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Solitosynthesis: Cosmological evolution of nontopological solitons

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We consider the thermal creation, fusion, evaporation, and destruction of nontopological solitons (NTS's) after a phase transition in the early Universe. By defining and following NTS statistical equilibrium and departures from it, we show that depending on particle-physics parameters one of three possible scenarios occurs. If reaction rates are high enough, a period of equilibrium occurs and relic abundances are determined by the "freeze-out" temperature. We show that equilibrium first drives most NTS's into their constituents (free ϕ particles) and then causes rapid fusion into large NTS's. If freeze-out occurs during the first phase, the NTS's are almost entirely destroyed, while if it occurs during the second phase, solitosynthesis occurs and NTS's may be cosmically relevant. For slow reaction rates the NTS's are "born frozen out" and have the abundance determined by the phase transition. We develop analytic approximations for determining the abundances and test them by numerically integrating a reaction network in an expanding Universe. Unfortunately, for most of the parameter space considered, solitodestruction and/or evaporation occurs.

I. INTRODUCTION

Nontopological solitons (NTS's) are classically stable field configurations which have been studied by many groups in recent years.[1] They exist in a variety of field theories, but most simply one considers a complex scalar field ϕ, carrying a conserved charge, coupled to a neutral scalar in such a way as to allow regions of false vacuum where the ϕ is massless and true vacuum where ϕ has a finite mass. NTS solutions occur when some number Q of the ϕ's are trapped in a region of a false vacuum and are unable to escape because their energy is lower than Qm_ϕ, the rest energy of Q free ϕ particles.

If they exist, nontopological solitons are interesting objects; they could have masses ranging from below a proton mass to above a galactic mass, with properties quite different from ordinary matter. One is lead to ask whether there is any mechanism for actually forming this kind of coherent state. One possibility, suggested by Frieman, Gelmini, Gleiser, and Kolb[2] (FGGK), is that during a phase transition in the early Universe when regions of false and true vacuums coexist, a certain number of ϕ's could be trapped in the false-vacuum regions and as these regions shrink they could become NTS's, perhaps surviving until today. FGGK estimated the relic abundance of NTS's, and under several assumptions found that $\Omega_{\rm NTS} \sim 1$ was possible. ($\Omega_{\rm NTS}$ is the ratio of NTS density to the critical density.) However, FGGK did not consider the actual fate of NTS's after the phase transition. Several possibilities exist: they could disassociate or evaporate into free ϕ particles; they could absorb free ϕ's, fuse, and become larger; or they could be created thermally by the fusion of free ϕ's. The object of this paper is to study these mechanisms and the resulting relic abundance of NTS's.

In many respects the problem of thermal creation and destruction of NTS's in the early Universe is similar to big-bang nucleosynthesis. There, light elements such as helium, deuterium, and lithium are synthesized out of protons and neutrons at a temperature of around 1 MeV. Borrowing ideas from big-bang nucleosynthesis we will take two general and complementary approaches: (1) solving a network of reactions involving the annihilation and fusion of a system of free ϕ's and NTS's in an expanding Universe and (2) analytically understanding the results of the network integration by defining and following NTS statistical equilibrium and departures from it.

By NTS statistical equilibrium (NTSSE) we refer to a state where all reactions involving creation and destruction of NTS's are proceeding faster than the expansion rate of the Universe, and the number densities of all species are determined by their binding energies and the entropy of the Universe. If NTSSE ever exists, knowledge of the abundance of NTS's created during a phase transition is lost and therefore irrelevant. At high enough temperatures and densities we expect NTSSE to obtain, but as the temperature drops, the number densities eventually fall so low that reaction rates for processes which establish NTSSE become less than the expansion rate and the relative abundances of NTS's "freeze-out." We denote the temperature at which this happens as T_F, the freeze-out temperature. Of course, NTS's can only exist after the phase transition finishes, so another important temperature is the Ginzburg temperature T_G, after which false-vacuum bubbles are unlikely to spontaneously flip into the true-vacuum state. If $T_G > T_F$ we expect to have a period of statistical equilibrium, while if $T_F > T_G$ we expect the relic abundances to be more or less determined by the phase transition, as discussed by FGGK. Finally, we note that for low enough temperature NTSSE drives all free ϕ particles into NTS's. This is in marked contrast to FGGK's phase transition where, for a relic density of NTS's near critical density, the relic density of free ϕ's was orders of magnitude larger. We solve for the temperature T_D at which the density of NTS's begins to dominate the density of free ϕ's and show that if $T_D > T_F$ we truly have solitosynthesis, the creation of significant numbers of large NTS's by the fusion of free ϕ particles. For the opposite case, $T_F > T_D$, we find that almost all NTS's disassociate and/or evaporate and it is unlikely that NTS's survive in significant numbers.

We will illustrate these various possibilities using the Lagrangian and phase-transition model of FGGK, and check the simple conclusions described above by running a network of reactions. We find good agreement between the network and the analytical approximation for most of the range of parameters we consider.

The plan of this paper is as follows. In Sec. II we describe the Lagrangian, review some necessary aspects of the NTS solution, and review the phase transition scenario of FGGK. In Sec. III we define and derive the formulas for NTS statistical equilibrium. We then develop the important features of NTSSE and find an approximate formula for T_D. In Sec. IV we list the reactions which go into maintaining NTSSE, write down the network equations, and solve them numerically for a truncated system. We also find an approximate solution for the freeze-out temperature T_F and compare this with the numerical results. In Sec. V we present our results, describing conditions under which we have solitosynthesis and conditions under which we have solitodestruction. Section VI sums up the paper.

II. REVIEW OF NONTOPOLOGICAL SOLITONS AND SOLITOGENESIS

We will use throughout the model and conventions of Frieman, Gelmini, Gleiser, and Kolb[2] (FGGK). More details concerning nontopological soliton solutions in general can be found in Ref. 1. The Lagrangian we consider is

$$\mathcal{L} = |\partial_\mu\phi|^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \frac{\lambda_1}{8}(\sigma^2-\sigma_0^2)^2 - h|\phi|^2(\sigma-\sigma_0)^2 - \frac{\lambda_2}{3}\sigma_0(\sigma-\sigma_0)^3 - \Lambda \ , \qquad (1)$$

where ϕ is a complex scalar field and σ is a real scalar field. The field ϕ has a conserved Noether charge Q, a mass of zero at the local minimum $\sigma=\sigma_0$, and a mass $m_\phi^2 = h(\sigma_- - \sigma_0)^2$ at the true minimum $\sigma=\sigma_-$.

By introducing a spherically symmetric trial solution of the equations of motion derived from Eq. (1), and minimizing the resultant energy with respect to the NTS radius, one can find the mass (energy) of an NTS of charge Q: $M_Q = 4\pi\sqrt{2}Q^{3/4}\Lambda^{1/4}/3$, where Λ is adjusted so that the value of the potential is zero at the true minimum. We define the binding energy of an NTS as the difference in mass between the NTS of charge Q, and Q massive ϕ particles:

$$B_Q = Qm_\phi - M_Q \ . \qquad (2)$$

We note that $B>0$ (implying that the NTS is classically stable) occurs as long as $Q>Q_{\rm min}$, where $Q_{\rm min} = 1231\Lambda h^{-2}(\sigma_- - \sigma_0)^{-4}$. If $Q<Q_{\rm min}$ the binding energy is negative and at zero temperature the NTS presumably flies apart into free ϕ particles.

For simplicity, we will consider only the value $\lambda_2 = 0.15\lambda_1$ used in FGGK, in which case the model is described by the parameters λ_1, $Q_{\rm min}$, and σ_0 alone. In this case we have $\Lambda = 0.6\lambda_1\sigma_0^4$, $h = 4.24(\lambda_1/Q_{\rm min})^{1/2}$, $M_Q = 5.15\sigma_0\lambda_1^{1/4}Q^{3/4}$, $m_\phi = 5.15\sigma_0\lambda_1^{1/4}/Q_{\rm min}{}^{1/4}$, and the NTS radius $R_Q = 0.8(Q/\lambda_1)^{1/4}/\sigma_0$.

In considering the development of the phase transition, FGGK define a critical temperature $T_C \approx 2\sigma_0$ after which the Universe divides up into domains of true and false vacuums separated by domain walls. Because the potential energy density of the false vacuum is higher, regions of false vacuum soon shrink, and in this scheme the regions of false vacuum which contain a net charge greater than $Q_{\rm min}$ eventually form nontopological solitons. However, not all regions of false vacuum remain false vacuum. Thermal fluctuations of the σ field can be large below T_C and regions of false vacuum can become true vacuum and vice versa. These fluctuations freeze-out at the Ginzburg temperature T_G which FGGK estimate as $T_G = 1.3\sigma_0/\lambda_1^{1/2}$. They estimate the relic density of NTS's as the density of regions (at $T=T_G$) which have charge greater than $Q_{\rm min}$. Defining the number density of free ϕ's as n_ϕ, the number density of free $\bar\phi$'s as $\bar n_\phi$, and the ϕ asymmetry, $\eta_\phi = (n_\phi - \bar n_\phi)/n_\gamma$, their estimate of the NTS relic number density is

$$\frac{n_{\rm NTS}}{n_\gamma} = \sqrt{2}\left[\frac{\eta_\phi}{\lambda_1 Q_{\rm min}}\right]^{3/2} \exp(-Q_{\rm min}\lambda_1^3/2\eta_\phi) \ , \qquad (3)$$

where $n_\gamma = 2\zeta(3)T^3/\pi^2$ is the photon number density and is related to the entropy density. We see from Eq. (3) that NTS's become exponentially rare as $Q_{\rm min}$ increases, and that in this scheme almost all NTS have charge $Q_{\rm min}$; those with $Q \gg Q_{\rm min}$ are suppressed, and, of course, there are none with $Q<Q_{\rm min}$. Equation (3) must break down as $\eta_\phi \to 0$ since the charge in a small region due to Poisson fluctuations will then be larger than the charge given by the asymmetry, but we will ignore this here.[3] The corresponding density of free ϕ's is roughly

$$\frac{n_\phi - \bar n_\phi}{n_\gamma} = \eta_\phi\left[1 - \frac{\exp(-r_{\rm min})}{r_{\rm min}^{3/2}}\right] , \qquad (4)$$

where $r_{\rm min}$ is the factor in the exponential in Eq. (3). Since $r_{\rm min}$ must be greater than one,[2] as long as $Q_{\rm min} > 1$ the free ϕ density dominates the NTS density. There is, therefore, a problem with having a closure density of NTS's unless the free ϕ particles (which are massive) can annihilate, or are allowed to decay and the ϕ's inside the NTS's (which are massless) are not. Even in this case, however, the ϕ's inside the NTS can "leak out" quantum mechanically and decay. The calculation of NTS decay and the subsequent restrictions on solitogenesis are discussed in Ref. 4 and will not be considered further here. This concludes our review of solitogenesis, but the interested reader is referred to Refs. 1 and 2 for more details.

III. NTS STATISTICAL EQUILIBRIUM (NTSSE)

If the system of free ϕ particles, free $\bar{\phi}$ particles, and NTS's is in statistical equilibrium, the number density of each species is determined by its binding energy and the temperature and entropy of the system. In kinetic equilibrium the number densities of NTS's of charge Q, and of free ϕ's and $\bar{\phi}$'s are given by

$$n_Q=\frac{1}{2\pi^2}TM_Q^2\exp(\mu_Q/T)K_2(M_Q/T)\ ,$$
$$n_\phi=\frac{1}{2\pi^2}Tm_\phi^2\exp(\mu_\phi/T)K_2(m_\phi/T)\ , \tag{5}$$
$$\bar{n}_\phi=\frac{1}{2\pi^2}Tm_\phi^2\exp(\bar{\mu}_\phi/T)K_2(m_\phi/T)\ ,$$

where T is the temperature, the μ's are chemical potentials, and K_2 is a modified Bessel function of the second kind and second order. We define statistical equilibrium to consist of kinetic equilibrium, defined above, plus chemical equilibrium which, if present, means reactions proceed fast enough to ensure relations among the chemical potentials. For example, in chemical equilibrium, the reaction $\phi+\bar{\phi}\leftrightarrow\sigma+\sigma$ proceeds fast enough to ensure $\mu_\phi+\bar{\mu}_\phi=0$, and chemical equilibrium of reactions such as $Q\phi\leftrightarrow\Phi_Q+X$, where X is some state with $Q=0$ and Φ_Q is an NTS of charge Q, implies $Q\mu_\phi=\mu_Q$. Other reactions imply $\bar{\mu}_Q=-\mu_Q$. These relations allow us to specify the system in terms of known binding energies, the temperature, and one chemical potential $\mu\equiv\mu_\phi$.

In order to find the number densities we must find μ, which may be done using charge conservation. The total charge in a comoving volume, R^3, will be conserved; $Q_{\rm TOT}=[n_\phi-\bar{n}_\phi+\sum_{Q=Q_{\rm min}}^{Q_{\rm max}}Q(n_Q-\bar{n}_Q)]R^3$, where $Q_{\rm max}$ is the largest charge NTS under consideration and in principle is infinity. Since the photon density is proportional to the entropy and scales as R^3 we can define the total charge asymmetry,

$$\eta=n_\gamma^{-1}\left[n_\phi-\bar{n}_\phi+\sum_{Q=Q_{\rm min}}^{Q_{\rm max}}Q(n_Q-\bar{n}_Q)\right]\ , \tag{6}$$

and remark that η is constant as long as entropy is conserved. The value of η is a free parameter and can range from zero (equal numbers of ϕ's and $\bar{\phi}$'s) to 10^{-9} (roughly the asymmetry of the baryons) to around $\frac{1}{2}$ (ϕ's as numerous as photons and very few $\bar{\phi}$'s).

For most of our work we will use a nonrelativistic expansion of $K_2(x)\approx e^{-x}(\pi/2x)^{1/2}$, although this is not necessary. Taking this limit, we find the number fractions $Y_i=n_i/\eta n_\gamma$ in statistical equilibrium to be

$$Y_Q^{\rm eq}=(c_1\eta)^{(Q-1)}\left[\frac{M_Q}{m_\phi}\right]^{3/2}\left[\frac{T}{m_\phi}\right]^{3(Q-1)/2}\times(Y_\phi^{\rm eq})^Q\exp(B_Q/T)\ ,$$
$$Y_\phi^{\rm eq}=(c_1\eta)^{-1}\left[\frac{m_\phi}{T}\right]^{3/2}\exp[(\mu-m_\phi)/T]\ , \tag{7}$$
$$\bar{Y}_\phi^{\rm eq}=Y_\phi^{\rm eq}\exp(-2\mu/T)\ ,$$

where $c_1=\zeta(3)\sqrt{32/\pi}\approx3.836$, and

$$B_Q=m_\phi Q-M_Q\approx\frac{4Q\lambda_1^{3/4}}{Q_{\rm min}^{1/4}}\left[1-\left[\frac{Q_{\rm min}}{Q}\right]^{1/4}\right]T_G \tag{8}$$

is the binding energy of an NTS of charge Q. The quantity $\bar{Y}_Q^{\rm eq}$ is the same as $Y_Q^{\rm eq}$ with Y_ϕ replaced by $\bar{Y}_\phi$. The charge-conservation condition becomes

$$Y_\phi^{\rm eq}-\bar{Y}_\phi^{\rm eq}+\sum_{Q_{\rm min}}^{Q_{\rm max}}Q(Y_Q^{\rm eq}-\bar{Y}_Q^{\rm eq})=1\ , \tag{9}$$

which, after specifying η and T, can be solved for μ and then all number densities can be found.

In Fig. 1 we display the equilibrium number fractions of ϕ's, $\bar{\phi}$'s, and NTS's for a system containing NTS's from $Q_{\rm min}=4$ to $Q_{\rm max}=5$ and their anti-NTS's. We set $\lambda_1=1,\eta=0.01$, and the temperature is divided by T_G so that σ_0 scales out. Figure 2(a) shows the total fraction in NTS's for the same system but several values of η. Fig-

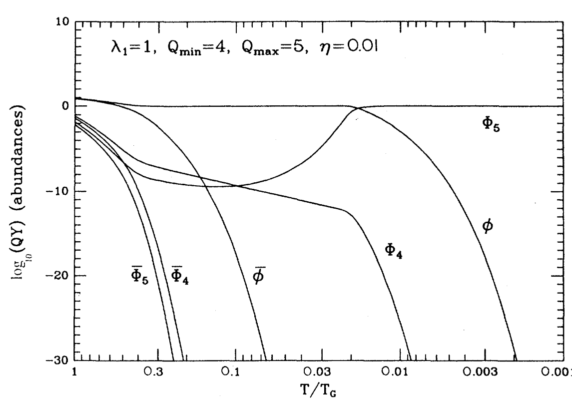

FIG. 1. Abundances as a function of temperature in NTS statistical equilibrium for a system consisting of $\phi,\bar{\phi},\Phi_4,\Phi_5,\bar{\Phi}_4,\bar{\Phi}_5$ ($Q_{\rm max}=5$). Parameter values $Q_{\rm min}=4$, $\lambda_1=1$, and $\eta=0.01$ were chosen. Temperatures are divided by T_G so the σ_0 dependence scales out. Since charge is conserved the algebraic sum of all the Y's (which are actually QY's) is unity.

ure 2(b) shows the effect of varying the size of the system, parametrized by $Q_{\rm max}$, and Fig. 2(c) shows the NTSSE values of μ for the same parameters values as Fig. 2(a). These plots show similarities with the corresponding plots of light element abundances during nucleosynthesis. At a given temperature, typically one isotope (charge) is favored, and as the temperature drops, species with different binding energies take over. Figures 1 and 2 can be understood qualitatively from the charge-conservation equation and from Eq. (7). Equation (9) can be approximated as $Y_\phi^{\rm eq}[1-e^{-2\mu/T}+(\eta T Y_\phi^{\rm eq})^{Q-1}e^{B/T}]=1$, where the first term is the contribution from $Y_\phi^{\rm eq}$, the second from $\bar{Y}_\phi^{\rm eq}$, and the third is the order-of-magnitude contribution from $Y_{\rm NTS}$.

Consider a relatively small value of η. Then at $T=T_G$, μ [Fig. 2(c)] is small, $Y_\phi^{\rm eq}\approx\bar{Y}_\phi^{\rm eq}\gg 1$ (just about cancelling each other) and $Y_{\rm NTS}$ is smaller than either. As the temperature drops, $Y_\phi^{\rm eq}\propto\exp[(\mu-m_\phi)/T]$ drops exponentially ($\mu\ll m_\phi$), as does $\bar{Y}_\phi^{\rm eq}\approx Y_\phi^{\rm eq}$. The $\exp(B/T)$ term increases, but the combination of the T^{Q-1} and $(Y_\phi^{\rm eq})^{Q-1}$ dominate at first and so $Y_{\rm NTS}$ drops also. As the temperature continues to drop, μ approaches m_ϕ, and $Y_\phi^{\rm eq}$ approaches 1, after which time $Y_\phi^{\rm eq}$ levels out and $\bar{Y}_\phi^{\rm eq}$ drops exponentially. (At this point $Y_{\rm NTS}$ is insignificant, so $Y_\phi^{\rm eq}$ cannot drop below unity and still satisfy the charge equation.) Since $Y_\phi^{\rm eq}$ is no longer dropping, the exponential $\exp(B/T)$ eventually overpowers the T^{Q-1} factor and $Y_{\rm NTS}$ begins to increase exponentially (at this point $\bar{Y}_Q^{\rm eq}$ is insignificant) still having little effect until it approaches unity. When $Y_{\rm NTS}$ gets close to unity, $(Y_\phi^{\rm eq})^{Q-1}$ must drop to balance the $\exp(B/T)$ factor, and so $Y_\phi^{\rm eq}$ drops away and we are left with $Y_{\rm NTS}\approx 1$ and negligible amounts of anything else. For larger asymmetry the same evolution occurs, except that the drop in $Y_{\rm NTS}$ is not as deep and the final rise in $Y_{\rm NTS}$ occurs sooner.

The temperature at which $Y_{\rm NTS}$ starts to dominate (defined as T_D in the Introduction) is important in understanding the evolution of the system, and can be fairly well approximated analytically. Since the binding energy of the highest charge NTS is most important, in a system with highest allowed charge $Q_{\rm max}$, the charge-conservation equation can be approximated as $Y_\phi^{\rm eq}-\bar{Y}_\phi^{\rm eq}+Y_{\rm NTS}=1$, where $Y_{\rm NTS}\sim Q_{\rm max}^2 Y_{Q_{\rm max}}$. Defining T_D as the temperature when $Y_{\rm NTS}=Y_\phi^{\rm eq}=\frac{1}{2}$ and noting that $\bar{Y}_\phi^{\rm eq}$ (and therefore $\bar{Y}_Q^{\rm eq}$) is typically very small at T_D, we can solve for the temperature:

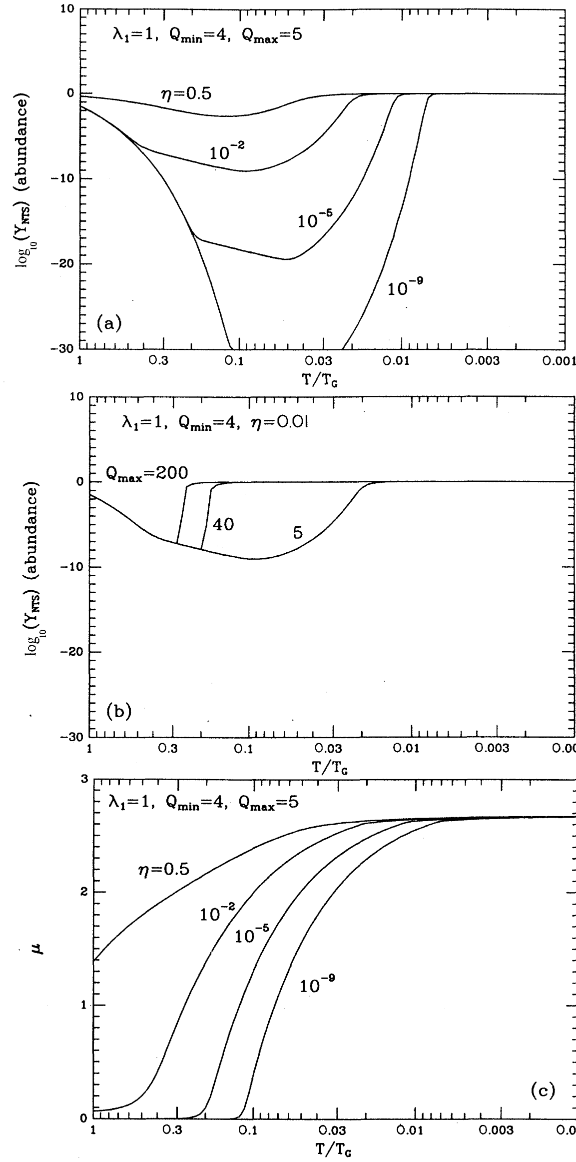

FIG. 2. Abundances of nontopological solitons in statistical equilibrium for a system with $Q_{\rm min}=4$ and $\lambda_1=1$. In (a), the total abundance of NTS's ($Y_{\rm NTS}$) is shown for $Q_{\rm max}=5$ and several values of the asymmetry ($\eta=0.5,10^{-2},10^{-5},10^{-9}$). In (b), $Y_{\rm NTS}$ is shown for $\eta=10^{-2}$ and several values of the system size ($Q_{\rm max}=5,40,200$). In (c), the value of the chemical potential μ is shown for $Q_{\rm max}=5$ and several values of η.

$$\frac{T_D(Q_{\rm max})}{T_G}=\frac{B_Q/T_G}{(Q_{\rm max}-1)\ln(4\lambda_1^{9/8}T_G^{3/2}/\eta T_D^{3/2})-\ln[(Q_{\rm min})^{3Q_{\rm max}/8}(Q_{\rm max})^{25/8}]}\ . \tag{10}$$

For the case we considered numerically, $\lambda_1=1$, $Q_{\rm min}=4$, and $Q_{\rm max}=5$, this becomes $T_D/T_G=-0.767/[2.08+4\ln\eta+6\ln(T_D/T_G)]$, which is in good agreement with the numerical results of Fig. 2. A plot of our approximate values of T_D for various values of η is shown in Fig. 3.

If we are interested in an infinite system, we note that, as $Q_{\rm max}\to\infty$,

$$\frac{T_D(\infty)}{T_G}=\frac{4\lambda_1^{3/4}Q_{\rm min}^{-1/4}}{\ln[4\lambda_1^{9/8}T_G^{3/2}/(\eta T_D^{3/2}Q_{\rm min}^{3/8})]}\ . \tag{11}$$

For the values of parameters used in Fig. 2 this gives

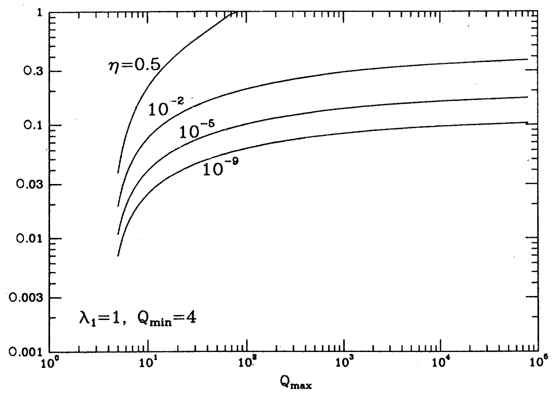

FIG. 3. The "dominance" temperature T_D after which NTSSE drives $Y_{\rm NTS} > \frac{1}{2}$, as a function of the effective size of the system $Q_{\rm max}$, and for several values of the asymmetry ($\eta = 0.5, 10^{-2}, 10^{-5}, 10^{-9}$).

$T_D/T_G \approx 0.4$, roughly a factor of 20 higher than the $Q_{\rm max}=5$ case and a factor of 2 higher than the $Q_{\rm max}=40$ case. In general $T_D(Q+1) > T_D(Q)$ for all Q, implying that the highest charge NTS allowed wants to take up all the free ϕ's. This contrasts with nuclear statistical equilibrium where, for example, ^{4}He dominates before ^{12}C. The difference lies in the slightly different pattern of binding energies in the nuclear and soliton cases.

In conclusion, we see that if statistical equilibrium is maintained until T_D we expect all ϕ's to be absorbed into NTS's, and in particular into the largest NTS possible. If NTSSE continued, eventually all ϕ's inside the horizon would be contained in one large NTS. In reality, the reactions which maintain NTSSE freeze-out, and the actual distribution of NTS's depends upon how many "reaction times" exist between T_D and T_F, and whether in fact, $T_D > T_F$ at all. We now turn to consider departure from statistical equilibrium and freeze-out.

IV. NTS NETWORK AND FREEZE-OUT

The statistical equilibrium described in the previous section is maintained at the microphysical level by the annihilation, creation, evaporation, and fusion of NTS's and ϕ particles. So long as these reactions proceed much faster than the expansion rate of the Universe, equilibrium can be maintained. The reaction rates are determined by cross sections and number densities, and in trying to decide whether or not equilibrium exists at a given temperature, one must specify these. If we consider only positive charge NTS's, a set of possible reactions is

$$\phi + \bar{\phi} \leftrightarrow \sigma + \sigma \ , \qquad (12a)$$

$$\Phi_{Q_{\rm min}} + \bar{\phi} \leftrightarrow (Q_{\rm min}-1)\phi + X \ , \qquad (12b)$$

$$\Phi_{Q_{\rm min}} + \phi \leftrightarrow \Phi_{Q_{\rm min}+1} + X \ , \qquad (12c)$$

$$\Phi_{Q_{\rm min}+1} + \bar{\phi} \leftrightarrow \Phi_{Q_{\rm min}} + X \ , \qquad (12d)$$

etc.,

where $\Phi_{Q_{\rm min}}$, etc., indicates a NTS with charge $Q_{\rm min}$, and X stands for anything else (e.g., $\gamma, e^+e^-, q\bar{q}$). This network would actually extend to $\Phi(Q_{\rm min}+\infty)$, but in any numerical integration we must truncate it. We find that many essential features are present in even this radically truncated system. In addition to the processes of Eq. (12), there will be processes such as $\Phi_{Q_{\rm min}} + \Phi_{Q_{\rm min}} \leftrightarrow \Phi(2Q_{\rm min}) + X$, which we will leave out since they are beyond the maximum charge of our truncated systems. Processes of this type would become important if equilibrium existed long after T_D. One might also question whether the reverse of reactions such as Eq. (12b) can take place and we will discuss later the effect of leaving out this reaction.

In order to follow the evolution of NTS and free ϕ number densities one translates the processes in Eq. (12) into a set of coupled Boltzmann equations. For example, the number density of free ϕ particles evolves according to

$$\frac{dn_\phi}{dt} = -3Hn_\phi + (-r_a + \bar{r}_a) + (Q_{\rm min}-1)(r_b - \bar{r}_b) - (r_c - \bar{r}_c) \ , \qquad (13)$$

where the $-3Hn_\phi$ is the reduction of n_ϕ due to the expansion of the Universe, the r_a, r_b, r_c are the forward reaction rates of Eqs. (12a), (12b), (12c) and the $\bar{r}_a, \bar{r}_b, \bar{r}_c$ are the reverse reaction rates. A similar equation can be written for $\bar{n}_\phi, n_Q, n_{Q_{\rm min}+1}$, etc., and these are displayed in Appendix A.

For a two-body process such as Eq. (12a), $r_a = \langle \sigma_a |v| \rangle n_\phi \bar{n}_\phi$, where $\langle \sigma_a |v| \rangle$ is the thermally averaged cross section. Likewise, $r_b = \langle \sigma_b |v| \rangle \bar{n}_\phi n_{Q_{\rm min}}$, $r_c = \langle \sigma_c |v| \rangle n_{Q_{\rm min}} n_\phi$, and $r_d = \langle \sigma_d |v| \rangle n_{Q_{\rm min}+1} \bar{n}_\phi$. For many-body reactions, the rate is also a matrix element averaged over phase space and we can still write $\bar{r}_b = \langle \bar{\sigma}_b \rangle (n_\phi)^{Q_{\rm min}-1} n_X$, where $\bar{\sigma}_b$ is no longer, however, a cross section.

Equations such as (13) can be simplified by noting that in equilibrium (in a nonexpanding Universe), $r_i^{\rm eq} - \bar{r}_i^{\rm eq} = 0$, that is, in equilibrium the forward and backward reactions cancel.[5] Using this detailed balance we have relations such as $\langle \sigma_c |v| \rangle n_{Q_{\rm min}}^{\rm eq} n_\phi^{\rm eq} = \langle \bar{\sigma}_c |v| \rangle n_{Q_{\rm min}+1}^{\rm eq} n_X^{\rm eq}$, where the "eq" superscript indicates the NTSSE abundance. This allows us to replace $\langle \bar{\sigma}_c |v| \rangle$ in Eq. (13), and pairs such as $r_c - \bar{r}_c$ can be written

$$r_c - \bar{r}_c = \langle \sigma_c |v| \rangle \left[n_{Q_{\rm min}} n_\phi - \frac{n_{Q_{\rm min}}^{\rm eq} n_\phi^{\rm eq} n_{Q_{\rm min}+1}}{n_{Q_{\rm min}+1}^{\rm eq}} \right] , \qquad (14)$$

where we have assumed that the X remains in equilibrium throughout the time of interest. We can simplify Eq. (13) further by changing variables[6] from n_i to $Y_i = n_i/(n_\gamma \eta)$ and from time to $x = T/T_G$. Then using detailed balance on all reactions, Eq. (13) becomes

$$\frac{dY_\phi}{dx}=c\eta\{-\langle\sigma_a|v|\rangle(Y_\phi\bar{Y}_\phi-Y_\phi^{\rm eq}\bar{Y}_\phi^{\rm eq})+\langle\sigma_b|v|\rangle(Q_{\rm min}-1)\times[\bar{Y}_\phi Y_Q-\bar{Y}_\phi^{\rm eq}Y_Q^{\rm eq}Y_\phi^{Q_{\rm min}-1}/(Y_\phi^{\rm eq})^{Q_{\rm min}-1}]-\langle\sigma_c|v|\rangle(Y_\phi Y_Q-Y_\phi^{\rm eq}Y_Q^{\rm eq}Y_{Q+1}/Y_{Q+1}^{\rm eq})\}\ , \tag{15}$$

where

$$c=-\frac{3\sqrt{5}\zeta(3)T_G m_{\rm Pl}}{\pi^{7/2}\sqrt{g_*}}\approx-0.014\,67T_G m_{\rm Pl}\ , \tag{16}$$

$m_{\rm Pl}$ is the Planck mass, and we took the number of degrees of freedom, $g_*\approx 100$. There are similar equations for $\bar{Y}_\phi, Y_Q$, etc., and these are displayed in Appendix A.

Next we need to consider the cross sections appearing in the equations. For the $\phi+\bar{\phi}\to\sigma+\sigma$ process we can use the interaction contained in the Lagrangian, Eq. (1):

$$\mathcal{L}_{\rm eff}\sim 1.06\sigma_0^2\lambda_1 x^2+h|\phi|^2x^2\ , \tag{17}$$

where x is the σ field shifted to the true minimum of the potential. Using this we find,[7] at the tree level,

$$\langle\sigma_a|v|\rangle=\frac{0.014\lambda_1^{1/2}}{Q_{\rm min}^{1/2}\sigma_0^2}\left[1-\frac{\lambda_1^{1/2}Q_{\rm min}^{1/2}}{12.5}\right]^{1/2}\ . \tag{18}$$

For processes such as $\phi+\Phi_{Q_{\rm min}}\to\Phi_{Q_{\rm min}+1}$ we are at a loss to calculate the cross section, so as a rough approximation, we will set $\langle\sigma|v|\rangle\sim\pi R_Q^2$, where R_Q is the radius of the NTS taking place in the process. Thus we have $\langle\sigma_b|v|\rangle\sim\langle\sigma_c|v|\rangle\sim 2Q_{\rm min}^{1/2}/(\lambda_1^{1/2}\sigma_0^2)$ and $\langle\sigma_d|v|\rangle\sim 2\sqrt{Q_{\rm min}+1}/(\lambda_1^{1/2}\sigma_0^2)$. Note that all the cross sections scale like σ_0^{-2}.

The last step before integrating the coupled set of ordinary differential equations (ODE's) is to specify the initial conditions. We will start at T_G in NTSSE, although it would probably be more correct to start with the distribution of $Y_{\rm NTS}$ [Eq. (3)] derived from the phase transition. However, the equations are extremely stiff and we either have a period of statistical equilibrium after T_G, in which case $Y_{\rm NTS}$ immediately evolves to $Y_{\rm NTS}^{\rm eq}$, or we do not have a period of equilibrium, in which case the reactions do nothing and we know that we are left with the initial $Y_{\rm NTS}$ produced in the phase transition.

Figure 4 shows an example of an integration of a network consisting of ϕ's, $\bar{\phi}$'s, and NTS's of charge $Q_{\rm min}=4$ through $Q_{\rm max}=5$. Values of $\lambda_1=1$, $\eta=10^{-2}$, and $\sigma_0=10^3$ TeV [Fig. 4(a)] or $\sigma_0=7\times10^7$ TeV (Fig. 4(b)], were used. Note in Fig. 4(a) that all abundances trace their NTSSE values (shown in Fig. 1) until around $T/T_G\sim0.1$, when $Y_{\rm NTS}$ (the sum of the charge four and five NTS abundances) "freezes out" and becomes constant. Y_ϕ and $\bar{Y}_\phi$ follow NTSSE for a good while longer in Fig. 4(a), but with the smaller cross section ($\propto\sigma_0^{-2}$) of Fig. 4(b) they too freeze out and become constant by $T/T_G\sim0.02$. This is the generic picture. Abundances follow their NTSSE values until freeze-out, which is determined by the cross sections, after which they become constant.

Since we are potentially interested in systems with large $Q_{\rm max}$, and since we cannot numerically integrate such systems,[8] we would like an analytic approximation for T_F just as we have for T_D. Toward this end we first note that ϕ and $\bar{\phi}$ typically stay in equilibrium longer than NTS's so that during freeze-out of the NTS's we can approximate Y_ϕ and $\bar{Y}_\phi$ by their NTSSE values. The key reaction is then Eq. (12b), which is our source of thermal NTS's, and the equivalent of Eq. (15) for Y_Q can be approximated as

$$\frac{dY_Q}{dx}\approx-c\eta\langle\sigma_b|v|\rangle\bar{Y}_\phi^{\rm eq}(Y_Q-Y_Q^{\rm eq})\ . \tag{19}$$

Defining $\Delta=Y_Q-Y_Q^{\rm eq}$, Eq. (19) can be rewritten as $d\Delta/dx=-c\eta\langle\sigma_b|v|\rangle\bar{Y}_\phi^{\rm eq}\Delta-dY_Q^{\rm eq}/dx$, and as long as the departure from equilibrium is small we can set $d\Delta/dx\approx0$ and solve for Δ:

$$\Delta\approx-\frac{dY_Q^{\rm eq}/dx}{c\eta\langle\sigma_c|v|\rangle\bar{Y}_\phi^{\rm eq}}\ . \tag{20}$$

Defining freeze-out as $\Delta=1.5Y_Q^{\rm eq}$ (see Ref. 6) and using Eq. (7) for $Y_Q^{\rm eq}$ and $\bar{Y}_\phi^{\rm eq}$, we find

$$\frac{T_F}{T_G}=\frac{(\mu/T_G)+4(\lambda_1^{3/4}/Q_{\rm min}^{1/4})}{\ln[(0.12\lambda_1^{1/8}Q_{\rm min}^{1/8}x_F^{1/2}m_{\rm Pl}/\sigma_0)/(-\frac{3}{2}x_F-Q\mu/T_G+4\lambda_1^{3/4}Q/Q_{\rm min}^{1/4}-B_Q/T_G)]}\ , \tag{21}$$

where $x_F=T_F/T_G$. For $\lambda_1=1$ and $Q=Q_{\rm min}=4$, this becomes

$$\frac{T_F}{T_G}\approx\frac{(\mu/T_G)+2.8}{\ln[(0.14x_F^{1/2}m_{\rm Pl}/\sigma_0)/(-\frac{3}{2}x_F-4\mu/T_G+11.3)]}\ , \tag{22}$$

where μ must be found from the NTSSE calculation. A plot of μ for several[9] values of η is given in Fig. 2(c).

The approximation, Eq. (21), agrees quite well with the results of the numerical integration (typically within 10–20 %) for small η, but becomes worse as η becomes large ($\eta\gtrsim0.1$). For example, with large η, $\lambda_1=1$, and $Q_{\rm min}=4$ the network gives $T_F/T_G\gtrsim0.3$ almost independent of the value of σ_0, while the approximation predicts smaller values which vary with σ_0. A plot of our approximate T_F for several values of σ_0 and η is given in Fig. 5.

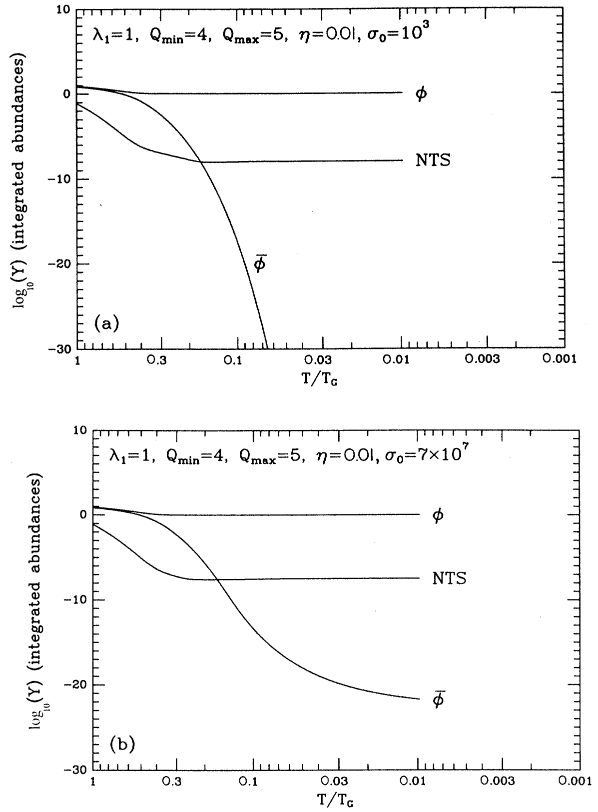

FIG. 4. Abundances of NTS's found by integrating a truncated network of reactions in an expanding universe (to be compared with the equilibrium values displayed in Fig. 1). Parameter values $Q_{\rm min}=4$, $Q_{\rm max}=5$, $\lambda_1=1$, and $\eta=10^{-2}$ were chosen. Note that anti-NTS's are not included. Two values of σ_0 [10^3 TeV in (a) and 7×10^7 TeV in (b)] are shown. The larger cross section ($\propto \sigma_0^{-2}$) in (a) allows equilibrium to be maintained to lower temperature, while in (b) freeze-out is seen for Y_ϕ, $\bar{Y}_\phi$, and $Y_{\rm NTS}$ (sum of charge four and five NTS abundances).

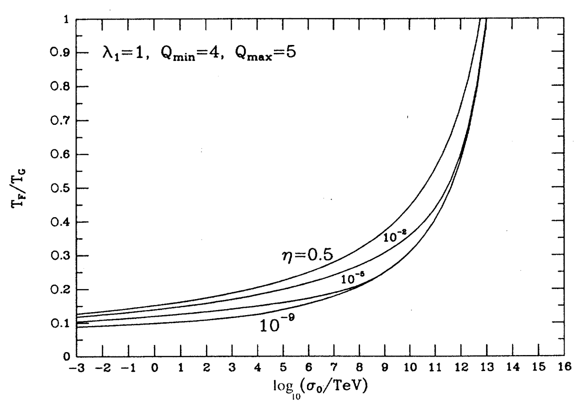

FIG. 5. The "freeze-out" temperature T_F defined in the text, as a function of σ_0 for several values of asymmetry ($\eta=0.5, 10^{-2}, 10^{-5}, 10^{-9}$). Parameter values $Q_{\rm min}=4$ and $\lambda_1=1$ were chosen and T_F is nearly independent of the size of the system, $Q_{\rm max}$.

V. RESULTS

We are now in position to try to decide quantitatively the relic abundance of NTS's as a function of our free parameters η, σ_0, λ_1, and $Q_{\rm min}$. We have discovered that there are really just three different possibilities, depending upon the relative order of three important events: the end of the phase transition (T_G), the dominance of the NTS's (T_D), and NTS freeze-out (T_F).

If $T_F > T_G$ then the NTS's are born frozen out, that is they never reach NTSSE. NTS's are neither thermally created, nor are they destroyed, and one is left today with roughly the same spectra of NTS's as was created by the phase transition. This was the case tacitly considered by FGGK, and so one expects to have $\Omega_{\rm NTS} \ll \Omega_\phi$, and with their assumptions, it is, therefore, difficult to have a significant density of NTS's extant today. In terms of the free parameters, we find that if σ_0 is greater than around 10^{13} TeV, then $T_F > T_G$, where σ_0 is roughly the mass of the free ϕ and also sets the scale for the ϕ+NTS cross section. Keep in mind that the Planck scale is 1.2×10^{16} TeV. For large asymmetry T_F/T_G is larger, but even the largest asymmetry we considered ($\eta=0.5$) does not allow $T_F > T_G$ for $\sigma_0 < 10^{12}$ TeV.

If $T_G > T_F$ then, at least for a while after the phase transition, we expect to have statistical equilibrium. In this case the number of NTS's created during the phase transition is irrelevant, the final density being determined purely by the temperature at which NTSSE ends, along with the asymmetry and NTS binding energiess. Typically the number fraction of NTS's is lower than Y_ϕ at T_G, drops exponentially for a while, levels off at some very low number fraction, and then rises exponentially until it reaches unity, after which time almost all free ϕ's disappear. The relation between the temperature of the final rise (T_D) and freeze-out (T_F) determines the NTS relic abundance. If the rise happens before freeze-out ($T_D > T_F$) then we have solitosynthesis. Large numbers of NTS's of various sizes will survive until today and the number of free ϕ particles will be small. If freeze-out happens first ($T_F > T_D$), the number fraction of NTS's will remain at the very low levels of the NTSSE dip, and we have soliton destruction and/or evaporation. In this case the number fraction of NTS's is insignificant and only free ϕ's and $\bar{\phi}$'s remain today.

The actual values of T_F and T_D depend upon the parameters of the model: σ_0, λ_1, $Q_{\rm min}$, and the asymmetry η. Consistency[10] forces λ_1 to be near 1 and $Q_{\rm min}$ to be between roughly 10^0 and 10^4. As mentioned, we allow σ_0 to vary up to $m_{\rm Pl}=1.2\times10^{16}$ TeV, while we consider η between 0 and $\frac{1}{2}$.

In general, as $Q_{\rm min}$ increases both T_F and T_D decrease. But, since T_D decreases faster, we find that as $Q_{\rm min}$ increases freeze-out occurs while $Y_{\rm NTS}$ is still low and solitodestruction is the result. Therefore, to find solitosynthesis we take $Q_{\rm min}$ as small as possible. Decreasing λ_1 also decreases both T_F and T_D, this time by roughly the same amount. Since there is very little range allowed in the choice of λ_1 we mainly consider only $\lambda_1=1$. The parameter σ_0 has no effect on T_D/T_G, but as it decreases,

T_F decreases, so for solitosynthesis a low value of σ_0 is desirable. (Cross sections are proportional to σ_0^{-2}, and a larger cross section will keep things in equilibrium longer.) Taking a very low value of σ_0 (and corresponding very large cross section) does not help much, however, because the dependence of T_F on σ_0 is logarithmic. For example, a cross section 10^{50} times larger than πR^2 would be needed to have $T_D > T_F$ for $Q_{\max}=40$ and $\eta = 10^{-5}$.

The value of the asymmetry is important. For values of η near the baryon asymmetry ($\eta \sim 10^{-9}$) we always have $T_F > T_D$ and cannot have a substantial relic abundance of NTS's. As η increases both T_F and T_D increase, but T_F increases to a smaller extent. Therefore, the best hope for solitosynthesis is when η is large.

Finally we must decide what value of $Q_{\max}$ to use. Since $T_D(Q+1) > T_D(Q)$, the larger the effective size of the system ($Q_{\max}$), the larger the area of parameter space in which solitosynthesis can occur. With the present analysis we have not been able definitively to decide the effective size of the system (we cannot numerically integrate a very large system), but we will discuss this question in some detail in Appendix B. In Fig. 6 we show the regions of parameter space for which each of the three scenarios, solitosynthesis, solitodestruction, and "born frozen out" takes place. We choose favorable (for solitosynthesis) parameter values of $Q_{\min}=4$ and $\lambda_1=1$ and plot the σ_0, η plane. The dotted-dashed line shows the boundary between solitodestruction and born frozen out, destruction occurring to the left of the line. The solid lines show the boundary between solitosynthesis and solitodestruction for two values of $Q_{\max}$ ($Q_{\max}=9$ and 44). Below these lines solitodestruction and/or evaporation takes place. As discussed in Appendix B, we favor a boundary with a low value of $Q_{\max}$ as it agrees with an estimate of the allowed region of solitosynthesis made using a different method. The dashed line shows the division of parameter space found by requiring enough time to generate an NTS of charge $Q_{\max}$ via one-body reactions. This is a necessary but not sufficient condition for solitosynthesis (see Appendix B). Taking the $Q_{\max}=9$ line as the boundary, we see that even for the favorable case of $Q_{\min}=4$, $\lambda_1=1$, and $\sigma_0=1$ TeV, we find $T_D > T_F$ only if $\eta > 0.1$. This is very near the degenerate limit and while such a value for η is not impossible, it is hard to imagine it arising in a natural way. Therefore, we conclude that for most all values of our parameters we will not have substantial numbers of NTS's extant today. Typically we either are left with the distribution created by the phase transition or we destroy even these, although there is a window of parameter space for which NTS's are naturally produced and could contribute, for example, $\Omega_{\rm NTS} \sim 1$.

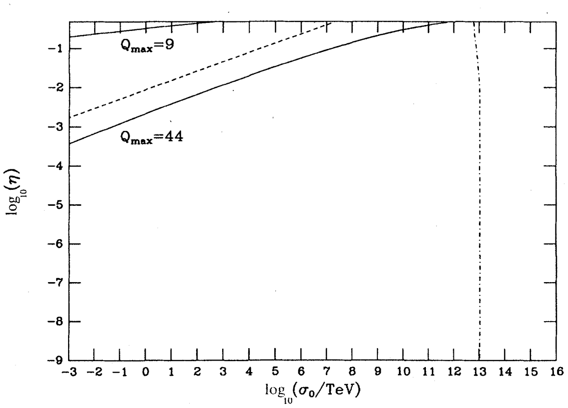

FIG. 6. Parameter space for solitosynthesis, the σ_0, η plane for $Q_{\min}=4$ and $\lambda_1=1$. To the right of the dotted-dashed line NTS's are "born frozen out" and equilibrium never obtains. Below the solid (and dashed) lines solitodestruction and/or evaporation occurs, that is, NTSSE erases knowledge of the phase transition, but freeze-out occurs when the abundance of NTS's is approximately zero. Above and to the left of the solid (and dashed) lines solitosynthesis occurs. Here, large numbers of NTS's are synthesized possibly leading to cosmically relevant abundances. Several methods of deciding the boundary between solitosynthesis and solitodestruction are displayed (see text and Appendix B) but the solid line labeled $Q_{\max}=9$ is probably the most relevant.

Now we briefly consider what happens if processes such as the reverse of Eq. (12b), $(Q_{\min}-1)\phi \to \Phi_{Q_{\min}} + \bar{\phi}$ are left out of the network. If NTS's can only be destroyed and not thermally created, the "equilibrium" state has $Y_{\rm NTS}=0$. Starting from an NTS distribution after a phase transition, or from actual statistical equilibrium one then finds the "evaporation rate" of NTS's by integrating such a network. The results of this integration are shown in Fig. 7. Comparing Fig. 7 with Fig. 4, we see that $Y_{\rm NTS}$ does drop very quickly to zero (and never rises again), while when thermal creation processes are allowed, the drop is more controlled and temporary. In this type of scenario, the only way to have any relic NTS's is to have $T_F > T_G$, that is, to have the NTS's born frozen out.

Finally, we mention that while the results presented above are in one sense very dependent upon the particu-

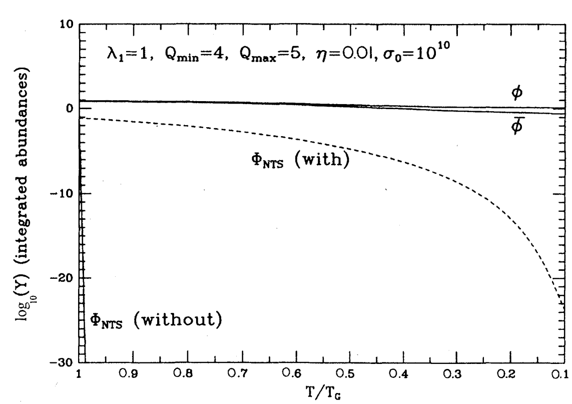

FIG. 7. Pure evaporation, the abundance of NTS's found by integrating the network when "creation" of NTS's is disallowed. The total abundance of NTS's drops quickly to zero. To be compared with the case when all reactions are allowed (dashed line and Fig. 4).

lar NTS model we used, in another sense they are quite model independent. The important temperature scale T_D depends primarily on the NTS binding energy and asymmetry; the freeze-out temperature depends on these as well as the cross sections. So while for different NTS models the precise regions of parameter space which give rise to solitosynthesis and solitodestruction will differ, we still expect the answers to be given by Eqs. (10) and (21), and to be qualitatively the same.

VI. CONCLUSIONS

In conclusion, we see that for the model of FGGK there are three generic outcomes, depending on the values of the parameters. If $T_F > T_G$, NTS's are born frozen out and $Y_{\rm NTS} \ll Y_\phi$ is determined by the phase transition. If $T_G > T_F$, a period of statistical equilibrium occurs, which erases all knowledge of NTS's formed during the phase transition. If $T_D > T_F$, which occurs only for fairly extreme values of the parameters, $Y_{\rm NTS} \sim 1$ and solitosynthesis gives rise to large, perhaps cosmically significant abundances of NTS's. If $T_F > T_D$, then all NTS's formed during the phase transition are destroyed, $Y_{\rm NTS} \ll Y_\phi$, and the relic abundance of NTS's is probably insignificant.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix we list the complete set of coupled Boltzmann equations used in running the solitosynthesis network. The symbols were all defined in Secs. III and IV:

$$\begin{aligned}
\frac{dY_\phi}{dx} &= c\eta\{-\sigma_c(Y_\phi Y_Q - Y_\phi^{\rm eq} Y_Q^{\rm eq} Y_{Q+1}/Y_{Q+1}^{\rm eq}) \\
&\quad +\sigma_b(Q-1) \\
&\qquad \times[\bar{Y}_\phi Y_Q - \bar{Y}_\phi^{\rm eq} Y_Q^{\rm eq}(Y_\phi)^{Q-1}/(Y_\phi^{\rm eq})^{Q-1}] \\
&\quad -\sigma_a(\bar{Y}_\phi Y_\phi - \bar{Y}_\phi^{\rm eq} Y_\phi^{\rm eq})\}\ , \\
\frac{d\bar{Y}_\phi}{dx} &= c\eta\{-\sigma_b[\bar{Y}_\phi Y_Q - \bar{Y}_\phi^{\rm eq} Y_Q^{\rm eq}(Y_\phi)^{Q-1}/(Y_\phi^{\rm eq})^{Q-1}] \\
&\quad -\sigma_a(Y_\phi \bar{Y}_\phi - Y_\phi^{\rm eq} \bar{Y}_\phi^{\rm eq}) \\
&\quad -\sigma_d(\bar{Y}_\phi Y_{Q+1} - \bar{Y}_\phi^{\rm eq} Y_{Q+1}^{\rm eq} Y_Q/Y_Q^{\rm eq})\}\ , \\
\frac{dY_Q}{dx} &= c\eta\{-\sigma_c(Y_\phi Y_Q - Y_\phi^{\rm eq} Y_Q^{\rm eq} Y_{Q+1}/Y_{Q+1}^{\rm eq}) \\
&\quad -\sigma_b[\bar{Y}_\phi Y_Q - \bar{Y}_\phi^{\rm eq} Y_Q^{\rm eq}(Y_\phi)^{Q-1}/(Y_\phi^{\rm eq})^{Q-1}] \\
&\quad +\sigma_d(\bar{Y}_\phi Y_{Q+1} - \bar{Y}_\phi^{\rm eq} Y_{Q+1}^{\rm eq} Y_Q/Y_Q^{\rm eq})\}\ , \\
\frac{dY_{Q+1}}{dx} &= c\eta[\sigma_c(Y_\phi Y_Q - Y_\phi^{\rm eq} Y_Q^{\rm eq} Y_{Q+1}/Y_{Q+1}^{\rm eq}) \\
&\quad -\sigma_d(\bar{Y}_\phi Y_{Q+1} - \bar{Y}_\phi^{\rm eq} Y_{Q+1}^{\rm eq} Y_Q/Y_Q^{\rm eq})]\ ,
\end{aligned} \tag{A1}$$

where σ_i stands for $\langle \sigma_i |v| \rangle$ and $Q = Q_{\rm min}$ throughout.

APPENDIX B

The discussion in the main body of the text which decided the boundary between solitosynthesis and solitodestruction and/or evaporation was very simplistic, but we feel probably adequate. In this appendix we discuss the caveats and our reservations in more detail.

In comparing T_D with T_F the question arises as to what size system (effective $Q_{\rm max}$) is relevant. It was seen that $T_D(Q+1) > T_D(Q)$, for all Q and therefore the larger $Q_{\rm max}$ the larger the region of parameter space which would allow solitosynthesis. This is basically a bottleneck question, and since it is exponentially sensitive, is difficult to answer with confidence. Consider a temperature between $T_D(Q_{\rm min}+1000)$ and $T_D(Q_{\rm min}+1)$. NTSSE would drive $Y_{\rm NTS}$ to unity in a system of size $Q_{\rm min}+1000$, but force $Y_{\rm NTS} \ll 1$ in a system of size $Q_{\rm min}+1$. Can the system generate the large number of NTS's needed to reach NTSSE when $Y_{Q_{\rm min}}^{\rm eq}$, $Y_{Q_{\rm min}+1}^{\rm eq}$, etc., are extremely small and when the $\Phi_{Q_{\rm max}}$'s are generated by a sequence of reactions such as $\Phi_{Q_{\rm min}} + \phi \to \Phi_{Q_{\rm min}+1}$?

One approximate way to answer this question is to consider the number of reactions such as $\Phi_{Q_{\rm min}} + \phi \to \Phi_{Q_{\rm min}+1}$ which could have taken place between T_G and $T_D(Q_{\rm max})$. This number N must be greater than or equal to $Q_{\rm max}$ for NTSSE to obtain. N is overestimated by $N \simeq \langle \sigma v \rangle n^{\rm eq}(Q_{\rm min}, T_D(Q_{\rm min}))t_D$, where t_D is the time which corresponds to T_D and $n^{\rm eq}(Q_{\rm min})$ is evaluated in a system of maximum charge $Q_{\rm min}$. (This overestimates N since $n_{Q_{\rm min}}^{\rm eq}$ would be smaller for a system with a larger maximum charge.) Setting $N = Q_{\rm max}$ and solving for $T_D(Q_{\rm max})$ one finds the effective size (maximum possible charge) of the NTSSE system as a function of the Lagrangian parameters. $Q_{\rm max} = N > Q_{\rm min}$ is a necessary (but not sufficient) condition for solitosynthesis. In Fig. 6 we show the line $Q_{\rm max} = Q_{\rm min}$ (for $Q_{\rm min} = 4$) in the η, σ_0 plane (dashed line). Note that it falls between the $Q_{\rm max} = 9$ and 44 solid lines found previously and argues for a small effective $Q_{\rm max}$.

This might have been anticipated. At temperatures where $Y_{\rm NTS} \ll 1$, high-Q NTS's are very much suppressed compared to NTS's of charge $Q_{\rm min}$, so one has an effective maximum charge of $Q_{\rm min}$ (or $Q_{\rm min}+\epsilon$, where ϵ is small). If the system freezes out before $T_D(Q_{\rm max} = Q_{\rm min}+\epsilon)$, then there was never a time when large numbers of NTS's of charge $Q_{\rm min}$, $Q_{\rm min}+1$, etc., existed, and so no way to generate large numbers of $Q_{\rm max}$ charge NTS's even if NTSSE would have liked it.

Another oversimplification of the discussion in the main body of the text was the use of T_F, the freeze-out temperature of reaction Eq. (12b), $(Q_{\rm min}-1)\phi \leftrightarrow \Phi_{Q_{\rm min}} + \bar{\phi}$. If any of the reactions, $\Phi_{Q_{\rm min}+Q-1} + \phi \to \Phi_{Q_{\rm min}+Q}$, freeze out then the system will stop building. Defining a freeze-out temperature T_F^Q (for example, for the reaction above) and using the same method as in Sec. IV we find

$$\frac{T_F^Q}{T_G}=\frac{(B_Q-B_{Q-1})/T_G}{\ln\{[0.057c_Q(Q-1)^{1/2}(x_F^Q)^{1/2}m_{\rm Pl}/\sigma_0]/[-\frac{3}{2}x_F^Q+Q(-\mu/T_G+4\lambda_1^{3/4}Q_{\rm min}^{-1/4})-B_Q/T_G]\}}\ ,$$

where $x_F^Q=T_F^Q/T_G$ and

$$c_Q=\frac{0.48Q_{\rm min}^{3/8}}{\lambda_1^{1/8}}\left[\frac{Q}{Q-1}\right]^{9/8}.$$

We find that $T_F^Q>T_F^{Q+1}$ for all Q, so the higher charge reactions freeze out first and are, therefore, the only ones which need be considered. However, in comparing T_F^Q with the previously defined T_F, we find T_F slightly larger or equal to T_F^Q for moderate values of Q. Since $T_F^{Q_{\rm min}}$, $T_F^{Q_{\rm min}+1}$, etc., freeze out later, the use of T_F, as was done in the body of the text, seems adequate. Other limitations of our analysis include neglect of the NTS surface energy (clearly not well founded with the small $Q_{\rm min}$'s considered here) and the use of the zero-temperature form of the potential.

Finally, a potentially serious flaw in our analysis is that the reactions of Eq. (12) are perhaps not the relevant ones. One might expect many-body reactions such as $Q_{\rm min}\phi\to\Phi_{Q_{\rm min}}+X$ to exist, as well as $\Phi_{Q_{\rm min}}+\phi+\phi\to\Phi_{Q_{\rm min}+2}$, etc. If these go at appreciable rates large NTS's could be built much faster and a larger region of parameter space might allow solitosynthesis.

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[2]J. A. Frieman, G. B. Gelmini, M. Gleiser, and E. W. Kolb, Phys. Rev. Lett. **60**, 2101 (1988).

[3]K. Griest, E. W. Kolb, and A. Massarotti, this issue, Phys. Rev. D **40**, 3529 (1989).

[4]K. Griest and P. Salati (in preparation).

[5]Actually in the full set of i equations one can only set $dn_i/dt=0$ (leaving out expansion) and solve the resulting set of algebraic equations. One can show that certain of the reactions obey detailed balance; for example, in the set given, $r_b^{\rm eq}=\bar{r}_b^{\rm eq}$. In what follows we will assume that all reactions individually obey detailed balance, while actually only the set as a whole must.

[6]K. Griest and D. Seckel, Nucl. Phys. **B283**, 681 (1987); **B296**, 1034(E) (1988).

[7]For this process to be kinematically allowed we need $m_{\sigma}<m_{\phi}$, which implies $Q_{\rm min}<150/\lambda_1$.

[8]In fact even this radically truncated system requires a sophisticated ODE integrator. Gear's method, with a tiny time step, and hand calculation of the Jacobian is necessary. The equations are extremely stiff, and abundances are differences between nearly equal large numbers, and a matrix involving these differences must be inverted at each time step.

[9]For parameter values not shown in Fig. 2(c), a reasonable approximation for the chemical potential is $\mu=\min(\mu_{\rm asy},\mu_1)$, where $\mu_{\rm asy}/T_G=4\lambda_1^{3/4}/Q_{\rm max}^{1/4}$ and

$$\mu_1/T_G=(T/T_G)\,\text{arcsinh}\{0.24\eta Q_{\rm min}^{3/8}\lambda_1^{-9/8}(T/T_G)^{3/2}\times\exp[4\lambda_1^{3/4}Q_{\rm min}^{-3/4}(T/T_G)]\}\ .$$

[10]In order for perturbation theory to be valid (whatever that means in a theory such as this) we expect $\lambda_1\le 4$ (roughly) and $h\le 1$. Following FGGK and requiring that $m_\phi\ge T_G$ so that ϕ's can be trapped at T_G, we find $0.16Q_{\rm min}^{1/3}\le\lambda_1\le 0.056Q_{\rm min}$. This constraint cannot be satisfied unless $Q_{\rm min}\ge 4$ and in this case $\lambda_1=0.27$. Using this with $\lambda_1\le 4$ implies $4\le Q_{\rm min}\le 15\,000$ and $0.27\le\lambda_1\le 4$. There is some uncertainty in these bounds since the value of T_G and input constraint inequalities are only approximate. Throughout we used $\lambda_2=0.15\lambda_1$, but there is not much latitude here either since requiring that the false-vacuum bubbles do not grow implies that λ_2 is not too small.