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Masses of Skyrmions

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When realistic enough for the description of mesons, Skyrme-type effective Lagrangians proposed so far predict masses that are too high for the nucleon. We attempt to solve this problem by investigating the effects of the quartic term originated by the ϵ meson ($J^{PC}=0^{++}, I=0$) when stabilized by a sixth-order term. The possibility of new types of solutions for the Euler-Lagrange equations giving rise to a phase boundary within the soliton is explored. They are found to have the nice property of lowering significantly the soliton mass and hence the baryon masses.

I. INTRODUCTION

The idea of associating the soliton solutions of effective chiral field theories of mesons with baryons has had some success over the past few years.[1] However, there is a major shortcoming. The models proposed so far, when they are made realistic enough for mesons, fail to reproduce the nucleon mass. For example, in a previous work[2] we have constructed an effective Lagrangian, the parameters of which are determined by fitting the low-energy meson observables. The soliton solutions found, when identified as baryons, satisfactorily predict their static properties except for the masses which are about 50% too large. Our Lagrangian contains the π, ρ, A_1, ω, and ϵ mesons. We believe that the ϵ meson, although it is a broad resonance, should be included in order to account for the pion-pion S-wave attraction. Moreover this is the only meson to have the desirable property of lowering the baryon masses. The counterpart of this good feature is that it can make the soliton unstable, the energy density is not always positive definite. To remove this unsatisfactory feature, we here introduce a stabilizing term and we investigate its effects. In the course of this study, we discover the possibility of new types of solution for the Euler-Lagrange equations which give rise to a discontinuity in the energy density. These new types of solutions have the nice property of lowering significantly the soliton mass and hence the baryon masses.

II. THE MODEL

Although we could start with the comprehensive Lagrangian of Ref. 2 we here adopt for simplicity the approximation obtained by eliminating the heavy-meson fields in their large-mass limit. In this approximation, the Lagrangian is of the form[3]

$$\mathcal{L}_0=\frac{F_\pi^2}{16}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)+\frac{1}{32e^2}\mathrm{Tr}[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2 +\frac{\gamma}{8e^2}[\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)]^2-\frac{b}{2F_\pi^2}B_\mu B^\mu \tag{1}$$

with

$$B_\mu(x)=\frac{\epsilon_{\mu\alpha\beta\gamma}}{24\pi^2}\mathrm{Tr}[(U^\dagger\partial^\alpha U)(U^\dagger\partial^\beta U)(U^\dagger\partial^\gamma U)]$$

and

$$U=e^{2i\tau\cdot\phi/F_\pi}$$

$\phi(x)$ being the pion field.

It should be noted that this Lagrangian can be regarded as the first terms of an expansion in powers of the derivatives of the pion field. It contains the quadratic term, the two quartic terms, and one of the possible sixth-order terms. The term $\gamma[\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)]^2$ destabilizes the soliton, an effect which cannot be compensated by the other terms. Using the hedgehog ansatz $U=e^{i\tau\cdot\mathbf{r}\theta(r)}$ the energy functional can be written as

$$\begin{aligned}E_0=4\pi\int_0^{+\infty}r^2\,dr\Bigg\{&\frac{F_\pi^2}{4}\left[\frac{1}{2}\left[\frac{d\theta}{dr}\right]^2+\frac{\sin^2\theta}{r^2}\right]\\&+\frac{1}{2e^2}\frac{\sin^2\theta}{r^2}\left[2\left[\frac{d\theta}{dr}\right]^2+\frac{\sin^2\theta}{r^2}\right]\\&-\frac{2\gamma}{e^2}\left[\frac{1}{2}\left[\frac{d\theta}{dr}\right]^2+\frac{\sin^2\theta}{r^2}\right]^2\\&+\frac{b}{8\pi^4F_\pi^2}\left[\frac{d\theta}{dr}\right]^2\frac{\sin^4\theta}{r^4}\Bigg\}\,.\end{aligned}\tag{2}$$

The instability is apparent in the γ term which is negative but contains the highest power of $(d\theta/dr)^2$. Thus, functions $\theta(r)$ which have rapid oscillations can make the mass negative. Strictly speaking there is no global minimum, although for small values of γ there are local minima. Of course, the effective Lagrangian is inadequate to cope with highly oscillating functions but it can be anticipated that higher-order terms in the derivative expansion, if properly included, would make the local minimum a good approximation to the true minimum.

We have *a priori* very little indication of the form of a stabilizing term to be added to $\mathcal{L}_0$. However, it must be positive and diverge faster than $(d\theta/dr)^4$ when $d\theta/dr$ becomes large. In the framework of the derivative expansion the lowest-order and simplest term which satisfies these criteria is a sixth-order term of the form

$$\mathcal{L}_{\mathrm{st}}=\frac{s^2}{32F_\pi^2}[\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)]^3 . \tag{3}$$

Its contribution to the soliton energy is

$$E_{\mathrm{st}}=\frac{8\pi s^2}{F_\pi^2}\int_0^\infty r^2 dr\left[\frac{1}{2}\left[\frac{d\theta}{dr}\right]^2+\frac{\sin^2\theta}{r^2}\right]^3 . \tag{4}$$

The full Lagrangian we consider from now on is

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\mathrm{st}} \tag{5}$$

and the soliton energy

$$E=E_0+E_{\mathrm{st}} . \tag{6}$$

The parameters appearing in Eq. (1) are related to those of Ref. 2 by

$$e=\sqrt{2}g,\quad b=(\beta_\omega F_\pi/m_\omega)^2 \tag{7}$$

which yield $e=5.68$, $b=4.89$ with $F_\pi=186$ MeV, $\beta_\omega=9.3$, $m_\omega=782.6$ MeV, and $g=4.1$ as in Ref. 2.

The parameter γ is also related to the meson parameters of Ref. 2. We however, consider here both γ and s to be free parameters but with the constraint

$$\gamma<se^2/\sqrt{2} \tag{8}$$

which ensures that the energy density is positive definite. This constraint is sufficient but not necessary since it was obtained by neglecting in Eq. (6) the terms with coefficients $1/e^2$ and b/F_π^2 which have positive contributions to the energy. The soliton mass is given by the chiral angle $\theta(r)$ which makes this energy functional a minimum. The masses and static properties of baryons are calculated by introducing rotational dynamics as in Ref. 4. The contribution of different terms of Eq. (2) to the moment of inertia can be found in Ref. 3, that of the stabilizer, Eq. (3), is

$$\lambda_{\mathrm{st}}=\frac{16\pi s^2}{F_\pi^2}\int_0^\infty r^2\sin^2\theta\left[\frac{1}{2}\left[\frac{d\theta}{dr}\right]^2+\frac{\sin^2\theta}{r^2}\right]^2 dr . \tag{9}$$

III. SMOOTH SOLUTIONS OF THE EULER-LAGRANGE EQUATIONS

Holding the parameters $e=5.68$, $b=4.89$ as given in the previous section we find, with $\gamma=0.50$ and $s=0.031$, the results listed in column 1 of Table I. The value of s was chosen so that the the contribution of the stabilizer (a sixth-order term) is no more than 25% of that due to the quadratic term and γ was given by its maximum value that yielded a smooth solution for $\theta(r)$. For comparison, we show in column 2 of the same table the results obtained with the same values for e and b but without the stabilizer ($s=0$), γ being again taken at its maximum value, $\gamma=0.17$. As can be seen, the presence of the stabilizer leads to a significant improvement for the nucleon mass. At the same time, it is satisfying to know that with the stabilizer the soliton mass corresponds to a global minimum of the energy. However, the improvement is not important enough to bring the masses into agreement with experimental data.

IV. BROKEN SOLUTIONS OF THE EULER-LAGRANGE EQUATIONS

As is explained below, during our search for a minimizing function $\theta(r)$, we realized that $d\theta/dr$ can be discontinuous without any harm to the physical observables. The chiral angle $\theta(r)$ which minimizes the energy functional

$$E=4\pi\int_0^\infty \rho(\theta,\dot\theta,r)dr \tag{10}$$

must be continuous but need not have a continuous

TABLE I. Static properties of baryons. The different columns correspond to the different sets of the parameter values F_π, e, b, s, and γ.

	1	2	3	Expt.
F_π (MeV)	186	186	186	186
e	5.68	5.68	5.68	
b	4.89	4.89	4.89	
s	0.031	0.0	0.031	
γ	0.50	0.17	0.64	
M (MeV)	1250	1394	929	
M_N (MeV)	1294	1440	991	939
M_Δ (MeV)	1471	1623	1240	1232
$\mu_p(e/2M_N)$	2.86	2.74	2.18	2.79
$\mu_n(e/2M_N)$	−2.44	−2.38	−1.59	−1.91
$\lvert\mu_p/\mu_n\rvert$	1.17	1.15	1.37	1.46
$\langle r^2\rangle_{I=0}^{1/2}$ (fm)	0.67	0.60	0.66	0.72
$\langle r^2\rangle_{M;I=0}^{1/2}$ (fm)	0.86	0.82	0.76	0.81
g_A	1.06	1.03	0.41	1.23

derivative $\dot{\theta}(r)$ $[\dot{\theta}(r)=d\theta/dr]$. The possibility of discontinuities in the solutions of minimization problems of this type has been remarked upon in the literature[5] but the conditions that must be satisfied at points of discontinuity are not well known. In the following we derive these conditions and describe the method that we have used to implement them.

First, making a small and smooth variation of a function $\theta(r)$ that minimizes E yields the Euler-Lagrange equation

$$\frac{\partial\rho}{\partial\theta}=\frac{d}{dr}\frac{\partial\rho}{\partial\dot{\theta}} \; . \tag{11}$$

If this equation has a solution $\theta(r)$ broken (i.e., continuous but with a discontinuous derivative) at a point $r=r_B$ it is still necessary that $d\rho/\partial\dot{\theta}$ be continuous at r_B, otherwise a Dirac δ function $\delta(r-r_B)$ would appear on the right-hand side and although the left-hand side can be discontinuous it will not have a compensating δ function and the equation will not be satisfied.

When comtemplating broken solutions the condition of continuity $d\rho/\partial\dot{\theta}$ is not sufficient to guarantee a stationary value of E with respect to small variations. It is also necessary to consider small variations $\delta\theta$ that move the point of break, say from r_B to $r_B+\delta r_B$. For such variations, $\delta\dot{\theta}$ is not small in the interval δr_B. Consider $\theta_1(r)$ to have a break at $r=r_B$ and $\theta_2(r)$ a break at $r=r_B+\delta r_B$, $\delta\theta=\theta_2-\theta_1$ is infinitesimal for all r, and, $\delta\dot{\theta}=\dot{\theta}_2-\dot{\theta}_1$ is also infinitesimal except for $r_B<r<r_B+\delta r_B$. If θ_1 satisfies the Euler-Lagrange equation then the infinitesimal change of E is, to first order,

$$\delta E=\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}\delta\theta(r_B)-\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+}\delta\theta(r_B+\delta r_B) \\ +(\rho_{-}-\rho_{+})\delta r_B \; . \tag{12}$$

The $-$ $(+)$ sign indicates the functions evaluated with $\theta(r)$ on the left- (right-)hand side of the break.

Now

$$\delta\theta(r_B+\delta r_B)=\theta_2(r_B+\delta r_B)-\theta_1(r_B+\delta r_B) \\ =\theta_2(r_B)+\delta r_B\dot{\theta}_{-}(r_B)-\theta_1(r_B)-\delta r_B\dot{\theta}_{+}(r_B) \tag{13}$$

to first order, and

$$\delta\theta(r_B+\delta r_B)=\delta\theta+\delta r_B(\dot{\theta}_{-}-\dot{\theta}_{+}) \; . \tag{14}$$

Hence,

$$\delta E=\left[\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}-\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+}\right]\delta\theta \\ +\left[\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+}(\dot{\theta}_{+}-\dot{\theta}_{-})+\rho_{-}-\rho_{+}\right]\delta r_B \; . \tag{15}$$

If δr_B is taken as zero, then for a stationary value of E we recover the condition

$$\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}=\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+} , \tag{16}$$

that is the continuity of $\partial\rho/\partial\dot{\theta}$. If this condition is satisfied Eq. (15) becomes

$$\delta E=\left[\left[\rho-\dot{\theta}\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}-\left[\rho-\dot{\theta}\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+}\right]\delta r_B \tag{17}$$

and a stationary value implies the second condition

$$\left[\rho-\dot{\theta}\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}=\left[\rho-\dot{\theta}\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+} , \tag{18}$$

that is the continuity of $\rho-\dot{\theta}\,\partial\rho/\partial\dot{\theta}$ at the break point.

As an application of our techniques we consider the parameters $e=5.68$, $b=4.89$, and $s=0.031$ as in column 1 of Table I but with varying γ. The stability condition (8) is satisfied for $\gamma\leq 0.70$. For γ larger than 0.5, the maximum value that yields a continuous solution, we observe that solutions still exist but with a discontinuity in $d\theta/dr$ and those with a break are found as follows. The equation was first integrated out from $r=0$ with the boundary values $\theta(0)=\pi$ and $\dot{\theta}(0)=D_1$ a trial number. The equation was then integrated in from a large $r=R$ where we took $\theta(R)$ to obey the asymptotic condition $\dot{\theta}(R)=2\theta(R)/R$ and $\dot{\theta}(R)=D_2$ taken as another trial number. Values of D_1 and D_2 for which these two solutions meet give a possible continuous solution but broken at the meeting point. To satisfy the other continuity conditions we computed

$$F(D_1,D_2)=\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}-\left[\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+} \tag{19}$$

and

$$G(D_1,D_2)=\left[\rho-\dot{\theta}\frac{\partial\rho}{\partial\dot{\theta}}\right]_{-}-\left[\rho-\dot{\theta}\frac{\partial\rho}{\partial\dot{\theta}}\right]_{+} \tag{20}$$

at the meeting point. The two conditions $F=0$ and $G=0$ were sufficient to fix D_1 and D_2 corresponding to a solution with a minimum mass. We then checked that the mass was in fact a true minimum by verifying that the equipartition relation which results from the minimization of the energy E was satisfied.

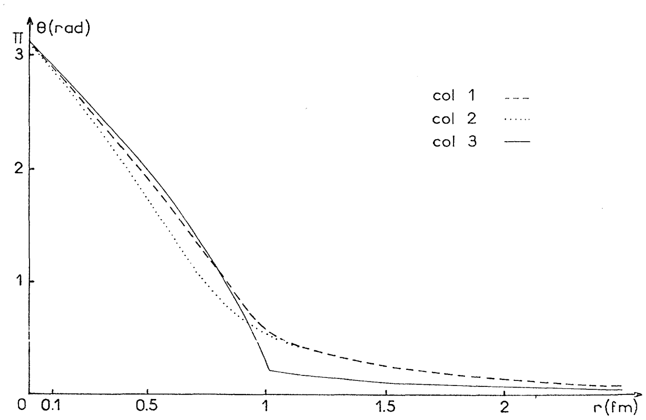

FIG. 1. Profiles of the chiral angle $\theta(r)$ corresponding to the different parameter sets used in the different columns of Table I.

We were thus able to find solutions for $0.50<\gamma<0.70$. No broken solutions were found for $\gamma<0.50$ and no solution at all for $\gamma>0.70$. As these broken solutions correspond to larger values of γ, they provide, as expected, lower values for the soliton mass. In column 3 we show the results obtained with $\gamma=0.64$ corresponding to the broken solution $\theta(r)$ shown in Fig. 1. The major success of this solution is that it gives good values for the nucleon and delta masses without any modification of the values of F_π, e, and b as given by the meson observables. However the other static properties of the nucleon and especially the axial-vector coupling constant g_A are less satisfactory.

In an apparently different context, it was shown[6] that scalar fields can be added to the Skyrme Lagrangian, which can be simply identified with the scale anomaly of QCD. Different models and solutions have been investigated in that reference, and it is interesting to note that, in their minimal model with only one scalar field, the solution referred to as the "deep bag" is able to give good baryon masses using the experimental value for F_π. This solution has several similarities with our model. First, the shape of the chiral angle $\theta(r)$, although not presenting a break like ours, does present a rapid fall in the region 0.8–1.1 fm giving a small "tail" like our curve (column 3). Second, their predictions for the mean-square radii of the nucleon and for g_A are very close to ours.

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[1]For an up-to-date list of references, see *Chiral Solitons,* edited by K. F. Liu (World Scientific, Singapore, 1987).

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