e^{\alpha_0}(t-t_0)$, and hence as $t\to\infty$ we have $\tau\to\infty$. Thus, if $\rho+P_1+P_2+P_3\geq 0$, the existence of a solution which expands for a half-infinite interval $[\tau_0,\infty)$ of τ time (i.e., proper time as measured by observers who move orthogonally to the homogeneous hypersurfaces) is equivalent to the existence of a solution which expands for a half-infinite interval in t time, i.e., for which $\dot\alpha\geq 0$ for all $t\in[t_0,\infty)$.

In this paper we shall prove that if the dominant energy condition holds and if the average pressure $\frac{1}{3}(P_1+P_2+P_3)$ is non-negative, then there does not exist a solution of Eqs. (1.11)–(1.14) which expands forever in this sense. The proof of this result simplifies considerably in the vacuum case and we shall give the vacuum proof in the next section. The generalization to the nonvacuum case will be given in Sec. III.

II. PROOF FOR THE VACUUM CASE

As previously mentioned, for vacuum Bianchi type-IX solutions, there is no loss of generality in restricting to the diagonal case. Einstein's equations (1.11)–(1.14) in the vacuum case become

$$\dot\alpha^2-\dot\beta_+^2-\dot\beta_-^2+\tfrac{1}{4}(1-V)=0 , \tag{2.1}$$

$$\ddot\alpha+2(\dot\beta_+^2+\dot\beta_-^2)=0 , \tag{2.2}$$

$$\ddot\beta_++2\dot\alpha\dot\beta_++\frac{1}{8}\frac{\partial V}{\partial\beta_+}=0 , \tag{2.3}$$

$$\ddot\beta_-+2\dot\alpha\dot\beta_-+\frac{1}{8}\frac{\partial V}{\partial\beta_-}=0 . \tag{2.4}$$

(As already noted above, only three of these equations are independent.) We seek solutions of (2.1)–(2.4) for which $\dot\alpha\geq 0$ for all $t\in[t_0,\infty)$. As pointed out at the end of the previous section, Eq. (2.2) implies that

$$\ddot\alpha\leq 0 \tag{2.5}$$

and, hence,

$$\dot\alpha\leq\dot\alpha_0 , \tag{2.6}$$

where $\dot\alpha_0=\dot\alpha(t_0)$. Hence, if $\dot\alpha\geq 0$, from Eq. (2.1), we have

$$\dot\beta_+^2+\dot\beta_-^2+\tfrac{1}{4}V\leq\tfrac{1}{4}+\dot\alpha_0^2 . \tag{2.7}$$

Since $V\geq 0$, this shows that for an expanding universe, $\dot\beta_+$ and $\dot\beta_-$, as well as V, are uniformly bounded in time. Hence, β_+ and β_- (as well as α), cannot diverge in any finite interval of t time. Equations (2.2)–(2.4) then show that the second (and higher) derivatives of α, β_+, and β_- also are bounded in any finite interval of t time. Consequently, for a solution defined on a finite, open interval of t time, the quantities α,β_+,β_- and their first time derivatives can be continuously extended to the future end point of that interval. Hence, the solution can be extended beyond that interval. This establishes that in the vacuum case, no singularities can occur during an expanding phase, as was claimed in the previous section.

The proof that there do not exist solutions of (2.1)–(2.4) which expand forever divides naturally into two steps. In the first step, we prove that for any solution which expands forever, the dynamical trajectory must "escape to infinity" in the β_+-β_- plane along one of the "channels" of the potential V (see Fig. 1). This step uses only Eq. (2.2), the bound (2.7), and the nonexistence of static solutions to the full set of equations. On the other hand, the second step of the argument uses detailed properties of the equations to show that such "escape along a channel" is impossible.

The first step is accomplished by means of the following lemma.

Lemma. Let $\alpha(t),\beta_+(t),\beta_-(t)$ be a solution of Eqs. (2.1)–(2.4) such that $\dot\alpha\geq 0$ for all $t\in[t_0,\infty)$. Let K be any compact subset of the β_+-β_- plane. Then at sufficiently late times the dynamical trajectory cannot enter K; i.e., there exists $t_1\in\mathbb{R}$ such that $(\beta_+(t),\beta_-(t))\notin K$ for all $t\geq t_1$.

Proof. Since α enters Eqs. (2.1)–(2.4) only in differentiated form, we may view the initial data space $\mathcal{S}$ for Eqs. (2.1)–(2.4) as consisting of the quantities $(\beta_+,\beta_-,\dot\beta_+,\dot\beta_-)$ subject to the condition

$$\dot\beta_+^2+\dot\beta_-^2+\tfrac{1}{4}[V(\beta_+,\beta_-)-1]\geq 0 . \tag{2.8}$$

[The initial value of $\dot\alpha$ then is determined by taking the positive square root in Eq. (2.1), and Eqs. (2.2)–(2.4) determine the subsequent evolution of $\dot\alpha$, β_+, and β_-.] Note that since V is continuous, $\mathcal{S}$ is a closed subset of

$\mathbb{R}^4$.

Define the function $f:\mathcal{S}\to\mathbb{R}$ by

$$f(p)=\int_{t_0}^{\infty}(\dot\beta_+^2+\dot\beta_-^2)dt \ . \tag{2.9}$$

Here the integral is taken over the dynamical trajectory determined by the initial condition $p\in\mathcal{S}$ at time $t=t_0$. [If recollapse occurs and the solution determined by p exists for only a finite time, the integral is taken only over that finite time; $f(p)$ may, of course, be infinite.] We claim, first, that given any compact subset $\tilde K\subset\mathcal{S}$, there exists a $c>0$ such that $f(p)\geq c$ for all $p\in\tilde K$. To prove this statement, choose $T>0$ to be sufficiently small that all solutions with initial data in $\tilde K$ exist for at least time $2T$. (That such a T exists follows from standard existence theorems for ordinary differential equations; see, e.g., theorem 8 of Chap. 5 of Ref. 6.) Define $f_T(p)$ by

$$f_T(p)=\int_{t_0}^{t_0+T}(\dot\beta_+^2+\dot\beta_-^2)dt \ . \tag{2.10}$$

Then f_T is a continuous, non-negative function on $\tilde K$ on account of the continuous dependence of solutions to ordinary differential equations on the initial data. Hence, f_T achieves its minimum value, $c\geq 0$, on $\tilde K$. However, $c=0$ is impossible since that would imply $\dot\beta_+=\dot\beta_-=0$ for all $t\in[t_0,t_0+T]$ which, in turn [by Eqs. (2.3) and (2.4)], implies $\beta_+=\beta_-=0$ for all $t\in[t_0,t_0+T]$; however, $\dot\beta_+=\dot\beta_-=\beta_+=\beta_-=0$ is incompatible with Eq. (2.1). Since, clearly $f(p)\geq f_T(p)$, we thus obtain

$$f(p)\geq f_T(p)\geq c>0 \quad \forall p\in\tilde K \tag{2.11}$$

as we claimed.

Now, consider any solution of Eqs. (2.1)–(2.4) for which $\dot\alpha\geq 0$ for all $t\in[t_0,\infty)$. By integrating Eq. (2.2) from t_0 to t, we find

$$2\int_{t_0}^{t}(\dot\beta_+^2+\dot\beta_-^2)dt=\dot\alpha_0-\dot\alpha(t)\leq\dot\alpha_0 \ . \tag{2.12}$$

Thus, taking the limit as $t\to\infty$, we find that the integral on the left side of Eq. (2.12) converges:

$$\int_{t_0}^{\infty}(\dot\beta_+^2+\dot\beta_-^2)dt<\infty \ . \tag{2.13}$$

Since the integrand is non-negative, this equation further implies that

$$\lim_{t\to\infty}\int_{t}^{\infty}(\dot\beta_+^2+\dot\beta_-^2)dt=0 \ ; \tag{2.14}$$

i.e., we have

$$\lim_{t\to\infty}f(p(t))=0 \ . \tag{2.15}$$

Now, let K be a compact subset of the β_+-β_- plane. Let $\tilde K$ be the intersection with $\mathcal{S}$ of the Cartesian product of K with the closed disk of radius $(\frac{1}{4}+\alpha_0^2)^{1/2}$ in the $\dot\beta_+$-$\dot\beta_-$ plane. Then $\tilde K$ is a compact subset of $\mathcal{S}$. By Eq. (2.7), if the dynamical trajectory enters K, it must enter $\tilde K$. As proven above, there exists a $c>0$ such that $f(p)\geq c$ for all $p\in\tilde K$. However, by Eq. (2.15) there exists $t_1\in\mathbb{R}$ such that $f(p(t))<c$ for all $t\geq t_1$. This fact implies that $p(t)\notin\tilde K$ and hence $(\beta_+(t),\beta_-(t))\notin K$ for all $t\geq t_1$, as we desired to show. □

The above lemma directly implies that for a Universe which expands forever we must have $\beta_+^2+\beta_-^2\to\infty$ as $t\to\infty$. However, by Eq. (2.7), V remains uniformly bounded for all time. Thus, the only possible behavior of the solution in the β_+-β_- plane as $t\to\infty$ is for it to "escape" to infinity along one of the three "channels" of V indicated in Fig. 1. On account of the threefold symmetry of V, we may assume, without loss of generality, that the dynamical evolution proceeds along the "right channel," so that $\beta_+\to\infty$ (and $\beta_-\to 0$) as $t\to\infty$. Thus, we may assume that given any $C\in\mathbb{R}$, there exists a $t_1\in\mathbb{R}$ such that $\beta_+(t)>C$ for all $t\geq t_1$. For reasons to be made clear below [see Eq. (2.18)], we choose $C=1$.

We shall now prove that there do not exist any solutions of Eqs. (2.1)–(2.4) which satisfy $\dot\alpha\geq 0$ and $\beta_+>1$ for all $t\geq t_1$ and for which $\beta_+\to\infty$ as $t\to\infty$. We shall structure the proof so that it can be taken over to the nonvacuum case (see Sec. III); a somewhat simplified proof, applicable only to the vacuum case, also could be given.

By multiplying Eq. (2.2) by 5 and adding it to Eq. (2.1), we obtain

$$5\ddot\alpha+\dot\alpha^2+9(\dot\beta_+^2+\dot\beta_-^2)+\tfrac{1}{4}(1-V)=0 \ . \tag{2.16}$$

Adding this equation to Eq. (2.3), we get

$$\ddot\beta_+ +5\ddot\alpha=-9(\dot\beta_+^2+\dot\beta_-^2)-2\dot\alpha\dot\beta_+-\dot\alpha^2 -\frac{1}{8}\left[\frac{\partial V}{\partial\beta_+}+2(1-V)\right] \ . \tag{2.17}$$

From Eq. (1.8), we have

$$-\frac{1}{8}\left[\frac{\partial V}{\partial\beta_+}+2(1-V)\right]=-\tfrac{2}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) -\tfrac{1}{6}e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-)-1] +\tfrac{5}{12}e^{-8\beta_+}<-\tfrac{1}{3}e^{-2\beta_+} \tag{2.18}$$

for all $\beta_+>1$.

We may write the remaining terms on the right side of Eq. (2.17) in the following two ways:

$$\begin{aligned}-9(\dot\beta_+^2+\dot\beta_-^2)-2\dot\alpha\dot\beta_+-\dot\alpha^2 &=-(\dot\beta_++\dot\alpha)^2-8\dot\beta_+^2-9\dot\beta_-^2\\ &=-\tfrac{6}{5}\dot\beta_+(\dot\beta_++5\dot\alpha)-(2\dot\beta_+-\dot\alpha)^2-\tfrac{19}{5}\dot\beta_+^2-9\dot\beta_-^2 \ .\end{aligned} \tag{2.19}$$

From Eqs. (2.17), (2.18), and the first equality of (2.19) we obtain, for all $t>t_1$,

$$\ddot\beta_++5\ddot\alpha<-\tfrac{1}{3}e^{-2\beta_+}. \tag{2.20}$$

In addition, from Eqs. (2.17), (2.18), and the second equality of (2.19), we obtain, for all $t>t_1$,

$$\ddot\beta_++5\ddot\alpha<-\tfrac{6}{5}\dot\beta_+(\dot\beta_++5\dot\alpha) \ . \tag{2.21}$$

The nonexistence of solutions now follows directly from Eqs. (2.20) and (2.21). Define $X(t)$ by

$$X=\dot{\beta}_{+}+5\dot{\alpha} \ . \tag{2.22}$$

Then Eqs. (2.20) and (2.21) become simply

$$\dot{X}<-\tfrac{1}{3}e^{-2\beta_{+}} \tag{2.23}$$

and

$$\dot{X}<-\tfrac{6}{5}X\dot{\beta}_{+} \ . \tag{2.24}$$

The first step of the argument is to show that $X>0$ for all $t\geq t_1$. To do so, we note that Eq. (2.23) implies that $\dot{X}<0$ for all $t\geq t_1$. Hence, if we had $X(t_2)\leq 0$ for any $t_2\in[t_1,\infty)$, it would follow that $X(t)<0$ for all $t>t_2$. Since $\dot{\alpha}>0$, this would imply, in turn, that $\dot{\beta}_{+}<0$ for all $t>t_2$. However, this is incompatible with the fact that $\beta_{+}\rightarrow\infty$ as $t\rightarrow\infty$. Thus, we conclude that $X>0$ for all $t\geq t_1$.

Next, we multiply Eq. (2.23) by X and integrate from any $t\in[t_1,\infty)$ to any $t_2>t$, thereby obtaining

$$\begin{aligned}\tfrac{1}{2}[X^2(t_2)-X^2(t)]&<-\tfrac{1}{3}\int_t^{t_2}e^{-2\beta_{+}}X\,dt\\&<-\tfrac{1}{3}\int_t^{t_2}e^{-2\beta_{+}}\dot{\beta}_{+}dt\\&=\tfrac{1}{6}(e^{-2\beta_{+}(t_2)}-e^{-2\beta_{+}(t)})\ ,\end{aligned} \tag{2.25}$$

where $X>0$ and $\dot{\alpha}>0$ were used. Thus, we find that

$$X^2(t)>\tfrac{1}{3}(e^{-2\beta_{+}(t)}-e^{-2\beta_{+}(t_2)}) \ . \tag{2.26}$$

Taking the limit as $t_2\rightarrow\infty$, we obtain, for all $t\geq t_1$,

$$X(t)>\frac{1}{\sqrt{3}}e^{-\beta_{+}(t)} \ . \tag{2.27}$$

On the other hand, if we divide Eq. (2.24) by X and integrate from t_1 to t, we obtain

$$\begin{aligned}\ln X(t)-\ln X(t_1)&<-\tfrac{6}{5}\int_{t_1}^{t}\dot{\beta}_{+}dt\\&=-\tfrac{6}{5}[\beta_{+}(t)-\beta_{+}(t_1)] \ .\end{aligned} \tag{2.28}$$

Hence, for all $t\geq t_1$, we have

$$X(t)<ce^{-6\beta_{+}(t)/5} \ , \tag{2.29}$$

where c is a constant. However, Eq. (2.29) is incompatible with Eq. (2.27), since $\beta_{+}\rightarrow\infty$ as $t\rightarrow\infty$. Thus, we have obtained a contradiction, thereby proving the nonexistence of solutions. Note that this argument depends very sensitively on the precise form of Eqs. (2.1)–(2.4); appropriate small changes in the numerical coefficients appearing in these equations would invalidate this argument.

We may summarize the results of this section, together with some results from the previous section, with the following theorem.

Theorem 1. There do not exist any vacuum Bianchi type-IX solutions which expand ($\dot{\alpha}\geq 0$) for an infinite amount of proper time τ as measured by observers moving orthogonally to the homogeneous hypersurfaces. Furthermore, since every expanding vacuum Bianchi type-IX solution on a finite proper time interval is extendible into the future, every initially expanding, inextendible, vacuum Bianchi type-IX must recollapse, i.e., $d\alpha/d\tau$ becomes negative within a finite proper time. The singularity theorems then imply existence of a singularity, which must be "all encompassing" (i.e., all timelike curves have finite length) if the surfaces of homogeneity are Cauchy surfaces.

III. THE NONVACUUM CASE

In this section we shall extend the proof of the nonexistence of vacuum Bianchi type-IX universes which expand for an infinite time, to the nonvacuum, diagonal case, under the assumption that the matter satisfies the dominant energy condition and has non-negative average principal pressure. Thus, we assume that

$$|P_i|\leq\rho, \quad P_1+P_2+P_3\geq 0 \ , \tag{3.1}$$

where ρ is the energy density and P_1,P_2,P_3 are the principal pressures. (As discussed in Sec. I, the stress-energy tensor T_{ab} must be diagonalizable in a diagonal Bianchi type-IX solution.) Recall that the proof for the vacuum case was composed of two main steps: First, we proved a lemma which showed that for an expanding universe, we must have $\beta_{+}^2+\beta_{-}^2\rightarrow\infty$ (with V bounded) as $t\rightarrow\infty$. Then we showed that such an "escape along a channel" is impossible. We shall extend the proof to the nonvacuum case by first showing that, for an expanding universe, the matter source terms vanish asymptotically as $t\rightarrow\infty$. This behavior will imply that the late time evolution must be sufficiently close to that of a vacuum solution that the conclusions of the lemma continue to hold. The second step of the argument may then be directly taken over to rule out the existence of solutions which satisfy Eq. (3.1) and expand for an infinite time.

First, we recall from the end of Sec. I that, when Eq. (3.1) holds, Eq. (1.12) implies

$$\ddot{\alpha}\leq 0 \tag{3.2}$$

and hence

$$\dot{\alpha}\leq\dot{\alpha}_0 \ . \tag{3.3}$$

Hence, if $\dot{\alpha}\geq 0$, Eq. (1.11) then yields

$$\dot{\beta}_{+}^2+\dot{\beta}_{-}^2+\tfrac{1}{4}V+\frac{8\pi}{3}e^{2\alpha}\rho\leq\tfrac{1}{4}+\dot{\alpha}_0^2 \ . \tag{3.4}$$

Thus, as in the vacuum case, $\dot{\beta}_{+}$, $\dot{\beta}_{-}$, and V are uniformly bounded in time. In addition, we find now that $e^{2\alpha}\rho$ also is uniformly bounded. The dominant energy condition then implies that $e^{2\alpha}P_i$ is uniformly bounded.

Now differentiate Eq. (1.11) with respect to t, and use Eqs. (1.12)–(1.14) to eliminate the second time derivatives of α, β_{+}, and β_{-}. (The resulting equation expresses conservation of stress energy, $\nabla^a T_{ab}=0$, which follows from the Bianchi identity $\nabla^a G_{ab}=0$.) We obtain

$$\frac{d}{dt}(e^{2\alpha}\rho) = -\dot{\alpha}e^{2\alpha}(\rho+P_1+P_2+P_3) - \dot{\beta}_+ e^{2\alpha}(P_1+P_2-2P_3) - \sqrt{3}\dot{\beta}_- e^{2\alpha}(P_1-P_2) \ . \tag{3.5}$$

Since all quantities on the right side of this equation are uniformly bounded, we find that the first time derivative of $e^{2\alpha}\rho$ is also uniformly bounded in time. Note, however, that our assumptions about the matter do not enable us to place bounds on the first and higher time derivatives of the P_i or the second and higher time derivatives of ρ. Thus, further conditions on the matter would be necessary to ensure that no singularities occur during an expanding phase. Note that if one wishes to have smooth (C^∞) solutions, the "matter regularity condition" of Ref. 1 would not be adequate for this purpose since it does not rule out singularities in the time derivatives of the matter variables and curvature; an example of a sufficient condition is an equation of state, $P_i = P_i(\rho)$, for which P_i is a smooth function of ρ (including at $\rho=0$).

We now prove that the matter source terms in Eqs. (1.11)–(1.14) vanish at asymptotically late times. By integrating Eq. (1.12) from time t to infinity, we obtain

$$\dot{\alpha} - \dot{\alpha}_\infty = 2\int_t^\infty (\dot{\beta}_+^2 + \dot{\beta}_-^2)dt + \frac{4\pi}{3}\int_t^\infty e^{2\alpha}(\rho+P_1+P_2+P_3)dt \ , \tag{3.6}$$

where $\dot{\alpha}_\infty = \lim_{t\to\infty}\dot{\alpha} \geq 0$. (This limit exists since $\dot{\alpha}$ is a monotone decreasing function, bounded from below by zero.) In particular, since ρ and $(P_1+P_2+P_3)$ are nonnegative, this equation implies that $\int_t^\infty e^{2\alpha}\rho\, dt$ converges. In general, for a non-negative, differentiable function F, the convergence of $\int_t^\infty F(t)dt$ does not imply that $F\to 0$ as $t\to\infty$, since F could "spike up" sporadically, with ever narrower "spikes," at arbitrarily large t. However, if, in addition, the derivative of F is uniformly bounded in t, i.e., $F' \leq C$, as in the case here, then F must go to zero because there is a lower bound on the width of any such "spikes." [More precisely, given $\epsilon > 0$, at any $T \in \mathbb{R}$ for which $F(T) \geq \epsilon$, the interval of size $\delta = 2\epsilon/C$ centered about $t = T$ will contribute at least $\epsilon\delta/2$ to the integral. Hence, there can be at most a finite number of such intervals. Consequently, we have $F(t) < \epsilon$ for sufficiently large t.] Thus, we obtain

$$\lim_{t\to\infty} e^{2\alpha}\rho = 0 \ . \tag{3.7}$$

The dominant energy condition then implies that we also have

$$\lim_{t\to\infty} e^{2\alpha}P_i = 0 \tag{3.8}$$

and thus the terms on the right sides of Eqs. (1.11)–(1.14) vanish asymptotically, as claimed.

Equations (3.7) and (3.8) together with the continuous dependence of solutions to ordinary differential equations on "source terms" allow us to extend the lemma of Sec. II to the case of matter. We outline now the steps needed to accomplish this extension.

Consider a solution of Eqs. (1.11)–(1.14) which expands for an infinite time, and for which the matter terms satisfy the dominant energy condition, with non-negative average pressure. Rewrite the equations as a first-order system, with $\dot{\alpha}$, β_+, $\dot{\beta}_+$, β_-, and $\dot{\beta}_-$ viewed as the "independent unknowns" and $e^{2\alpha}\rho$, $e^{2\alpha}P_i$ viewed as prescribed source terms. Let K be a compact subset of the β_+-β_- plane as in the lemma of Sec. II. We shall show that the conclusion of this lemma continues to hold by comparing the given solution with matter to the solution of the vacuum equations ($\rho=0$, $P_i=0$) which, at a specified time, has the same initial data for $\dot{\alpha}$, β_+, $\dot{\beta}_+$, and β_-, and has $\dot{\beta}_-$ adjusted so as to satisfy Eq. (1.11) with $\rho=0$. As in the lemma, choose $T>0$ to be sufficiently small that any vacuum solution starting with $(\beta_+,\beta_-)\in K$ and with $\dot{\alpha} \leq \dot{\alpha}_0$ cannot develop a singularity within time $2T$. Let $c>0$ be a lower bound for $f_T(p)$, defined by Eq. (2.10), for vacuum solutions with initial data satisfying these conditions. Theorems on the continuous dependence of solutions to ordinary differential equations on initial data and "source terms" (see, e.g., theorem 3 of Chap. V of Ref. 6) then imply that for any time t at which the given solution to the equations (with matter sources) satisfies $(\beta_+(t),\beta_-(t))\in K$, the difference between the given solution and the corresponding vacuum solution in the time interval $[t, t+T]$ can be bounded in terms of the maximum value of $e^{2\alpha}\rho$ and $e^{2\alpha}P_i$ in that time interval. Consequently, by virtue of Eqs. (3.7) and (3.8), we can choose a sufficiently large that for all $t \geq a$ satisfying $(\beta_+(t),\beta_-(t))\in K$, we have

$$\left| \int_t^{t+T} (\dot{\beta}_+^2 + \dot{\beta}_-^2)dt - f_T(p(t)) \right| < c/2 \tag{3.9}$$

and, hence,

$$\int_t^{t+T} (\dot{\beta}_+^2 + \dot{\beta}_-^2)dt > c/2 \ . \tag{3.10}$$

[Here the integral is taken over the solution with matter; $f_T(p(t))$ is the corresponding integral computed for vacuum solutions with the corresponding initial data at time t.] By the same argument as used in the vacuum case, the existence of the lower bound (3.10) implies the desired conclusion of the lemma that there exists a t_1 such that $(\beta_+(t),\beta_-(t))\notin K$ for all $t \geq t_1$.

As in the vacuum case, the validity of the lemma, together with the bound on V obtained from Eq. (3.4), implies that the only possible behavior of the solution in the β_+-β_- plane at late times is "escape along a channel," which we again assume, without loss of generality, is the one with $\beta_+\to\infty$ as $t\to\infty$. Thus, we have completed the extension to solutions with matter of the first step of the vacuum proof.

Taking the same combinations of equations as used to derive Eq. (2.17) in the vacuum case, we now obtain

$$\ddot{\beta}_+ + 5\ddot{\alpha} = -9(\dot{\beta}_+^2 + \dot{\beta}_-^2) - 2\dot{\alpha}\dot{\beta}_+ - \dot{\alpha}^2 - \frac{1}{8}\left[\frac{\partial V}{\partial \beta_+} + 2(1-V)\right] - 4\pi e^{2\alpha}(\rho+P_3) - \frac{16\pi}{3}e^{2\alpha}(P_1+P_2+P_3) \ . \tag{3.11}$$

By Eq. (3.1), the matter terms in Eq. (3.11) are nonpositive. Hence, Eqs. (2.20) and (2.21) continue to hold. The proof of the nonexistence of solutions then follows as in the vacuum case. Thus, we have proven the following theorem.

Theorem 2. There do not exist any diagonal Bianchi type-IX solutions with matter satisfying the dominant energy condition with non-negative average principal pressure [see Eq. (3.1)] which expand for an infinite amount of proper time τ as measured by observers moving orthogonally to the homogeneous hypersurfaces.

We conclude by commenting on possible generalizations of our results with regard to weakening the conditions imposed upon the matter and with regard to other homogeneous cosmological models. With regard to the conditions imposed upon the matter, for the extension of the lemma to the nonvacuum case, the conditions that $\rho+P_1+P_2+P_3\geq 0$ and $\rho\geq 0$ [both of which follow from Eq. (3.1)] played a crucial role [see Eqs. (3.2) and (3.4)]. In addition, the positivity of $P_1+P_2+P_3$ was used in the argument below Eq. (3.6) and the bounding of P_i via the dominant energy condition was used to bound the right side of Eq. (3.5) and to derive Eq. (3.8). Hence, for the proof of the lemma, some weakening of condition (3.1) could be achieved; e.g., it would suffice to assume that there is a constant $C>0$ such that

$$\rho+P_1+P_2+P_3\geq C\rho\geq 0 \tag{3.12}$$

and that there is a continuous function h with $h(\rho)\to 0$ as $\rho\to 0$, such that

$$|P_i|\leq h(\rho)\ . \tag{3.13}$$

For the remainder of the proof, all that was used was the positivity of the matter terms on the right-hand side of Eq. (3.11) (as well as the similar conditions obtained by the cyclically permitting P_1,P_2,P_3 to account for the possibility of "escape" along a different channel). In fact, however, the conditions thereby obtained can be generalized somewhat: By taking different linear combinations of Eqs. (1.11)–(1.13), it is possible to give a similar proof of the nonexistence of solutions provided that the matter satisfies

$$A(P_1+P_2+P_3)+B\rho+P_i\geq 0\ , \tag{3.14}$$

where A and B are any real numbers satisfying

$$-2(\sqrt{2}-1)<B<2(1+\sqrt{2}) \tag{3.15}$$

and

$$p<A\leq B+1\ , \tag{3.16}$$

where

$$p=\max\left(\tfrac{1}{3}(B-1)+\tfrac{2}{3}\sqrt{B^2+1/3}\ ,\ \tfrac{1}{2}(B-1)+\tfrac{1}{2}\sqrt{3B^2-2B+1}\right)\ . \tag{3.17}$$

(Our proof corresponds to $B=1$, $A=\frac{4}{3}$.) Thus, the conditions we have imposed upon the matter could be weakened to requiring that Eqs. (3.12)–(3.14) hold. Note, however, that some non-negativity requirement must be imposed on the pressures, since without any such conditions one can easily obtain static solutions to Eqs. (1.11)–(1.14) which satisfy the dominant energy condition.

With regard to generalizations to other homogeneous closed universe models, Burnett[7] has proven that Kantowski-Sachs models (which encompass those with spatial topology $S^1\times S^2$) cannot expand for an infinite time provided only that the matter satisfies $\rho+P_i\geq 0$ and $\sum_i P_i\geq 0$. Since the Bianchi type-IX models encompass all the spatially homogeneous models with S^3 topology, the closed-universe-recollapse conjecture holds for all vacuum homogeneous cosmologies. Thus, for homogeneous cosmologies, the only case which remains is that of nondiagonal Bianchi type-IX solutions with matter. While we foresee no difficulty in extending the conclusions of the lemma of Sec. II to this case, the remainder of the proof made use of the detailed structure of the equations, so it is not clear that a simple generalization of our argument can be given for the nondiagonal case.

Note added in proof. It appears that we now have succeeded in obtaining a generalization of our argument to the nondiagonal case, thereby completing the proof of the closed-universe-recollapse conjecture for homogeneous cosmologies.

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