Renormalization of constrained $SU(2)_L \times U(1)_Y \times \tilde{U}(1)$ models

Giuseppe Degrassi and Alberto Sirlin
Department of Physics, New York University, New York, New York 10003
(Received 6 March 1989)

A simple renormalization framework for constrained $SU(2)_L \times U(1)_Y \times \tilde{U}(1)$ theories is presented. If the relation $\tan^2\phi = (m_W^2/\cos^2\theta_W - m_{Z_1}^2)/(m_{Z_2}^2 - m_W^2/\cos^2\theta_W)$ is regarded as exact (the m's are the physical masses and ϕ is the mixing angle of the neutral vector bosons), it is found that the definition of $\sin^2\theta_W$ must be carefully chosen to ensure consistency and avoid the emergence of radiative corrections of $O(\alpha/\phi)$, i.e., nonanalytic terms in the neighborhood of $\phi=0$. The formulation developed in this paper prevents the occurrence of such potentially catastrophic terms and leads to a definition of $\sin^2\theta_W$ in terms of G_μ, α, and m_W^2 which is very close to the corresponding $SU(2)_L \times U(1)_Y$ expression. A strategy to incorporate approximately the $O(\alpha)$ terms in the neutral currents of these theories is outlined. The discussion identifies in a simple way the mathematical origin of the potential nonanalytic terms and emphasizes the role that the m_t dependence of the radiative corrections may have in the future in determining the tenability of these theories.

I. INTRODUCTION

During the last several years electroweak models based on $SU(2)_L \times U(1)_Y \times \tilde{U}(1)$ have received considerable attention.[1-3] Indeed, such models often arise as the low-energy limit of interesting grand unified theories (GUT's) and superstring theories. It is understood that the presently known or postulated matter fields, that is, gluons and quarks and leptons of the three generations, have the same transformation properties under $SU(2)_L \times U(1)_Y$ as in the standard model (SM). There may exist, of course, additional exotic underlying fields. The assignment of the $\tilde{U}(1)$ quantum number is left free initially. We will see later on that in the cases of interest an orthogonality constraint between the $\tilde{U}(1)$ and $U(1)_Y$ quantum numbers exists, which can be traced to the fact that such models are descendants of gauge theories associated with simple Lie groups.

In this paper we discuss the renormalization and some relevant radiative corrections for a class of these models, characterized by the fact that the Higgs bosons transform as doublets or singlets of $SU(2)_L$. They are referred to as constrained models. As is well known, because of the mixing between the neutral bosons, the lighter mass m_{Z_1} is somewhat smaller than the value attained by m_Z in the SM and, as a consequence, $\rho \equiv m_W^2/m_{Z_1}^2\cos^2\theta_W > 1$. At the tree level there is however an important relation:

$$\tan^2\phi = \frac{m_W^2/\cos^2\theta_W - m_{Z_1}^2}{m_{Z_2}^2 - m_W^2/\cos^2\theta_W} \tag{1}$$

or, equivalently,

$$\rho - 1 = \sin^2\phi \left[\frac{m_{Z_2}^2}{m_{Z_1}^2} - 1 \right] , \tag{2}$$

where ϕ is the mixing angle relating the mass eigenstates Z_1 and Z_2 to the Z and C vector bosons associated with $SU(2)_L \times U(1)_Y$ and $\tilde{U}(1)$, respectively. The gauge sector of the constrained models is characterized, in general, by six parameters that can be identified with e, m_W, m_{Z_1}, m_{Z_2}, $\sin^2\theta_W$, and the $\tilde{U}(1)$ gauge coupling $\tilde{g}$; ϕ is then a dependent parameter specified by Eq. (1). In the cases of interest renormalization-group arguments relate the value of $\tilde{g}$ and the $SU(2)_L$ coupling $g = e/\sin\theta_W$ so that effectively the gauge sector is described by the first five parameters.

When radiative corrections are considered, two possible strategies come to mind: (i) keep Eq. (1) as an exact relation between renormalized parameters or (ii) allow for the possibility that Eq. (1) is corrected by finite terms of $O(\alpha)$. Clearly the distinction between (i) and (ii) would reflect different ways of choosing the renormalization conditions. In this paper we adopt strategy (i). In our view this has obvious advantages: the dependence of ϕ on the other parameters is particularly simple and transparent; recent analyses[2] which incorporate the effect of the radiative corrections employ Eq. (1) as an exact equation; finally, as shown in Sec. II, Eq. (1) follows from the natural requirement that ϕ be the angle that diagonalizes the renormalized mass matrix.

It is also clearly advantageous to identify m_W, m_{Z_1}, m_{Z_2} in (1) with the physical masses. As present experiments are compatible with $\phi=0$, we should allow a null value in the range of variability of ϕ. On the other hand, in some interesting models the experimentally allowed values of m_{Z_2} are as low as $\simeq 110$ GeV. This places an important constraint: in order for (1) to hold for finite m_{Z_2} and arbitrarily small ϕ, $\cos^2\theta_W$ must be defined in such a way that

$$\lim_{\phi\to 0} \cos^2\theta_W = m_W^2/m_{Z_1}^2 ; \tag{3}$$

that is, as $\phi \to 0$, $\cos^2\theta_W$ must approach the SM definition in term of m_W and m_{Z_1}. We will see in the course of our analysis that if (3) is not satisfied, the response of the radiative corrections is catastrophic: terms of $O(\alpha/\phi)$ arise so that one loses the validity of the perturbation expansion in the neighborhood of $\phi=0$. The mathematical origin of such potential nonanalytic terms is explained in Sec. V. Notice that Eq. (3) is a consequence of (a) our insistence that (1) be an exact relation valid in the presence of radiative corrections and (b) our identification of m_W and m_{Z_1} with the physical masses. If (a) and (b) hold we cannot define for instance $\cos^2\theta_W$ by minimal subtraction, because (3) would not be true to $O(\alpha)$ and, as a consequence, (1) would not be satisfied as $\phi\rightarrow 0$ for finite m_{Z_2}. On the other hand, one could define consistently the three parameters m_W, m_{Z_1}, and $\cos^2\theta_W$ by minimal subtraction. A mixed convention, in which m_W, m_{Z_1} are identified with the physical masses and $\cos^2\theta_W$ is defined by minimal subtraction can be only employed if we relax (a) so that (1) is corrected by terms of $O(\alpha)$. In this paper we accept premises (a) and (b). In the next sections we show how $\cos^2\theta_W$ can be defined in ways compatible with (3). As we will see the analysis leads to interesting generalizations of the SM definition of $\cos^2\theta_W$ and the basic radiative correction Δr. Our discussion follows a pattern parallel to that developed for the SM in Ref. 4. In particular, the emphasis is focused on developing a renormalization scheme at the S-matrix level, so that radiative corrections for physical processes can be evaluated in an effective and relatively simple manner.

II. MASS MATRIX OF THE VECTOR MESONS

In this section we discuss the mass terms of the vector bosons and generate the corresponding counterterms. In the theories under consideration, after spontaneous symmetry breaking the mass terms for the vector mesons are of the form

$$\mathcal{L}_M^{\rm VB}=m_{0W}^2W_\mu^\dagger W^\mu+\tfrac{1}{2}m_{0Z}^2Z_\mu Z^\mu+\tfrac{1}{2}m_{0C}^2C_\mu C^\mu+m_{0ZC}^2Z_\mu C^\mu\ , \tag{4}$$

where m_{0W}^2, m_{0Z}^2, m_{0C}^2, and m_{0ZC}^2 are known combinations of coupling constants and vacuum expectation values. All fields and coupling constants in (4) are unrenormalized. In particular, we have

$$Z_\mu=c_0W_\mu^3-s_0B_\mu,\quad A_\mu=s_0W_\mu^3+c_0B_\mu\ , \tag{5}$$

where W_μ^3 and B_μ are neutral gauge fields associated with $SU(2)_L$ and $U(1)_Y$, respectively; A_μ and Z_μ are then identified with the photon and the neutral $SU(2)_L$ vector-boson fields, respectively. In (5) $c_0\equiv\cos\theta_{0W}$, $s_0\equiv\sin\theta_{0W}$ with $\tan\theta_{0W}\equiv g_0'/g_0$, g_0 and g_0' being the $SU(2)_L$ and $U(1)_Y$ coupling constants.

In constrained models, we have the further relation

$$m_{0Z}^2=\frac{m_{0W}^2}{c_0^2}\ . \tag{6}$$

The first two terms in (4) are then as in the SM.

In order to generate counterterms we write

$$m_{0W}^2=m_W^2-\delta m_W^2,\quad m_{0C}^2=m_C^2-\delta m_C^2\ ,$$
$$m_{0ZC}^2=m_{ZC}^2-\delta m_{ZC}^2,\quad c_0^2=c^2-\delta c^2\ ,$$

where m_W^2, m_C^2, m_{ZC}^2, and $c^2\equiv\cos^2\theta_W$ are regarded as renormalized parameters. In this way we obtain

$$\begin{aligned}\mathcal{L}_M^{\rm VB}&=m_W^2W_\mu^\dagger W^\mu+\frac{1}{2}\frac{m_W^2}{c^2}Z_\mu Z^\mu\\&+\tfrac{1}{2}m_C^2C_\mu C^\mu+m_{ZC}^2Z_\mu C^\mu\\&-\delta m_W^2W_\mu^\dagger W^\mu-\frac{1}{2}\frac{m_W^2}{c^2}\left[\frac{\delta m_W^2}{m_W^2}-\frac{\delta c^2}{c^2}\right]Z_\mu Z^\mu\\&-\tfrac{1}{2}\delta m_C^2C_\mu C^\mu-\delta m_{ZC}^2Z_\mu C^\mu+\cdots\ ,\end{aligned} \tag{7}$$

where the ellipses stand for terms of second and higher order in the counterterms.

The first four terms in (7) represent the renormalized mass matrix. The contribution of the neutral vector bosons to this matrix can be expressed as

$$\tfrac{1}{2}(Z^\mu\ \ C^\mu)\begin{bmatrix}\dfrac{m_W^2}{c^2} & m_{ZC}^2\\ m_{ZC}^2 & m_C^2\end{bmatrix}\begin{bmatrix}Z_\mu\\ C_\mu\end{bmatrix}\ . \tag{8}$$

In (7) and (8), m_W^2 and m_C^2 are positive but m_{ZC}^2, although real, can have either sign depending on the model. The Hermitian matrix can be diagonalized by means of an orthogonal transformation of angle ϕ:

$$Z^\mu=\chi Z_1^\mu-\sigma Z_2^\mu,\quad C^\mu=\sigma Z_1^\mu+\chi Z_2^\mu\ , \tag{9}$$

where $\chi\equiv\cos\phi$, $\sigma\equiv\sin\phi$. Z_1^μ and Z_2^μ stand for the mass eigenfields. The mixing angle ϕ and the eigenvalues are given by

$$\begin{aligned}\tan(2\phi)&=\frac{2m_{ZC}^2}{\dfrac{m_W^2}{c^2}-m_C^2}\ ,\\ 2m_{Z_1}^2&=\frac{m_W^2}{c^2}+m_C^2-\left[\left[m_C^2-\frac{m_W^2}{c^2}\right]^2+4m_{ZC}^4\right]^{1/2}\end{aligned} \tag{10}$$

with the sign of the square root reversed for $2m_{Z_2}^2$. As in all cases of interest, $m_C^2>m_W^2/c^2$, we see that the sign of ϕ is opposite to that of m_{ZC}^2. One can also eliminate m_{ZC}^2 and m_C^2 and obtain Eq. (1).

In terms of the mass eigenfields (7) becomes

$$\begin{aligned}\mathcal{L}_M^{\rm VB}&=m_W^2W_\mu^\dagger W^\mu+\tfrac{1}{2}m_{Z_1}^2Z_{1\mu}Z_1^\mu+\tfrac{1}{2}m_{Z_2}^2Z_{2\mu}Z_2^\mu\\&-\delta m_W^2W_\mu^\dagger W^\mu-\tfrac{1}{2}\delta m_{Z_1}^2Z_{1\mu}Z_1^\mu-\tfrac{1}{2}\delta m_{Z_2}^2Z_{2\mu}Z_2^\mu\\&-\delta m_{Z_1Z_2}^2Z_{1\mu}Z_2^\mu\ ,\end{aligned} \tag{11}$$

where

$$\delta m^2_{Z_1}=\chi^2\delta m^2_Z+\sigma^2\delta m^2_C+2\chi\sigma\delta m^2_{ZC}\ ,$$
$$\delta m^2_{Z_2}=\sigma^2\delta m^2_Z+\chi^2\delta m^2_C-2\chi\sigma\delta m^2_{ZC}\ , \quad (12)$$
$$\delta m^2_{Z_1Z_2}=-\chi\sigma\delta m^2_Z+\chi\sigma\delta m^2_C+(\chi^2-\sigma^2)\delta m^2_{ZC}\ ,$$

$$\delta m^2_Z\equiv\frac{m^2_W}{c^2}\left[\frac{\delta m^2_W}{m^2_W}-\frac{\delta c^2}{c^2}\right]. \quad (13)$$

Eliminating δm^2_{ZC} and δm^2_C we have the alternative expression

$$\delta m^2_{Z_1Z_2}=\frac{\chi}{2\sigma}\delta m^2_{Z_1}+\frac{\sigma}{2\chi}\delta m^2_{Z_2}-\frac{1}{2\chi\sigma}\delta m^2_Z\ . \quad (14)$$

In order to identify m_W, m_{Z_1}, and m_{Z_2} with the physical masses we set

$$\delta m^2_W=\mathrm{Re}\,A_{WW}(m^2_W)+t_{WW}\ ,$$
$$\delta m^2_{Z_1}=\mathrm{Re}\,A_{Z_1Z_1}(m^2_{Z_1})+t_{Z_1Z_1}\ , \quad (15)$$
$$\delta m^2_{Z_2}=\mathrm{Re}\,A_{Z_2Z_2}(m^2_{Z_2})+t_{Z_2Z_2}\ .$$

Here the A's are the contributions proportional to $g^{\mu\nu}$ in the unrenormalized (but regularized) self-energies:

$$\Pi^{\mu\nu}_{ij}(q)=A_{ij}(q^2)g^{\mu\nu}+B_{ij}(q^2)q^\mu q^\nu\ . \quad (16)$$

In turn the $\Pi^{\mu\nu}_{ij}$ equal $(-i)$ times the corresponding self-energy diagrams with the external legs extracted. The t's in (15) stand for the contributions of tadpole and tadpole counterterms to the self-energies. A simple calculation shows that to one-loop order the tadpole contribution to the Z_1Z_2 self-energy is given by

$$t_{Z_1Z_2}=\frac{\chi}{2\sigma}t_{Z_1Z_1}+\frac{\sigma}{2\chi}t_{Z_2Z_2}-\frac{1}{2\chi\sigma}(t_{WW}/c^2)\ , \quad (17)$$

which exhibits a structure similar to that of (14).

Inserting (13) and (15) into (14) leads to

$$\delta m^2_{Z_1Z_2}=\frac{\chi}{2\sigma}[\mathrm{Re}\,A_{Z_1Z_1}(m^2_{Z_1})+t_{Z_1Z_1}]+\frac{\sigma}{2\chi}[\mathrm{Re}\,A_{Z_2Z_2}(m^2_{Z_2})+t_{Z_2Z_2}]-\frac{1}{2\sigma\chi c^2}\left[\mathrm{Re}\,A_{WW}(m^2_W)+t_{WW}-\frac{m^2_W}{c^2}\delta c^2\right]. \quad (18)$$

It is important to note that (18) contains terms that potentially diverge as $\phi\to 0$ (or equivalently $\sigma\to 0$). To prevent $\delta m^2_{Z_1Z_2}$ from becoming singular in this limit, we choose $\delta c^2/c^2$ to be of the form

$$\frac{\delta c^2}{c^2}=\frac{\mathrm{Re}\,A_{WW}(m^2_W)+t_{WW}}{m^2_W}-\frac{c^2}{m^2_W}[\mathrm{Re}\,A_{Z_1Z_1}(m^2_{Z_1})+t_{Z_1Z_1}]+f(\sigma)\ , \quad (19)$$

where $f(\sigma)$ is a function to be determined later, which however satisfies

$$f(\sigma)\sim\sigma \quad \text{as } \sigma\to 0\ . \quad (20)$$

Inserting (19) into (18) and recalling (17) we find

$$\delta m^2_{Z_1Z_2}=\frac{\sigma}{2\chi}[\mathrm{Re}\,A_{Z_2Z_2}(m^2_{Z_2})-\mathrm{Re}\,A_{Z_1Z_1}(m^2_{Z_1})]+\frac{1}{2\chi\sigma}\left[\frac{t_{WW}}{c^2}-t_{Z_1Z_1}\right]+\frac{m^2_W}{2\sigma\chi c^2}f(\sigma)+t_{Z_1Z_2}\ . \quad (21)$$

As $\sigma\to 0$, $t_{Z_1Z_1}\to t_{ZZ}$ and furthermore $t_{WW}/c^2\to t_{ZZ}$. Thus, each term in (21) is regular as $\sigma\to 0$, and so is the overall Z_1Z_2 self-energy which is proportional to $A_{Z_1Z_2}(q^2)+t_{Z_1Z_2}-\delta m^2_{Z_1Z_2}$. In order to determine completely $\delta c^2/c^2$ and $\delta m^2_{Z_1Z_2}$ we must still choose $f(\sigma)$. This will be carried out in Sec. III where we study the radiative corrections to μ decay.

Finally, it is worth mentioning that in theories involving two U(1) groups, local gauge invariance allows in principle the existence of counterterms proportional to $(\partial_\mu B_\nu-\partial_\nu B_\mu)(\partial^\mu C^\nu-\partial^\nu C^\mu)$. They are in general necessary to remove divergent contributions to the B-C self-energy arising before spontaneous symmetry breaking from fermions loops. As mentioned in Sec. I, in the models of interest the $\mathrm{U}(1)_Y$ and $\tilde{\mathrm{U}}(1)$ charges obey an orthogonality condition; indeed, this condition prevents the mixing divergences from occurring at the one-loop level. For this reason, we will not consider these possible counterterms hereafter.

III. INTERACTIONS OF VECTOR MESONS AND MATTER FIELDS

In the theories under consideration the interactions of the vector bosons with leptons and quarks is given by

$$\mathcal{L}_{\rm int}=-\frac{g_0}{\sqrt{2}}(W^\dagger_\mu J^\mu_W+\mathrm{H.c.})-\frac{g_0}{c_0}Z_\mu(J^\mu_Z)_0-g_0s_0A_\mu J^\mu_\gamma-\tilde{g}_0J^\mu_C C_\mu\ . \quad (22)$$

In (22) J^μ_γ is the electromagnetic current of the matter fields and J^μ_W and J^μ_Z the corresponding currents coupled to W and Z. The first three terms in (22) are exactly the same as in the SM and the detailed form of the currents is given, for instance, in Eqs. (17a)–(18c) of Ref. 4. In particular we recall that J^μ_Z is of the form

$$(J^\mu_Z)_0=\tfrac{1}{2}J^\mu_3-s^2_0J^\mu_\gamma\ , \quad (23)$$

where J^μ_3 is the third current of the weak isospin. The fourth term in (22) describes the interaction of the gauge boson C_μ associated with the additional $\tilde{\mathrm{U}}(1)$ factor. Specifically, we will write

$$J^\mu_C=\bar{f}\gamma^\mu(\tilde{Y}_La_-+\tilde{Y}_Ra_+)f\ ; \quad (24)$$

here $a_\mp\equiv(1\mp\gamma_5)/2$, $\tilde{Y}_L$ and $\tilde{Y}_R$ are diagonal matrices and f is a column vector whose entries are the mass eigenfields of quarks and leptons. We again generate counterterms writing $g_0=g-\delta g$, $c_0=c-\delta c$, $\tilde{g}_0=\tilde{g}-\delta\tilde{g}$ and regarding g, c, and $\tilde{g}$ as renormalized parameters. This leads to

$$\mathcal{L}_{\rm int}=\hat{\mathcal{L}}_{\rm int}+\delta\mathcal{L} , \tag{25}$$

where

$$\hat{\mathcal{L}}_{\rm int}=-\frac{g}{\sqrt{2}}(W^{\dagger}_{\mu}J^{\mu}_{W}+\text{H.c.})-\frac{g}{c}Z_{\mu}J^{\mu}_{Z}-gsA_{\mu}J^{\mu}_{\gamma}-\tilde{g}J^{\mu}_{C}C_{\mu} , \tag{26}$$

$$\delta\mathcal{L}=\frac{\delta g}{\sqrt{2}}(W^{\dagger}_{\mu}J^{\mu}_{W}+\text{H.c.})+\frac{g}{c}\left[\frac{\delta g}{g}-\frac{\delta c}{c}\right]Z_{\mu}J^{\mu}_{Z}-\frac{g}{c}\delta s^2 J^{\mu}_{\gamma}Z_{\mu}+\delta e A_{\mu}J^{\mu}_{\gamma}+\delta\tilde{g}J^{\mu}_{C}C_{\mu} , \tag{27}$$

$$\delta e\equiv s\delta g+g\delta s , \tag{28}$$

$$J^{\mu}_{Z}=\tfrac{1}{2}J^{\mu}_{3}-s^2J^{\mu}_{\gamma} . \tag{29}$$

Again we have neglected terms of second and higher order in the counterterms. It is convenient to express $\hat{\mathcal{L}}_{\rm int}$ and $\delta\mathcal{L}$ in terms of the mass eigenfields Z^{μ}_1 and Z^{μ}_2 introduced in (9). Defining

$$J^{\mu}_{Z_1}\equiv\chi J^{\mu}_{Z}+\frac{\tilde{g}c}{g}\sigma J^{\mu}_{C},\quad J^{\mu}_{Z_2}\equiv-\sigma J^{\mu}_{Z}+\frac{\tilde{g}c}{g}\chi J^{\mu}_{C} \tag{30}$$

we find

$$\hat{\mathcal{L}}_{\rm int}=-\frac{g}{\sqrt{2}}(W^{\dagger}_{\mu}J^{\mu}_{W}+\text{H.c.})-gsJ^{\mu}_{\gamma}A_{\mu}-\frac{g}{c}(J^{\mu}_{Z_1}Z_{1\mu}+J^{\mu}_{Z_2}Z_{2\mu}) , \tag{31}$$

$$\delta\mathcal{L}=\frac{\delta g}{\sqrt{2}}(W^{\dagger}_{\mu}J^{\mu}_{W}+\text{H.c.})+\delta eJ^{\mu}_{\gamma}A_{\mu}+\frac{g}{c}Z_{1\mu}(\delta pJ^{\mu}_{Z_1}-\delta tJ^{\mu}_{Z_2}-\chi\delta s^2J^{\mu}_{\gamma})+\frac{g}{c}Z_{2\mu}(-\delta tJ^{\mu}_{Z_1}+\delta qJ^{\mu}_{Z_2}+\sigma\delta s^2J^{\mu}_{\gamma}) , \tag{32}$$

where

$$\delta p\equiv\chi^2\left[\frac{\delta g}{g}-\frac{\delta c}{c}\right]+\sigma^2\frac{\delta\tilde{g}}{\tilde{g}} , \tag{33}$$

$$\delta q\equiv\sigma^2\left[\frac{\delta g}{g}-\frac{\delta c}{c}\right]+\chi^2\frac{\delta\tilde{g}}{\tilde{g}} , \tag{34}$$

$$\delta t\equiv\chi\sigma\left[\frac{\delta g}{g}-\frac{\delta c}{c}-\frac{\delta\tilde{g}}{\tilde{g}}\right] . \tag{35}$$

We now identify $e\equiv gs$ in (31) with the conventionally defined charge of the positron. As shown in Ref. 4, this is achieved by setting

$$\frac{2\delta e}{e}=-\Pi_{\gamma\gamma}(0)-\frac{\alpha}{\pi}\left[\frac{2}{n-4}+\gamma-\ln4\pi+\ln\frac{m^2_W}{\mu^2}\right] \tag{36}$$

where $\Pi_{\gamma\gamma}(q^2)$ is related to the photon self-energy by $A_{\gamma\gamma}(q^2)=-q^2\Pi_{\gamma\gamma}(q^2)$. In order to define precisely g^2, or alternatively $\sin^2\theta_W$, we consider the most accurately measured charged-current process: namely, μ decay. As explained in Ref. 4, the traditional photonic corrections of the $V-A$ theory can be separated out. The remainder leads to a radiatively corrected amplitude of the form

$$\mathcal{M}=\mathcal{M}^0/(1-\Delta\tilde{r}) , \tag{37}$$

$$\Delta\tilde{r}=\frac{\text{Re}A_{WW}(q^2)-\text{Re}A_{WW}(m^2_W)}{q^2-m^2_W}+V+B-\frac{2\delta g}{g} . \tag{38}$$

Here $A_{WW}(q^2)$ arises from WW self-energy diagrams at q^2 values characteristic of μ decay ($q^2\simeq0$), V and B represent vertex and box diagrams after separation of the $V-A$ photonic corrections, and $\text{Re}A_{WW}(m^2_W)$ arises from the mass renormalization counterterm δm^2_W [cf. Eq. (15)]. The last term is the contribution from the δg counterterm in (32) and $\mathcal{M}^0$ is the zeroth-order amplitude:

$$\mathcal{M}^0=\frac{ig^2}{2(q^2-m^2_W)}(\bar{u}_{\nu_\mu}\gamma_{\mu}a_{-}u_{\mu})(\bar{u}_e\gamma^{u}a_{-}u_{\nu_e}) . \tag{39}$$

As shown in Ref. 5, the insertion of $\Delta\tilde{r}$ in the denominator rather than in the numerator of (37), includes automatically the effect of large logarithmic contributions associated with mass singularities to higher order in α.

From (28) we find

$$\frac{2\delta g}{g}=\frac{2\delta e}{e}-\frac{2\delta s}{s}=\frac{2\delta e}{e}+\frac{\delta c^2}{s^2} \tag{40}$$

and, recalling (19),

$$\frac{2\delta g}{g}=\frac{2\delta e}{e}+\frac{c^2}{s^2}\left[\frac{\text{Re}A_{WW}(m^2_W)+t_{WW}}{m^2_W}-\frac{c^2}{m^2_W}[\text{Re}A_{Z_1Z_1}(m^2_{Z_1})+t_{Z_1Z_1}]+f(\sigma)\right] . \tag{41}$$

Inserting the above expression into (38) and setting $q^2=0$,

$$\Delta\tilde{r}=\Delta r_1(\sigma)-\frac{c^2}{s^2}f(\sigma) , \tag{42}$$

$$\Delta r_1(\sigma)=\frac{\text{Re}A_{WW}(m^2_W)-A_{WW}(0)}{m^2_W}-\frac{2\delta e}{e}+V+B+\frac{c^2}{s^2}\left[c^2\frac{\text{Re}A_{Z_1Z_1}(m^2_{Z_1})+t_{Z_1Z_1}}{m^2_W}-\frac{\text{Re}A_{WW}(m^2_W)+t_{WW}}{m^2_W}\right] . \tag{43}$$

As explained in Sec. II, in the limit $\phi=\sigma=0$ we have $f(0)=0$, $m^2_{Z_1}=m^2_W/c^2$, and $c^2t_{Z_1Z_1}/m^2_W=t_{WW}/m^2_W$. In this limit the terms involving the self-energies as well as $\delta e/e$ become identical with the corresponding ones in the $SU(2)_L\times U(1)_Y$ correction Δr [Eq. (34b) of Ref. 4], with the understanding that one must include the contributions of additional particles arising from the extended Higgs sector or fermion structure of the various models. The additional Z^{μ}_2 becomes identical to C^{μ}; using the Ward identities of the current algebra and noting that J^{μ}_W commutes with J^{μ}_C, it easy to see that to $O(G_F\alpha)$ this vector boson does not contribute to the vertex diagrams V in μ decay. It does however contribute to the box diagrams B leading to a small and finite additional contribution[3]

$$(B)_{Z_2} = -\frac{3}{8\pi^2}\tilde{g}^2(\tilde{Y}_L)_l^2 \frac{\ln x}{x-1} , \tag{44}$$

where $(\tilde{Y}_L)_l$ is the $\tilde{Y}_L$ quantum number of the e or μ lepton doublets and $x \equiv m_{Z_2}^2/m_W^2$. We conclude that

$$\Delta r_1(0) = \Delta r + (B)_{Z_2} . \tag{45}$$

The simplest strategy is then to choose

$$\frac{c^2}{s^2} f(\sigma) = \Delta r_1(\sigma) - \Delta r_1(0) , \tag{46}$$

which satisfies the requirement $f(0)=0$ [cf. Eq. (20)] and leads to

$$\Delta\tilde{r} = \Delta r_1(0) = \Delta r + (B)_{Z_2} . \tag{47}$$

Following a line of argument identical to that of Ref. 4, Eq. (37) implies

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2(1-\Delta\tilde{r})} \tag{48}$$

or, recalling $g \equiv e/\sin\theta_W$,

$$\sin^2\theta_W = \frac{(37.281 \text{ GeV})^2}{m_W^2(1-\Delta\tilde{r})} , \tag{49}$$

where we used $\pi\alpha/(G_\mu\sqrt{2}) = (37.281 \text{ GeV})^2$. Equations (47) and (49) are important in our formulation as they provide the definition of the fundamental parameter $\sin^2\theta_W$. It is easy to see that this definition is consistent with (1) and (3). Indeed, in the limit $\phi=0$ the radiative corrections to μ decay must be those of the $SU(2)_L \times U(1)_Y$ theory, evaluated with $\cos^2\theta_W \equiv m_W^2/m_{Z_1}^2$, plus the additional contribution of the new boson, the $C \equiv Z_2$ field. But these are precisely the terms Δr and $(B)_{Z_2}$ in Eq. (47).

In the models of interest one has[2,3]

$$\tilde{Y}_{L,R} = \cos\beta Q_{L,R}^\chi + \sin\beta Q_{L,R}^\psi , \tag{50}$$

$$\tilde{g} = (\tfrac{3}{8})^{1/2} g \tan\theta_W \sqrt{\lambda} , \tag{51}$$

where β is a mixing-angle characteristic of the model, λ is a renormalized parameter $\simeq 1$ and

$$\begin{aligned}
&(Q_L^\chi)_e = (Q_L^\chi)_\nu = -3(Q_L^\chi)_u = -3(Q_L^\chi)_d = 1 , \\
&3(Q_R^\chi)_e = 3(Q_R^\chi)_u = -(Q_R^\chi)_d = 1 , \\
&(Q_L^\psi)_e = (Q_L^\psi)_\nu = (Q_L^\psi)_u = (Q_L^\psi)_d = \sqrt{\tfrac{5}{27}} , \\
&(Q_R^\psi)_e = (Q_R^\psi)_u = (Q_R^\psi)_d = -\sqrt{\tfrac{5}{27}} .
\end{aligned} \tag{52}$$

It is easy to verify that the Q^χ and Q^ψ charges are orthogonal to the Y hypercharges of $U(1)_Y$. Equation (51) is obtained by following the evolution of the coupling constants from the GUT scale to the m_W scale. From these equations we see that

$$(B)_{Z_2} = -\frac{9\alpha\lambda}{16\pi\cos^2\theta_W}[\cos\beta + \sin\beta(\tfrac{5}{27})^{1/2}]^2 \frac{\ln x}{x-1} . \tag{53}$$

Using $\lambda=1$, $\cos^2\theta_W=0.77$, $m_W=81$ GeV and noting that $[\cos\beta+\sin\beta(\frac{5}{27})^{1/2}]^2 \le 1.185$ we find that $|(B)_{Z_2}| \le 1.6\times10^{-3}$, 1.0×10^{-3}, 7×10^{-4} for $m_{Z_2}=100$, 150, and 200 GeV, respectively. For the Z_η boson which occurs in the breaking $E_6 \to SU(3)\times SU(2)_L \times U(1)_Y \times U(1)_\eta$ expected in some superstring models, we have $\cos\beta = (\frac{3}{8})^{1/2}$, $\sin\beta = -(\frac{5}{8})^{1/2}$, and $(B)_{Z_2}$ is greatly reduced: $(B)_{Z_2} = -1.0\times10^{-4}$, -6.3×10^{-5}, -4×10^{-5} for the same m_{Z_2} values. We conclude that the correction to Δr arising from the Z_2 exchange is indeed very small. As a consequence, (49) is very close to the $SU(2)_L \times U(1)_Y$ expression for $\sin^2\theta_W$ in terms of G_μ, α, and m_W^2.

IV. NEUTRINO-INDUCED NEUTRAL-CURRENT PHENOMENA

In this section we illustrate some interesting aspects of the cancellation of the divergences in the one-loop correction to ν-induced neutral-current phenomena and make some general remarks concerning the finite parts.

At the tree level these processes involve the sum of the two amplitudes depicted in Fig. 1: namely,

$$\begin{aligned}
\mathcal{M}^0 &= i\frac{g^2}{c^2}\left[\frac{\langle f|J_{Z_1}^\mu|i\rangle\langle\nu_f|J_{Z_1\mu}|\nu_i\rangle}{q^2-m_{Z_1}^2} + \frac{\langle f|J_{Z_2}^\mu|i\rangle\langle\nu_f|J_{Z_2\mu}|\nu_i\rangle}{q^2-m_{Z_2}^2}\right] \\
&\equiv \mathcal{M}_{Z_1}^0 + \mathcal{M}_{Z_2}^0 ,
\end{aligned} \tag{54}$$

where $|i\rangle$ ($|\nu_i\rangle$) and $|f\rangle$ ($|\nu_f\rangle$) represent initial and final hadron (neutrino) states. In the case of leptonic processes, i and f represent the initial and final charged-lepton states.

The $O(\alpha)$ electroweak corrections can be separated into three different parts: loop corrections to Fig. 1(a), to

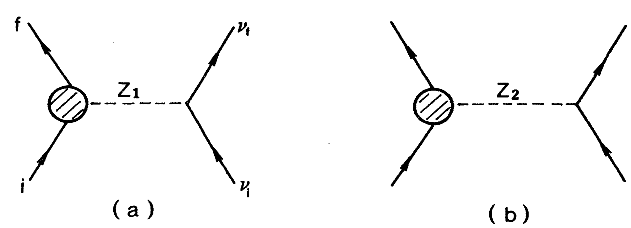

FIG. 1. Zeroth-order diagrams in ν-induced neutral-current processes. The shaded circles in this figure indicate that i and f may be hadronic states. In that case f represents the final hadron state and may involve many particles. The amplitudes corresponding to (a) and (b) are denoted by $\mathcal{M}_{Z_1}^0$ and $\mathcal{M}_{Z_2}^0$, respectively.

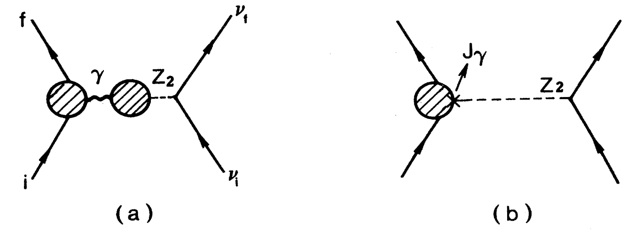

FIG. 2. γ-Z_2 mixing amplitude and associated counterterm diagrams. The cross in (b) represents the insertion of the counterterm proportional to $J^{\mu}_{\gamma}Z_{2\mu}$ from Eq. (32).

Fig. 1(b) and mixing γ-Z_2 and Z_1-Z_2 amplitudes, some of which are illustrated in Figs. 2 and 3 together with relevant counterterm diagrams. By convention we include the γ-Z_1 mixing diagrams among the loop corrections to Fig. 1(a) so that they are not depicted separately.

We consider first the diagrams of Fig. 2 and focus on the fermionic contributions to the γ-Z_2 mixing. The sum of Figs. 2(a) and 2(b) reads

$$\mathcal{M}_{\gamma Z_2} = i\frac{ge}{c}\frac{\langle f|J_{\gamma\mu}|i\rangle\langle \nu_f|J^{\mu}_{Z_2}|\nu_i\rangle}{q^2-m^2_{Z_2}}\left[\frac{A_{\gamma Z_2}}{q^2}-\frac{\sigma\delta s^2}{sc}\right]. \tag{55}$$

The counterterm $\delta s^2 = -\delta c^2$ can be obtained from (19), (43), and (46). Evaluation of the fermionic contribution to the large parentheses gives

$$\left[\frac{A_{\gamma Z_2}}{q^2}-\frac{\sigma\delta s^2}{sc}\right]_f = \frac{\tilde{g}e\chi}{12\pi^2}\sum_i Q^i_{\rm el}(\tilde{Y}^i_L+\tilde{Y}^i_R)\times\left[\frac{1}{n-4}+Pf\right], \tag{56}$$

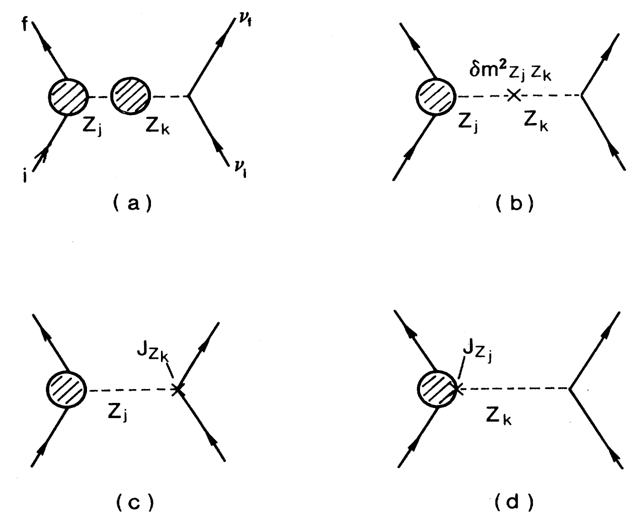

FIG. 3. Self-energy and relevant counterterm diagrams. (a) is assumed to include tadpole and tadpole counterterm contributions. Here j, $k=1,2$; $\delta m^2_{Z_jZ_j} \equiv \delta m^2_{Z_j}$ where $\delta m^2_{Z_j}$ $(j=1,2)$ are the counterterms defined in Eq. (15), and $\delta m^2_{Z_1Z_2}$ is given by Eqs. (21), (43), and (46). The crosses in (c) and (d) represent insertions from relevant counterterms from Eq. (32).

where the sum runs over all the fermions in the theory (a sum over the color degrees of freedom is understood), $Q^i_{\rm el}$ is the electric charge of the ith fermion, Pf means "finite part" and the subscript f reminds us that we are considering the fermionic loops. In the models of interest, which are the low-energy limit of GUT's described by simple Lie groups, the orthogonality condition

$$\sum_i Q^i_{\rm el}(\tilde{Y}^i_L+\tilde{Y}^i_R)=0 \tag{57}$$

is satisfied automatically. In fact, in such cases the third component T_{3L} of the weak isospin and the hypercharges Y, $\tilde{Y}$ are proportional to neutral generators of the higher group which, as is well known, obey an orthogonality condition of the form $\mathrm{Tr}(T_iT_j)\propto\delta_{ij}$. Equation (57) implies the automatic cancellation of the divergent part of (56). The analogue of (57) when the sum runs over the scalars present in the theory applies to the Higgs sector. In models which are not the descendants of GUT's described by simple Lie groups, (57) may not be valid. In such cases the divergent part of (56) would not vanish automatically even in the unbroken theory and one must appeal to mixing counterterms of the type described at the end of Sec. II.

We now turn our attention to the loop correction of Fig. 3 with $j=k$. We note that both the divergent and the finite parts of all the counterterms have been determined in Secs. II and III, with the exception of $\delta\tilde{g}$. Consideration of the above diagrams shows that, after taking into account (57), the remaining divergences can be canceled by a judicious choice of $\delta\tilde{g}$. Specifically the divergent part of $\delta\tilde{g}$ is seen to be the term proportional to

$$\sigma^2\left[\frac{1}{n-4}+\frac{\gamma-\ln(4\pi)}{2}\right]$$

in $dA_{Z_1Z_1}(q^2)/dq^2$ or, equivalently, to

$$\chi^2\left[\frac{1}{n-4}+\frac{\gamma-\ln(4\pi)}{2}\right]$$

in $dA_{Z_2Z_2}(q^2)/dq^2$. As to the finite part, a convenient strategy is to simply define $\tilde{g}$ by the modified minimal subtraction scheme ($\overline{\rm MS}$) at the mass scale $\mu=m_W$. Finally we consider the diagrams of Fig. 3 with $j=2$, $k=1$ (the case $j=1$, $k=2$ is completely analogous). For this case the sum of the amplitudes of Fig. 3 can be written as

$$\mathcal{M}_{Z_1Z_2} = i\frac{g^2}{c^2}\frac{\langle f|J^{\mu}_{Z_2}|i\rangle\langle \nu_f|J_{Z_1\mu}|\nu_i\rangle}{(q^2-m^2_{Z_1})(q^2-m^2_{Z_2})}\times\left[C_{Z_2Z_1}(q^2)-\frac{m^2_W}{2c^2}\frac{f(\sigma)}{\sigma\chi}\right], \tag{58}$$

where

$$C_{Z_2Z_1}(q^2)=A_{Z_1Z_2}(q^2)+\delta t(q^2-m^2_{Z_1})+\delta t(q^2-m^2_{Z_2}) - \frac{\sigma}{2\chi}[\mathrm{Re}A_{Z_2Z_2}(m^2_{Z_2})-\mathrm{Re}A_{Z_1Z_1}(m^2_{Z_1})] \tag{59}$$

and $f(\sigma)$ is defined by Eqs. (46) and (43). Assuming the validity of (57), evaluation of the fermionic contributions leads to

$$C_{Z_2Z_1}(q^2) = -\frac{g^2}{4\pi c^2}\sum_i m_i^2\left[-\frac{\chi\sigma}{4}+\frac{\tilde{g}c}{g}(\chi^2-\sigma^2)T_{3i}(\tilde{Y}^i_L-\tilde{Y}^i_R) + \chi\sigma\frac{\tilde{g}^2c^2}{g^2}(\tilde{Y}^i_L-\tilde{Y}^i_R)^2\right] \times\frac{1}{n-4}+Pf\ , \qquad (60)$$

where m_i is the mass of the ith fermion. One can verify that the divergent part of (60) is canceled by the term proportional to $f(\sigma)/\sigma$ in (58). In particular, it is worth noticing that the divergent part of (60) survives in the limit $\sigma\to 0$. However, the term $(c^2/m_W^2)\mathrm{Re}A_{Z_1Z_1}(m_{Z_1}^2)$ in $f(\sigma)$ [cf. Eqs. (46) and (43)] contains a divergent part linear in σ and involving the m_i^2; it survives in $(m_W^2/2c^2)[f(\sigma)/\sigma\chi]$ as $\sigma\to 0$ and cancels the divergent part of $C_{Z_2Z_1}$ in the same limit. We also note that the second and third terms between the large parentheses in (60) arise from the axial-vector part of J_C^μ. Indeed, if J_C^μ were vectorial, it would be conserved because it is diagonal in flavor space. As a consequence, in that case the divergent part of the self-energies would not be proportional to the m_i^2.

In summary we conclude that all the divergent parts in the one-loop corrections to ν-induced neutral-current phenomena are canceled by the counterterms determined in Secs. II–IV when (57) is taken into account.

Regarding the finite parts of the electroweak corrections we note that the experimental upper bounds on ϕ and the magnitude of the amplitude $\mathcal{M}^0_{Z_2}$ are quite small. The latter is a consequence of either $m_{Z_2}^2 \gg m_{Z_1}^2$ and or the fact that the Z_2 couplings to the ordinary fermions are quite weak. A sensible approximate strategy is then to neglect the loop corrections to Fig. 1(b) and the mixing contributions exemplified by Figs. 2 and 3 (with $j\neq k$) and evaluate the electroweak corrections to the dominant amplitude $\mathcal{M}^0_{Z_1}$ in the limit $\phi\to 0$. The latter become then the corrections of the $SU(2)_L\times U(1)_Y$ theory evaluated with $m_{Z_1}^2=m_W^2/c^2$. (See Ref. 6). This is essentially the approximation employed in recent phenomenological studies of the $SU(2)_L\times U(1)_Y\times\tilde{U}(1)$ models.[2] There is a small difference with the present work: while the analyses of Ref. 2 maintain Eq. (1) as an exact relation, they approximate $\Delta\tilde{r}\to\Delta r$ in the definition of $\sin^2\theta_W$ [cf. Eq. (49)]. In most cases of interest $(B)_{Z_2}\ll 1$ and, as a consequence, the difference is negligible.

V. DISCUSSION

In the preceding sections we developed a renormalization scheme for the constrained $SU(2)_L\times U(1)_Y\times\tilde{U}(1)$ theories that satisfies two main objectives: (a) it is consistent with Eq. (1) even for arbitrarily small ϕ and any mass $m_{Z_2}>m_{Z_1}$ and (b) it prevents the emergence of nonanalytic terms of $O(\alpha/\phi)$ in the neighborhood of $\sigma=0$ (or, equivalently, $\phi=0$). These two requirements are indeed related and are implemented by the constraint $f(\sigma)\sim\sigma$ as $\sigma\to 0$ [Eq. (20)]. This strategy led us in turn to a definition of $\sin^2\theta_W$ [Eq. (49)] which, as pointed out in Sec. III, is very close to the $SU(2)_L\times U(1)_Y$ expression obtained in Ref. 4 in terms of G_μ, α, and m_W^2. In principle, by altering our choice of the finite part of $f(\sigma)$ we could slightly shift $\Delta\tilde{r}$ and correspondingly the definition of $\sin^2\theta_W$ by terms of $O(\sigma)$. Our specific choice in (46) is probably the simplest one consistent with (20). Aside from this freedom, the choice in the definition of $\sin^2\theta_W$ is greatly restricted. Suppose that an attempt were made to redefine $\sin^2\theta_W$ so that it differs from (49) by terms of $O(\alpha)$ but of zeroth order in σ. This requires a corresponding change in $\Delta\tilde{r}$ and, because of (42), it would demand $f(0)\neq 0$. In turn this would lead to a catastrophic consequence: because of (21) the counterterm $\delta m^2_{Z_1Z_2}$ would contain terms of $O(\alpha/\sigma)$ and the feasibility of the perturbation expansion in the neighborhood of $\sigma=0$ would be lost.

How is one to understand the potential emergence of such nonanalytic terms? The answer turns out to be very simple: because of Z-C mixing, the zeroth-order amplitudes contain terms proportional to $\sigma\chi J_Z^\mu J_{C_\mu}$, i.e., $\sim\sigma$ as $\sigma\to 0$. For fixed values of m_W, m_{Z_1}, and m_{Z_2}, a variation $\delta\cos^2\theta_W$ of $O(\alpha)$ but of zeroth order in σ would induce according to (1) a shift

$$\delta\sigma=\frac{-(m_W^2/c^4)\delta c^2}{2\sigma(m_{Z_2}^2-m_W^2/c^2)}$$

for small σ, which is clearly of $O(\alpha/\sigma)$. As a consequence, such change in $\cos^2\theta_W$ would result in a shift of $O(\alpha/\sigma)$ in the Born contributions and this must be compensated by an identical and opposite modification of the $O(\alpha)$ corrections. Thus, the response of the radiative corrections to such a change would be sharp and catastrophic in the neighborhood of $\sigma=0$. Of course, this would not be important if ϕ was known to be a finite angle $\gg\alpha$ because in that case it would not be necessary to consider the domain $\phi\sim 0$. However, the fact that present experiments are perfectly consistent with the $SU(2)_L\times U(1)_Y$ theory requires that this be included in the allowed region of variability of σ.

It is also important to emphasize that the Δr that appears in (47) and (49) is the complete Δr of the $SU(2)_L\times U(1)_Y$ theory, evaluated with $\cos^2\theta_W=m_W^2/m_{Z_1}^2$. This includes the effects of the new Higgs multiplets and fermions whose masses are, of course, unknown. Even if we assume that the effect of the new particles on Δr is negligible, one faces the problem that at present two of the important parameters of the minimal $SU(2)_L\times U(1)_Y$ theory, namely, m_t and m_H, are unknown. In current analyses of the $SU(2)_L\times U(1)_Y\times\tilde{U}(1)$ theories one essentially assumes that m_t is not very large: namely, $m_t<100$ GeV. However, if one considers the full range $m_t<180$ GeV presently allowed by experiments, it is possible that the m_t dependence of Δr may play a significant role in deciding whether the constrained $SU(2)_L \times U(1)_Y \times \tilde{U}(1)$ models are tenable, especially once the m_W and m_Z masses become accurately known. The point is that according to Eq. (1) we must have $m_W^2/\cos^2\theta_W \geq m_{Z_1}^2$ and because of Eq. (49) this may be valid only for some range of values of m_t. As an example, let us assume that m_W and m_{Z_1} were known to equal 81 and 92 GeV, respectively, with great precision. This would imply that $\cos^2\theta_W \leq 0.775$. Inserting the value $\Delta r = 0.0713$ corresponding to $m_t = 45$ GeV and $m_H = 100$ GeV, (49) gives $\sin^2\theta_W = 0.228$ or $\cos^2\theta_W = 0.772$ which is certainly compatible. On the other hand, for $m_t = 150$ GeV, $\Delta r = 0.0412$ and (49) would lead to $\sin^2\theta_W = 0.221$ or $\cos^2\theta_W = 0.779$ which would lie outside the allowed range.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation under Grant No. PHY-8715995. One of us (G.D.) would also like to acknowledge the kind support of the Fondazione A. Della Riccia.

[1]L. S. Durkin and P. Langacker, Phys. Lett. **166B**, 431 (1986); D. London and J. Rosner, Phys. Rev. D **34**, 1530 (1986); J. Rosner, Comm. Nucl. Part. Phys. **14**, 229 (1985); E. Cohen *et al.*, Phys. Lett. **165B**, 76 (1985); J. Ellis *et al.*, Nucl. Phys. **B276**, 14 (1986); V. Barger, N. G. Deshpande, and K. Whisnant, Phys. Rev. Lett. **56**, 30 (1986); Phys. Rev. D **35**, 1005 (1987); V. Barger *et al.*, *ibid.* **35**, 2893 (1987); F. del Aguila *et al.*, Nucl. Phys. **B283**, 50 (1987); **B284**, 530 (1987); P. J. Franzini and F. Gilman, Phys. Rev. D **35**, 855 (1987); M. Cvetič and B. W. Lynn, *ibid.* **35**, 51 (1987).

[2]U. Amaldi *et al.*, Phys. Rev. D **36**, 1385 (1987); G. Costa, *et al.*, Nucl. Phys. **B297**, 244 (1988).

[3]W. J. Marciano and A. Sirlin, Phys. Rev. D **35**, 1672 (1987).

[4]A. Sirlin, Phys. Rev. D **22**, 971 (1980).

[5]A. Sirlin, Phys. Rev. D **29**, 89 (1984).

[6]W. J. Marciano and A. Sirlin, Phys. Rev. D **22**, 2695 (1980).