Gaussian-effective-potential method for SU(2)×U(1) gauge theory and bounds on the Higgs-boson mass

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By applying the Gaussian-effective-potential method to SU(2)×U(1) gauge theory, the quantum version of the Higgs mechanism is examined in detail. The stability condition for a broken vacuum leads to an estimation of upper and lower bounds on the mass of the Higgs boson: $0.925m_W < m_H < 2.07m_W$. An autonomous gauge theory is also achieved after removing the momentum cutoff.

I. INTRODUCTION

The Higgs mechanism plays an important role in standard SU(2)×U(1) electroweak theory.[1] However, there is some worry about its basis. Some authors showed that the pure quantum $\lambda\phi^4$ model may be trivial (i.e., $\lambda_R \to 0$, no interaction exists at all) in four space-time dimensions.[2] Recently, aiming at the revival of $\lambda\phi^4$ theory, some effort has been made using the nonperturbative Gaussian-effective-potential (GEP) approach.[3-9] One method is to introduce explicitly a large but finite moment cutoff Λ and treat the $\lambda\phi^4$ model as an effective model at low energy.[6-8] On the other hand, in the so-called autonomous theory,[9] after performing a special type of wave-function renormalization while keeping the bare coupling parameter λ_B positive but infinitesimal ($\lambda_R \to 0^+$), one can let $\Lambda \to \infty$ and regain a meaningful $\lambda\phi^4$ model. From the practical point of view, there is a benefit in the former kind of theory (with finite cutoff Λ) as some bounds on the mass of the elusive Higgs boson could be found when the gauge fields are included,[7,10-12] whereas no observable restriction exists in the latter kind of theory (with $\Lambda \to \infty$). Moreover, the latter approach has not been used in gauge theory. Both kinds of theories will be discussed in this paper.

The organization of this paper is as follows. In Sec. II, keeping a large but finite cutoff Λ, we obtain the criteria for the existence of the broken or symmetric phase in the SU(2)×U(1) model. Then in Sec. III, after carefully examining the stability condition of a broken vacuum, the upper and lower bounds on the Higgs-boson mass are found within some approximation. The so-called autonomous theory is discussed in Sec. IV. The final section contains a summary and discussion.

II. THE GEP AND THE EXISTENCE CRITERION FOR BROKEN AND/OR SYMMETRIC PHASE

Let us begin with the Lagrangian density

$$\mathcal{L}(x) = -\tfrac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + \sigma_0\phi^\dagger\phi - \frac{1}{3!}\lambda_0(\phi^\dagger\phi)^2 , \quad (1)$$

where

$$D_\mu\phi(x) = \left[\partial_\mu + igA_{\mu a}(x)\frac{\tau_a}{2} - i\frac{g'}{2}B_\mu(x)\right]\phi(x) , \quad (2)$$

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g\epsilon_{abc}A_b^\mu A_c^\nu ,$$
$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu , \quad (3)$$

and

$$\phi(x) = \begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix} = \frac{1}{\sqrt{2}}e^{ig\eta_a(x)\tau_a}\begin{bmatrix}0\\ \xi(x)\end{bmatrix} ,$$
$$\phi^\dagger(x) = (\phi_1^*(x), \phi_2^*(x)) , \quad (4)$$

where τ_a ($a = 1,2,3$) are Pauli matrices.

As is well known, one redefines

$$W^\mu = \frac{1}{\sqrt{2}}(A_1^\mu + iA_2^\mu) ,$$
$$Z^\mu = \frac{1}{(g^2+g'^2)^{1/2}}(gA_3^\mu + g'B^\mu) ,$$
$$A^\mu = \frac{1}{(g^2+g'^2)^{1/2}}(-g'A_3^\mu + gB^\mu) , \quad (5)$$

to represent the charged, neutral massive boson and photon, respectively. The quantization procedure is completely similar to that in Ref. 7, i.e., to evaluate the expectation value of Hamiltonian density in a Gaussian wave functional (ansatz):

$$\Psi=\exp\left[-\tfrac{1}{2}\int_{xy}\{[\xi(x)-\bar{\xi}(x)]F_{xy}(\bar{\xi})[\xi(y)-\bar{\xi}(y)]-[W^{\mu}(x)F_{xy}(\overline{W})W^{*}_{\mu}(y)+W^{*}_{\mu}(x)F_{xy}(\overline{W})W^{\mu}(y)]\right.$$
$$\left.-Z^{\mu}(x)F_{xy}(\overline{Z})Z_{\mu}(y)+A_{\mu}(x)F^{\mu\nu}_{xy}(\overline{A})A_{\nu}(x)\}\right] . \tag{6}$$

For convenience in calculation we choose the temporal gauge $W^0=Z^0=A^0=0$. Because of the time independence of the effective potential (which is defined as a functional expectation in the time-independent ground state), we can also choose $\eta_a(\pi)=0$ as a consequence of the time-independent gauge transformations which are still allowed in the temporal gauge.[13]

In Eq. (6), $F^{ij}_{xy}(\overline{A})$ has been fixed as

$$F^{ij}_{xy}(\overline{A})=\int\frac{d^3p}{(2\pi)^3}\frac{p^2\delta_{ij}-p_ip_j}{|p|}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

such that U(1) gauge invariance survives[13] and the mass of the photon m_A is zero.

After evaluating the expectation value of the Hamiltonian density in the Gaussian ground state (6) and minimizing the result with respect to $F_{xy}(\bar{\xi})$, $F_{xy}(\overline{W})$, and $F_{xy}(\overline{Z})$ as in Ref. 7, we arrive finally at the GEP:

$$\begin{aligned}V_G=&3CI_0(\mu_W^2)-\frac{3C}{2}\mu_W^2I_1(\mu_W^2)+\tfrac{1}{2}I_0(\mu_\xi^2)-\tfrac{1}{4}\mu_\xi^2I_1(\mu_\xi^2)+\frac{3C}{2}I_0(\mu_Z^2)\\&-\tfrac{3}{4}C\mu_Z^2I_1(\mu_Z^2)+\tfrac{3}{2}F^{11}_{xx}(\overline{A})+\frac{3}{2}C^2\frac{g^4}{g^2+g'^2}I_1(\mu_W^2)I_1(\mu_Z^2)\\&+\tfrac{3}{4}g^2C^2[I_1(\mu_W^2)]^2+\frac{3}{2}\frac{g^2g'^2C}{g^2+g'^2}I_1(\mu_W^2)G^{11}_{xx}(\overline{A})-\frac{\sigma_0}{2}[\bar{\xi}^2+\tfrac{1}{2}I_1(\mu_\xi^2)]\\&+\frac{3C}{8}g^2\left[I_1(\mu_W^2)+\frac{g^2+g'^2}{2g^2}I_1(\mu_Z^2)\right][\bar{\xi}^2+\tfrac{1}{2}I_1(\mu_\xi^2)]+\frac{\lambda_0}{4!}\{\bar{\xi}^4+3I_1(\mu_\xi^2)\bar{\xi}^2+\tfrac{3}{4}[I_1(\mu_\xi^2)]^2\}\end{aligned} \tag{7}$$

with $C=(\frac{3}{2})^{3/2}$, and the notation $I_n(\mu^2)$ is defined by[5,7,8]

$$I_n(\mu^2)\equiv\int\frac{d^3p}{(2\pi)^3}(p^2+\mu^2)^{1/2-n} . \tag{8}$$

The relations among $I_n(\mu^2)$, $F_{xy}(\overline{B})$, and $F^{-1}_{xy}(\overline{B})$ ($\overline{B}=\bar{\xi},\overline{W},\overline{Z}$) read

$$F_{xx}(\bar{\xi})=I_0(\mu_\xi^2),\quad F^{-1}_{xx}(\bar{\xi})=I_1(\mu_\xi^2) , \tag{9}$$

$$F_{xx}(\overline{B})=C^3I_0(\mu_B^2),\quad F^{-1}_{xx}(\overline{B})=C^3I_1(\mu_B^2)\quad(\overline{B}=\overline{W},\overline{Z}) , \tag{10}$$

$$G^{ij}_{xy}(\overline{A})=\int\frac{d^3p}{(2\pi)^3}\frac{|p|}{p^2\delta_{ij}-p_ip_j}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} ,$$

and

$$\int_y F_{xy}(\overline{B})F^{-1}_{yz}(\overline{B})=\delta(\mathbf{x}-\mathbf{z})\quad(\overline{B}=\overline{W},\overline{Z},\bar{\xi}) . \tag{11}$$

In Eq. (7), the mass parameters μ_B have been determined by $\partial V_G/\partial\mu_B^2=0$: i.e.,

$$\mu_W^2=g^2CI_1(\mu_W^2)+\frac{g^4C}{g^2+g'^2}I_1(\mu_Z^2)+\frac{g^2g'^2}{g^2+g'^2}G^{11}_{xx}(\overline{A})+\frac{g^2}{4}[\bar{\xi}^2+\tfrac{1}{2}I_1(\mu_\xi^2)] . \tag{12}$$

$$\mu_Z^2=\frac{2g^4C}{g^2+g'^2}I_1(\mu_W^2)+\frac{g^2+g'^2}{4}[\bar{\xi}^2+\tfrac{1}{2}I_1(\mu_\xi^2)] , \tag{13}$$

and

$$\mu_\xi^2=-\sigma_0+\frac{\lambda_0}{2}[\bar{\xi}^2+\tfrac{1}{2}I_1(\mu_\xi^2)]+\tfrac{3}{4}g^2CI_1(\mu_W^2)+\frac{g^2+g'^2}{4}\frac{3}{2}CI_1(\mu_Z^2) . \tag{14}$$

Differentiating V_G with respect to ξ, one is able to see that the symmetric phase is located at $\bar{\xi}_s=0$, while the broken phase is located at

$$\bar{\xi}|^2_{\rm vac}\equiv\bar{\xi}_0^2=\frac{3}{\lambda_0}\mu^2_{\xi_0} . \tag{15}$$

Now we turn to the criterion for existence of the symmetric or broken phase. We write Eq. (14) as

$$\begin{aligned}b\equiv\frac{\bar{\xi}^2}{\mu_\xi^2}=&\frac{2}{\lambda_0}\left[1+\eta\frac{\Lambda^2}{\mu_\xi^2}\right]\\&-\frac{3g^2C}{2\lambda_0\mu_\xi^2}\left[I_1(\mu_W^2)+\frac{g^2+g'^2}{2g^2}I_1(\mu_Z^2)\right]\\&-\frac{I_1(\mu_\xi^2)}{2\mu_\xi^2} ,\end{aligned} \tag{16}$$

where $\eta=\sigma_0/\Lambda^2$. Noticing (15), the existence of the broken phase is equivalent to the solubility of the equation

$$b_{\text{vac}}=\frac{3}{\lambda_0}$$
$$=\frac{2}{\lambda_0}\left[1+\eta\frac{\Lambda^2}{\mu_{\xi_0}^2}\right]$$
$$-\frac{3g^2C}{2\lambda_0\mu_{\xi_0}^2}\left[I_1(\mu_W^2|_{\bar\xi=\bar\xi_0})+\frac{g^2+g'^2}{2g^2}I_1(\mu_Z^2|_{\bar\xi=\bar\xi_0})\right]$$
$$-\frac{I_1(\mu_{\xi_0}^2)}{2\mu_{\xi_0}^2}\ . \tag{17}$$

Similar to Ref. 8, by examining the extremum behavior of the function $b(x)\equiv b(\Lambda/\mu_\xi)$, one sees that there is a critical value of σ_0 or η, say, η_{cr}; Eq. (17) will have no solution for $\mu_{\xi_0}^2$ when $\eta<\eta_{\text{cr}}$: i.e., the broken vacuum cannot exist when $\sigma_0<\eta_{\text{cr}}\Lambda^2$. It is easy to show that

$$\eta_{\text{cr}}\simeq\frac{\lambda_0}{16\pi^2}+\frac{9g^2+3g'^2}{32\pi^2}C\ . \tag{18}$$

III. THE CRITERION FOR STABILITY OF THE BROKEN VACUUM AND THE BOUNDS ON THE HIGGS-BOSON MASS

Now we turn to the detailed analysis of the stability condition for a broken vacuum which will lead to some constraint on the value λ_0/g^2 and then on the mass of the Higgs boson. The condition for stabilizing the broken phase reads

$$\left.\frac{\partial^2V_G}{\partial\bar\xi^2}\right|_{\bar\xi=\bar\xi_0}>0\ . \tag{19}$$

For calculating $\partial^2V_G/\partial\bar\xi^2$, the following expressions are useful:

$$\frac{\partial\mu_Z^2}{\partial\bar\xi}=\frac{4(g^2+g'^2)[2-2Cg^6I_2(\mu_W^2)/(g^2+g'^2)^2+g^2CI_2(\mu_W^2)]}{8\gamma+(\lambda_0\gamma+S)I_2(\mu_\xi^2)}\bar\xi\ , \tag{20}$$

$$\frac{\partial\mu_W^2}{\partial\bar\xi}=\frac{4g^2[2-g^2CI_2(\mu_Z^2)]}{8\gamma+(\lambda_0\gamma+S)I_2(\mu_\xi^2)}\bar\xi\ , \tag{21}$$

$$\frac{\partial\mu_\xi^2}{\partial\bar\xi}=\frac{8(\lambda_0\gamma+S)}{8\gamma+(\lambda_0\gamma+S)I_2(\mu_\xi^2)}\bar\xi \tag{22}$$

with

$$\gamma=2+g^2CI_2(\mu_W^2)-g^8C^2I_2(\mu_W^2)I_2(\mu_Z^2)/(g^2+g'^2)^2\ , \tag{23}$$

$$S=\tfrac{3}{8}(g^2+g'^2)^2\left[\left[\frac{g^4}{(g^2+g'^2)^2}-\frac{1}{4}\right]\times g^2C^2I_2(\mu_W^2)I_2(\mu_Z^2)-\frac{g^4CI_2(\mu_W^2)}{(g^2+g'^2)^2}-\frac{C}{2}I_2(\mu_Z^2)\right]\ . \tag{24}$$

By means of (20)–(22) we get

$$\frac{\partial^2V_G}{\partial\bar\xi^2}=\frac{2\bar\xi^2[4(\lambda_0\gamma+3S)-\lambda_0(\lambda_0\gamma+S)I_2(\mu_\xi^2)]}{3[8\gamma+(\lambda_0\gamma+S)I_2(\mu_\xi^2)]}\ . \tag{25}$$

Aiming at finding some information from (19) and (25), we resort to the approximation

$$g^2+g'^2\sim g^2 \tag{26}$$

which implies that in generating the masses of W and Z bosons, the SU(2) gauge fields play a dominant role whereas the U(1) gauge field only plays a minor one.

Considering the difference of (13) and (12) $\mu_Z^2-\mu_W^2$, which is a nonzero factor and thus can be erased, we obtain after using (26) that

$$N\equiv g^2CI_2(\mu_W^2)\simeq 2+O\left[\frac{1}{I_2(\mu_W^2)}\right]\ . \tag{27}$$

It is reasonable to make a further approximation:

$$I_2(\mu_W^2)\simeq I_2(\mu_Z^2)\simeq I_2(\mu_\xi^2)\simeq I_2(\mu^2)\ . \tag{28}$$

Then from (19) and (25) one finds

$$\tfrac{2}{3}g^2\bar\xi_{\text{vac}}^2\left[\frac{\lambda_0}{g^2}-3.22\right](0.642-\lambda_0/g^2)/(7.16+\lambda_0/g^2)>0\ ; \tag{29}$$

i.e.,

$$0.642<\lambda_0/g^2<3.22\ . \tag{30}$$

On the other hand, there is a well-known relation between m_H^2 and m_W^2 at the tree level:

$$\frac{m_H^2}{m_W^2}=\frac{4}{3}\frac{\lambda_0}{g^2}\ . \tag{31}$$

Hence,

$$0.925m_W<m_H<2.07m_W\ . \tag{32}$$

If $m_W=82$ GeV, one obtains the lower and upper bounds for the mass of the Higgs particle:

$$76<m_H<170\ \text{GeV}\ . \tag{33}$$

IV. AUTONOMOUS GAUGE THEORY

In previous sections, while treating the model described by (1) as a low-energy effective theory, we allow a large but finite cutoff Λ appearing explicitly in our calculation. This section will be devoted to an alternative approach—the so-called autonomous theory, in which one manages to remove the cutoff by setting $\Lambda \to \infty$.

We know that the masses of W and Z fields should be generated at the broken phase and not at the symmetric phase. So we have to renormalize μ_W^2 and μ_Z^2 [as shown in Eqs. (12) and (13)] such that μ_{WR}^2 and μ_{ZR}^2 are zero at symmetric phase. That is to say, all the μ_W^2 and μ_Z^2 in the effective potential (7) should be replaced by μ_{WR}^2 and μ_{ZR}^2; the results read

$$\mu_{WR}^2 = \tfrac{1}{4} g^2 \bar{\xi}^2 - \tfrac{1}{4} g^4 \frac{C\mu_{ZR}^2}{g^2+g'^2}\left[I_2(\mu_{ZR}^2) + \frac{1}{4\pi^2}\right] - \frac{g^2 C}{2}\mu_{WR}^2 \left[I_2(\mu_{WR}^2) + \frac{1}{4\pi^2}\right]$$
$$+ \frac{g^2}{8}\left\{ \tfrac{1}{2}(\mu_{\xi_0}^2 - \mu_\xi^2) I_2(\mu_\xi^2) - \frac{\mu_\xi^2}{8\pi^2}\left[\frac{\mu_{\xi_0}^2}{\mu_\xi^2}\ln\frac{\mu_{\xi_0}^2}{\mu_\xi^2} - \left[\frac{\mu_{\xi_0}^2}{\mu_\xi^2} - 1\right]\right]\right\}, \quad (34)$$

$$\mu_{ZR}^2 = \frac{g^2+g'^2}{16}\left\{ (\mu_{\xi_0}^2 - \mu_\xi^2) I_2(\mu_\xi^2) - \frac{\mu_\xi^2}{4\pi^2}\left[\frac{\mu_{\xi_0}^2}{\mu_\xi^2} - \left[\frac{\mu_{\xi_0}^2}{\mu_\xi^2} - 1\right]\right]\right\}$$
$$+ \frac{1}{4}(g^2+g'^2)\bar{\xi}^2 - g^4 \frac{C\mu_{WR}^2}{g^2+g'^2}\left[I_2(\mu_{WR}^2) + \frac{1}{4\pi^2}\right], \quad (35)$$

and

$$\mu_\xi^2 = -\sigma_0 + \frac{\lambda_0}{2}[\bar{\xi}^2 + \tfrac{1}{2} I_1(\mu_\xi^2)] + \tfrac{3}{4} g^2 C I_1(\mu_{WR}^2)$$
$$+ \tfrac{3}{8}(g^2+g'^2) C I_1(\mu_{ZR}^2) \,. \quad (36)$$

From (34) and (35) we know that g^2, g'^2, and λ_0 should be the order of $1/I_2(\mu_{WR}^2)$ to guarantee the finite masses of the W and Z particles. On the other hand, from Eq. (15), $\mu_{\xi_0}^2 = (\lambda_0/3)\bar{\xi}_0^2$, in order to prevent the value of the Higgs field from tending to infinity when removing the cutoff ($\Lambda \to \infty$), a renormalization procedure of $\bar{\xi}$ is necessary, say, $\lambda_0 \bar{\xi} \sim \Phi$. Actually, we will take

$$\lambda_0 = \frac{a}{I_2(\mu)}, \quad g^2 = \frac{b}{I_2(\mu)}, \quad g'^2 = \frac{b'}{I_2(\mu)},$$
$$\bar{\xi}^2 = \tfrac{1}{2} I_2(\mu)\Phi^2, \quad (37)$$

and

$$-\sigma_0 + \frac{\lambda_0}{4} I_1(0) + \tfrac{3}{4} g^2 C \left[I_1(0) + \frac{g^2+g'^2}{2g^2} I_1(0)\right]$$
$$= \sigma_1 - \frac{\sigma_2}{I_2(\mu)} \,. \quad (38)$$

The constants a, b, b', σ_1 and σ_2 in (37) and (38) are adjusted such that $V_G(\Phi)$ remains finite when $\Lambda \to \infty$. Once this is done, we can simply recast the $\partial V_G(\Phi)/\partial\Phi$ into the form

$$\frac{\partial V_G(\Phi)}{\partial \Phi} = \frac{I_2(\mu)}{2}\left[\mu_\xi^2 - \frac{a}{6}\Phi^2 \right]\Phi \,. \quad (39)$$

Substituting (37) and (38) into (34)–(36), one has

$$\mu_{ZR}^2 = \frac{b+b'}{8}\Phi^2 - \frac{b^2 C}{b+b'}\mu_{WR}^2 + \frac{b+b'}{16}(\mu_{\xi_0}^2 - \mu_\xi^2)$$
$$+ \frac{1}{I_2(\mu)}\left[\frac{b^2 C \mu_{WR}^2}{4\pi^2(b+b')}\left[\ln\frac{\mu_{WR}^2}{\mu^2} - 1\right] - \frac{b+b'}{64\pi^2}\left[\mu_{\xi_0}^2 \ln\frac{\mu_{\xi_0}^2}{\mu^2} + \mu_\xi^2 \ln\frac{\mu^2}{\mu_\xi^2} + (\mu_\xi^2 - \mu_{\xi_0}^2)\right]\right], \quad (40)$$

$$\mu_{WR}^2 = \frac{b}{8}\Phi^2 - \frac{b^2 C}{2(b+b')}\mu_{ZR}^2 - \frac{bC}{2}\mu_{WR}^2 + \frac{b}{16}(\mu_{\xi_0}^2 - \mu_\xi^2)$$
$$+ \frac{1}{I_2(\mu)}\left[\frac{b^2 C \mu_{ZR}^2}{8\pi^2(b+b')}\left[\ln\frac{\mu_{ZR}^2}{\mu^2} - 1\right] + \frac{bC\mu_{WR}^2}{8\pi^2}\left[\ln\frac{\mu_{WR}^2}{\mu^2} - 1\right] - \frac{b}{64\pi^2}\left[\mu_{\xi_0}^2 \ln\frac{\mu_{\xi_0}^2}{\mu^2} + \mu_\xi^2 \ln\frac{\mu^2}{\mu_\xi^2} + (\mu_\xi^2 - \mu_{\xi_0}^2)\right]\right], \quad (41)$$

and

$$\mu_\xi^2=\frac{8}{8+a}\left[\sigma_1+\frac{a}{4}\Phi^2-\frac{3bC}{8}\left[\mu_{WR}^2+\frac{b+b'}{2b}\mu_{ZR}^2\right]\right]$$
$$+\frac{8}{8+a}\frac{1}{I_2(\mu)}\left\{-\sigma_2+\frac{a}{32\pi^2}\mu_\xi^2\left[\ln\frac{\mu_\xi^2}{\mu^2}-1\right]\right.$$
$$\left.+\frac{3bC}{32\pi^2}\left[\mu_{WR}^2\left[\ln\frac{\mu_{WR}^2}{\mu}-1\right]+\frac{b+b'}{2b}\mu_{ZR}^2\left[\ln\frac{\mu_{ZR}^2}{\mu^2}-1\right]\right]\right\}. \tag{42}$$

One can see from (39) and (42) that as $I_2(\mu)\sim\ln(\Lambda/\mu)$ the finite part of μ_ξ^2 will provide an infinite contribution to $\partial V_G/\partial\Phi$ and thereby to V_G, whereas the infinitesimal part of order $1/I_2(\mu)$ in μ_ξ^2 will provide a finite one. The infinitesimal part of μ_ξ^2 of the order of $1/I_2^2(\mu)$ only makes a contribution to $\partial V_G/\partial\Phi$ of the order of $1/I_2(\mu)$ and thus could be neglected. Keeping this in mind and performing the tedious calculation, we arrive at

$$\mu_\xi^2=\frac{a}{6}\Phi^2+O\left[\frac{1}{I_2(\mu)}\right], \tag{43}$$

$$\mu_{WR}^2=\frac{b(2-bC)(b+b')^2(12-a)\Phi^2}{96[(b+b')^2(2+bC)-b^4C^2]}+O\left[\frac{1}{I_2(\mu)}\right]$$
$$\equiv f_W\Phi^2+O\left[\frac{1}{I_2(\mu)}\right], \tag{44}$$

$$\mu_{ZR}^2=\frac{(b+b')[(b+b')^2(2+bC)-2b^3C](12-a)}{96[(b+b')^2(2+bC)-b^4C^2]}\Phi^2+O\left[\frac{1}{I_2(\mu)}\right]$$
$$\equiv f_Z\Phi^2+O\left[\frac{1}{I_2(\mu)}\right] \tag{45}$$

together with the Gaussian effective potential for Φ,

$$V_G=V_{\text{vac}}-\frac{\sigma_3}{2}\Phi^2+\frac{\sigma_3}{4}\frac{\Phi^4}{v^2}+\frac{K}{4}\Phi^4\left[\ln\frac{\Phi^2}{v^2}-\frac{1}{2}\right], \tag{46}$$

where

$$\sigma_3=\frac{8\sigma_2}{16+2a-3bC[f_W+(b+b')f_Z/(2b)]} \tag{47}$$

and

$$K=\pi^{-2}\{16+2a-3bC[f_W+(b+b')f_Z/(2b)]\}^{-1}\left\{\frac{1}{4}\left[\frac{a}{6}+[3bCf_W+\tfrac{3}{2}(b+b')Cf_Z]\left[1-\frac{a}{12}\right]\right]\right.$$
$$-3bCf_W\left[\left[\frac{bCf_Z}{b+b'}+Cf_W\right]\left[1-\frac{a}{12}\right]+\frac{a}{48}\right]$$
$$\left.-3Cf_Z\left[\frac{b^2C}{b+b'}f_W\left[1-\frac{a}{12}\right]+\frac{b+b'}{96}a\right]\right\}. \tag{48}$$

In (46), v is the vacuum expectation value of Φ and the divergent parts have been canceled by the conditions

$$\sigma_1=\mu_{\xi_0}^2=0 \tag{49}$$

and

$$\frac{4a-6bC[f_W+(b+b')f_Z/(2b)]}{2(8+a)-3bC[f_W+(b+b')f_Z/(2b)]}=\frac{a}{6}. \tag{50}$$

It is easy to verify that, for $b=b'=0$, i.e., when removing the gauge coupling [see (37)], Eq. (46) reduces to the case of pure $\lambda\phi^4$ model:

$$V_G=V_{\text{vac}}-\frac{\sigma_3}{2}\Phi^2+\frac{\sigma_3}{4}\frac{\Phi^4}{v^2}+\frac{\Phi^4}{144\pi^2}\left[\ln\frac{\Phi^2}{v^2}-\tfrac{1}{2}\right]. \tag{51}$$

Furthermore, if $\sigma_3=0$, i.e., for massless $\lambda\phi^2$ theory, Eq. (51) is equivalent to the well-known result first derived by Coleman and Weinberg.[14]

V. SUMMARY AND DISCUSSION

(1) We have generalized the GEP method to the SU(2)×U(1) gauge field theory. Just as in the case of pure $\lambda\phi^4$ theory,[8] we get a broken phase only when $\sigma_0>\eta_{\text{cr}}\Lambda^2$. In addition to this, the stability condition of this vacuum leads to the lower and upper bounds on the mass of the Higgs boson: $0.925m_W<m_H<2.07m_W$ (see also Ref. 7) or $76<m_H<170$ GeV for $m_W=82$ GeV. Besides (28), an extra approximation (26) has been used. The upper bound of the Higgs particle has been studied by various authors. It ranges from 125 GeV to 1 TeV. Some recent typical values based on the triviality problem of $\lambda\Phi^4$ are 640 GeV (Ref. 15),

$(1/\sqrt{2})(3+\sqrt{6})^{1/2}m_W=135$ GeV (Callaway and coworkers[10]) and 125 GeV if $m_t<80$ GeV, or $65<m_H<175$ GeV if $80<m_t<168$ GeV (m_t is the mass of the top quark).[11,12] Our lower and upper bounds are near the results in Refs. 11 and 12 for $80<m_t<168$ GeV.

In our point of view, while the bounds on the Higgs-boson mass are constrainted by the vacuum stability, for a stability analysis in quantum field theory, the Λ dependence is inevitable whereas the perturbative theory may lose its way. If we agree to keep a finite cutoff Λ, then the trivality problem renders the allowed region of λ/g^2 quite narrow. As an ambitious speculation, it could even be determined completely.[16]

If in Eq. (19), we take the weak gauge coupling limit $g^2, g^2 \ll \lambda$ instead of Eq. (26) ($g^2+g'^2 \sim g^2$), then we can only get the cutoff-dependent upper bound on the mass of the Higgs boson similar to that in Ref. 15:

$$\overline{M}'_H \simeq 832\chi^{-1/2} \text{ GeV} \quad (\Lambda/\langle\phi\rangle \equiv 10^{\chi}) . \tag{52}$$

In Ref. 15, a low cutoff $\Lambda \simeq 2\pi\overline{M}_H$ is taken, then a rather high upper bound $\overline{M}_H=640$ GeV is found. Similarly, our expression (52) will give $\overline{M}'_H \simeq 700$ GeV when $\Lambda \sim 2\pi\overline{M}'_H$, whereas $\overline{M}'_H=170$ GeV would lead to a much higher value of Λ which exceeds the Planck scale.

(2) Furthermore, we consider the effects of the fermions on the bounds of the mass of the Higgs boson. The stability condition for the broken vacuum, Eq. (29), will then be modified to

$$y \equiv \left[\frac{\lambda_0}{g^2}-3.22\right]\left[0.642-\frac{\lambda_0}{g^2}\right]\Big/\left[7.16+\frac{\lambda_0}{g^2}\right] > y_1 , \tag{53}$$

where y_1 is a small positive constant depending on the mass of the top quark. So the fermion effect makes the interval between the lower and upper bounds of the Higgs-boson mass shrink more than estimated above.

(3) We have also established the autonomous theory for $SU(2)\times U(1)$ gauge theory. It is worthwhile to mention that all the arguments of I_1 and I_2 in Eqs. (37) and (38) may all be different constants, but all the results except (49) are independent of them. Since after renormalization the parameters a, b, and b' are connected by (50), then two of them together with V^2 and σ_3 constitute four free parameters in the automomous theory as the replacement of four classical constants λ_0, g^2, g'^2, and σ_0 before renormalization. Notice that, however, there still exists some constraint. If we demand $b>0$ in (37) as it should be, a too large parameter a is not allowed, otherwise the condition $\mu^2_{WR}>0$ would be spoiled. Indeed, if $b'=0$ [SU(2) gauge theory], Eq. (44) with (50) becomes

$$\mu^2_{WR}=\frac{4-a}{27bC}\Phi^2 ,$$

so $a<4$ is required. The case $a=4$ corresponds to the pure $\lambda\phi^4$ theory.

(4) In pure $\lambda\phi^4$ theory, there exists another type of phase in the so-called "precarious theory"[3] where the bare coupling constant λ_0 is infinitesimal and negative, say, $\lambda_0=-8/I_2(\mu)$. We could also find such a kind of phase in $SU(2)\times U(1)$ gauge theory if $a=-8$ together with b and b' satisfying

$$(b+b')^2\{b^2(2-bC)+\tfrac{1}{2}[(b+b')^2(2+bC)-2b^3C]\}=0 . \tag{54}$$

However, we believe that this phase is neither reliable nor capable of generating the mass of gauge fields, so we will not discuss it any longer.

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