Profile function of the chiral quantum baryon

M. Iwasaki and H. Ohyama
Department of Physics, Kochi University, Kochi 780, Japan
(Received 22 May 1989)

We investigate the Skyrme soliton in a simple chiral model of pions without the Skyrme term which is stabilized by quantum fluctuations. It is proved that the profile function $F(r)$ with the usual boundary conditions $[F(0)=n\pi$ and $F(\infty)=0]$ in the hedgehog ansatz does not exist. It is also shown that the solution with the asymptotic form r^{-2} has a singularity at the origin.

In a recent paper,[1] Jain, Schechter, and Sorkin have pointed out that a putative Skyrme soliton in a simple chiral model of pions without the Skyrme term can be stabilized against collapse by quantum fluctuations. Mignaco and Wulck[2] have also shown the same fact in a more detailed treatment of the profile function $F(r)$ which appears in the Skyrme ansatz.

They start with the nonlinear σ-model Lagrangian[3]

$$L=-\frac{f_\pi^2}{4}\int \mathrm{Tr}(\partial_\mu U\partial_\mu U^\dagger)d^3x \ , \tag{1}$$

where $f_\pi=132$ MeV is the pion decay constant and U is a 2×2 unitary unimodular matrix describing the pion fields. The classical equation of motion derived from the Lagrangian (1) can be solved by the hedgehog ansatz:[3]

$$U_0=\exp[i\tau\cdot\hat{\mathbf{r}}F(r)] \ , \tag{2}$$

where τ_k represent the usual Pauli matrices for SU(2) and $\hat{\mathbf{r}}\equiv\mathbf{r}/r$ $(r\equiv|\mathbf{r}|)$. The profile function $F(r)$ must satisfy the equation

$$\frac{d^2F}{dr^2}+\frac{2}{r}\frac{dF}{dr}=\frac{1}{r^2}\sin(2F) \ . \tag{3}$$

It should be noted that this equation has scale invariance. If $F(r)$ is a solution of (3) with boundary conditions[3]

$$F(0)=n\pi \ , \tag{4}$$

$$F(r)\to 0 \quad (r\to\infty) \ , \tag{5}$$

then $F(cr)$ (c is an arbitrary constant) is also another solution of (3) with the same boundary conditions. This fact led to collapse of the classical soliton $F(r)$ by Derrick's theorem[4] and the necessity of the introduction of the Skyrme term. However it was the main conclusion of the above-mentioned authors that if we quantize the scale parameter $c(t)$ as a collective variable, the Skyrme soliton without the Skyrme term can be stabilized against the collapse by virtue of the collective kinetic energy.

The purpose of this paper is to investigate the solutions of (3) in detail and in particular to prove that the solution with the boundary conditions (4) and (5) does not exist mathematically.

In order to do this we assume only the boundary condition (4) at the origin. Let us expand $F(r)$ around $r=0$:

$$F(r)=n\pi+c_1r+c_2r^2+\cdots \ . \tag{6}$$

Substituting this equation into (3) we obtain the solution

$$c_3=-\tfrac{2}{15}c_1^3, \quad c_5=\tfrac{1}{35}c_1^5,\ldots \ , \tag{7a}$$

$$c_2=c_4=c_6=\cdots=0 \ , \tag{7b}$$

where c_1 is not determined and becomes an arbitrary constant (scale invariance). We have carried out numerical calculations of $F(r)$ with the initial conditions (7). The asymptotic behavior of $F(r)$ seems to be

$$F(r\to\infty)\simeq\begin{cases} n\pi+\pi/2 & (c_1>0) \ , \\ n\pi & (c_1=0) \ , \\ n\pi-\pi/2 & (c_1<0) \ . \end{cases} \tag{8}$$

This conjecture apparently means that the solution with both boundary conditions (4) and (5) does *not* exist except for the trivial solution $F(r)=0$. An example with $n=1$ and $c_1=-1$ is shown by the solid line in Fig. 1.

Now let us consider the same problem analytically. If we introduce a new variable $y\equiv 1/r$, Eq. (3) can be transformed into

$$\frac{d^2F}{dy^2}=\frac{1}{y^2}\sin(2F) \ . \tag{9}$$

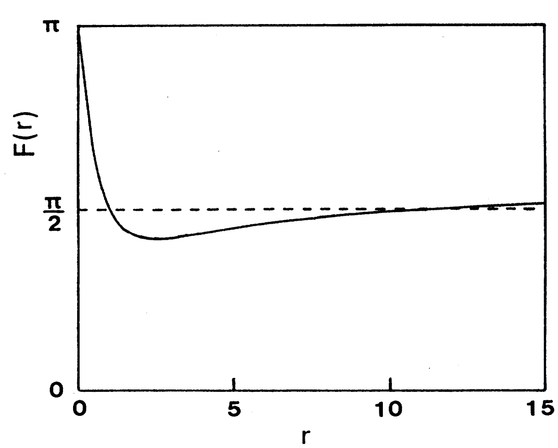

FIG. 1. The numerical solution of (3) with the boundary condition (4): $n=1$ and $c_1=-1$.

We obtain two kinds of solutions of Eq. (9) around $y=0$ [it is called a *regular singular point* if $\sin(2F)\simeq 2F$] as

$$\text{(a)}\quad F(y)=\frac{m\pi}{2}+cy^2+\cdots \quad (m=\text{even})\ , \tag{10a}$$

$$\text{(b)}\quad F(y)=\frac{m\pi}{2}+\sqrt{cy}\,\cos\left[\frac{\sqrt{7}}{2}\ln(cy)+\alpha\right]+\cdots \quad (m=\text{odd})\ , \tag{10b}$$

where m is an even [odd] integer in (a) [(b)] and c is an arbitrary constant. The α in (10b) is a constant to be determined adequately. Usually the solution (a) is preferred but the other solution (b) cannot be excluded mathematically.

Which does the solution with the boundary condition (4) approach—(a) or (b)? Multiplying (9) by $y^2F'(y)$ and integrating with respect to y from y to ∞, we get

$$\int_y^\infty yF'^2dy=\left[\frac{y^2}{2}F'^2\right]_y^\infty+\left[\frac{\cos(2F)}{2}\right]_y^\infty \tag{11}$$

$[F'(y)\equiv dF/dy]$. Since our solution has the asymptotic form (6), Eq. (11) is reduced to

$$y^2F'^2(y)+\int_y^\infty 2yF'^2dy=1-\cos[2F(y)]\ , \tag{12}$$

where use is made of (4). Noting that the quantity of the left-hand side is always positive, the value of $F(y)$ is limited to the interval

$$n\pi-\pi<F(y)<n\pi+\pi\ . \tag{13}$$

Moreover taking the limit $y\to 0$, (12) is reduced to

$$\int_0^\infty 2yF'^2dy=1-(-1)^m\ , \tag{14}$$

with the help of (10). Since the quantity of the left-hand side is positive, we must take $m=$odd. This and Eq. (7) mean that *the solution with the boundary condition (4) approaches (10b) with $m=2n\pm 1$ and cannot satisfy the asymptotic condition (5).* This fact is nothing but the conjecture (8) obtained by numerical calculations.

What behavior will the other solution (10a) have in the asymptotic region $y\to\infty$ $(r\to 0)$? In order to see that, let us multiply (9) by $F'(y)$ and integrate from 0 to y. The result is

$$F'^2(y)=\frac{2\sin^2F(y)}{y^2}+\int_0^y\frac{2\sin^2F}{y^3}dy\ , \tag{15}$$

with the use of (10a). Thus we obtain

$$F'(y)=\text{const} \tag{16}$$

in the asymptotic region $y\to\infty$ which means $F\propto 1/r$ $(r\to 0)$. *This solution has a singularity at the origin and cannot satisfy the usual boundary condition (4).*

In conclusion, we have proved that the profile function of (3) with the usual boundary conditions (4) and (5) does not exist in the simple chiral model without the Skyrme term. It has been also shown that the solution with the usual asymptotic form (10a) has a singularity at the origin. This singularity may suggest the necessity of the degree of freedom of quarks at the origin such as the chiral bag model.[5] It will be interesting to investigate whether the chiral bag without the Skyrme term can be stabilized by quantum fluctuations.

The authors would like to express their thanks to Professor S. Nishiyama and Professor S. Niizeki (Department of Mathematics) for helpful discussions.

[1]P. Jain, J. Schechter, and R. Sorkin, Phys. Rev. D **39**, 998 (1989).

[2]J. A. Mignaco and S. Wulck, Phys. Rev. Lett. **62**, 1449 (1989).

[3]N. K. Pak and H. C. Tze, Ann. Phys. (N.Y.) **117**, 164 (1979); I. Zahed and G. E. Brown, Phys. Rep. **142**, 1 (1986); T. H. R. Skyrme, Proc. R. Soc. London **A260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

[4]G. H. Derrick, J. Math. Phys. **5**, 1252 (1964).

[5]G. E. Brown and M. Rho, Phys. Lett. **82B**, 177 (1979); G. E. Brown, M. Rho, and V. Vento, *ibid.* **94B**, 383 (1979); M. Rho, A. S. Goldhaber, and G. E. Brown, Phys. Rev. Lett. **51**, 747 (1983).