Field theory and the nonrelativistic quark model: A parametrization of the baryon magnetic moments and masses

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We show that the results of calculations of physical quantities (e.g., the magnetic moments and the masses) in a relativistic field theory can be parametrized in a way typical of the nonrelativistic quark model (NRQM). For a relativistic field theory that, as QCD, satisfies the two properties that (a) the electromagnetic current is carried only by the quarks and (b) flavor breaking is due only to the mass difference between the quarks, the most general expression of the magnetic moments of the baryon octet contains ten different types of terms and therefore ten parameters. If one neglects terms that break flavor at an order higher than the first, the ten terms reduce to seven and these are determined from the seven known magnetic moments. We find that (1) among the seven parameters determined by fitting exactly the data, the two parameters characteristic of the simplest NRQM description are indeed the largest ones (their values, in spite of the presence of five more parameters, are almost unchanged with respect to the NRQM), and (2) the $\Sigma^0\Lambda\gamma$ rate (to first order in flavor breaking) is related to the magnetic moments of the octet baryons; the prediction is consistent with the present data on the rate. For the masses, in the flavor-breaking approximation to first order, we obtain a five-parameter formula containing the Gell-Mann–Okubo relation of the octet and the two equal-spacing relations of the decuplet. Comparing this general mass formula with that obtained with the NRQM (with the potential between quarks used by De Rújula, Georgi, and Glashow) it turns out that (1) one of the above five parameters does not intervene in the NRQM (it happens to be by far the smallest among the five) and (2) noting that in the NRQM the ratio between two of our parameters is related to the mass difference Δm between the λ and $\mathcal{P}$ quarks, the ratio $\Delta m/m_\lambda$ determined in this way is 0.31; that obtained from the magnetic moments is 0.35. These results, and others on the $M1$ and $E2$ $\Delta\to P\gamma$ transition, show that the NRQM provides an approximation to the exact solution correct to $\simeq 15\%$.

I. INTRODUCTION

The nonrelativistic quark model[1] (NRQM) gives a fair fit of the magnetic moments of the baryons (Fig. 1).

The fit is based on assigning to the constituent quarks $\mathcal{P}$ and $\mathcal{N}$ magnetic moments $+\frac{2}{3}\mu\sigma^{\mathcal{P}}$ and $-\frac{1}{3}\mu\sigma^{\mathcal{N}}$ (in the same ratio as the charges) and to the strange quark λ a magnetic moment $-(a/3)\mu\sigma^{\lambda}$ where $a<1$ may be due to the λ quark mass being larger than that of $\mathcal{P}$ and $\mathcal{N}$; the magnetic moment operator of a baryon is then written as the following two-parameter expression:

$$\mathbf{M}=\mu\sum(\tfrac{2}{3}\boldsymbol{\sigma}^{\mathcal{P}}-\tfrac{1}{3}\boldsymbol{\sigma}^{\mathcal{N}}-\tfrac{1}{3}\boldsymbol{\sigma}^{\lambda})+\frac{A}{3}\sum\boldsymbol{\sigma}^{\lambda}\ . \tag{1}$$

We have used in (1), instead of a, the parameter $A=\mu(1-a)$ to separate (last term) the effect of flavor violation by the λ quark. To obtain the magnetic moments of the baryons of the octet one calculates the expectation value of (1) using the wave functions of the NRQM; the spin-flavor factor of these functions has the SU(6) structure; this is a consequence of the orbital angular momentum $L=0$ and space symmetrical wave function in the NRQM, not an additional assumption. Clearly the magnetic moments from (1) for the octet baryons depend only on the two parameters μ and A (or, if one prefers μ and a); the fit in Fig. 1 is with $\mu=2.79$, $a=0.65$ (Ref. 2).

The problem is to compare expression (1) (used in conjunction with the $L=0$ NRQM wave functions) with what one gets from a relativistic field theory of quarks and gluons. A calculation of the magnetic moments from field theory, say, from QCD, is hard (QCD lattice results are still preliminary). In this paper we shall (a) give a general procedure to parametrize in a relativistic field theory the results of the calculations of physical quantities (e.g., the magnetic moments or the masses) in a way leading to the "nuclear physics" expressions typical of the NRQM, and (b) show that if the underlying field theory satisfies two properties, these parametrizations become rather simple. The two properties are (1) that the electromagnetic current is carried only by the quarks and (2) that the Lagrangian is flavor independent except for the mass terms; QCD satisfies these properties, although they are more general than QCD. These properties are sufficient to get the simple results presented in the following; how necessary they are will not be explored here.

For a theory with the above properties, the most general expression of the octet-baryon magnetic moments turns out to be the sum of ten terms. If the flavor breaking is taken into account only to first order, these reduce to seven. In terms of them we get the following expression for the magnetic moments of the octet baryons [in

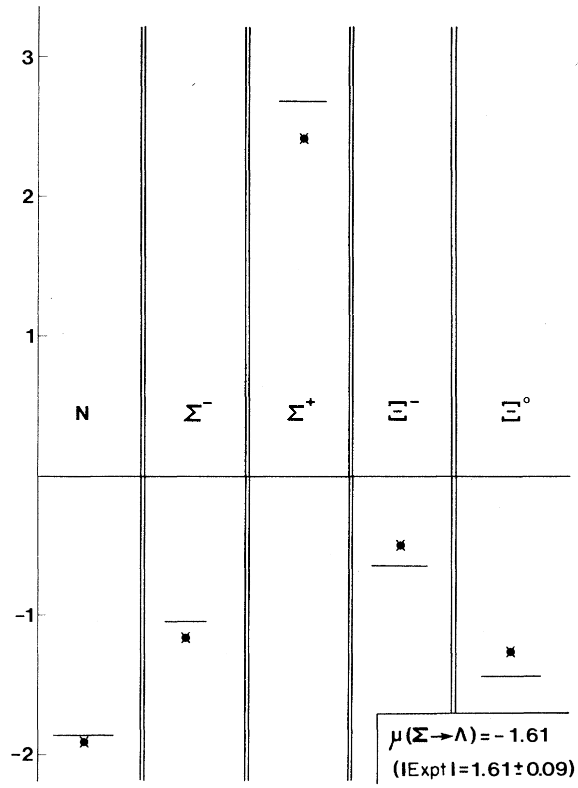

FIG. 1. The measured magnetic moments of the baryons compared with the values (solid lines) calculated with the simple two-parameter formula (1) of the NRQM with $\mu=2.793$ and $A=0.96$ (that is $a=0.65$) having used as input only the proton $(+2.793)$ and Λ $(=-0.613\pm0.004)$ magnetic moments. [Calculated and (measured) values are neutron $=-1.86$ (-1.913); $\Sigma^-=-1.04$ (-1.16 ± 0.025); $\Sigma^+=+2.68$ $(+2.42\pm0.05)$; $\Xi^-=-0.50$ (-0.65 ± 0.03); $\Xi^0=-1.43$ (-1.25 ± 0.014); $\mu(\Sigma\to\Lambda)=-1.61$ $(|\text{expt}|=1.61\pm0.09)$. The experimental value 2.42 ± 0.05 for Σ^+ is the average of two measurements, one giving 2.38 and the other 2.48, both with stated errors near ±0.02 (the calculations in Sec. VII of the text have been done separately for each of the above two values); the experimental value of -0.65 ± 0.03 given above for Ξ^- includes the recent preliminary data of C. Newsome *et al.* [Fermilab Report No. E761 (unpublished)] and of K. Johns *et al.* [Fermilab Report No. E756 (unpublished)] quoted in Ref. 2; otherwise it should be -0.69 ± 0.04.]

(2), S=strangeness, Q=charge, $\mathbf{J}$=total angular momentum]:

$$\mathbf{M}=(\mu+KS)\sum\left(\tfrac{2}{3}\sigma^{\mathcal{P}}-\tfrac{1}{3}\sigma^{\mathcal{N}}-\tfrac{1}{3}\sigma^{\lambda}\right)+\frac{A}{3}\sum\sigma^{\lambda}+F(2\mathbf{J})Q+H(2\mathbf{J})S+\frac{L}{3}Q\sum\sigma^{\lambda}+G(2\mathbf{J})QS\ . \tag{2}$$

A fit with (2) of the magnetic moments of the seven baryons $P,N,\Lambda,\Sigma^{+,-},\Xi^{0,-}$ produces (Sec. VII) the following values for μ, A,K,H,G,L (all in proton magnetons): $\mu=2.869$, $A=+1.005$, $K=+0.289$, $F=-0.076$, $H=+0.086$, $G=-0.15$, $L=-0.17$. It appears that all the new terms F,K,H,G,L are smaller than A; this is the reason for the approximate validity of the NRQM formula (1), and is the fact that a complete field-theoretical calculation should explain; at the same time it emerges that there is no reason to expect (1) to give a very accurate fit to the magnetic moments. We will also show that (2) can be used to calculate the $\Sigma^0\to\Lambda\gamma$ rate to first order in flavor breaking.

At this point the following remark is necessary: our parametrization is a "restricted" one in the sense that, for instance, the values of the coefficients in (2) given above refer only to the lowest baryon octet. If, for example, we knew experimentally the magnetic moments of an octet of radially excited baryons, we should not expect, after fitting them, to find for the coefficients in (2) *exactly* the same values as those given above.

One final remark: the procedure to be used for the magnetic moments can be extended to other cases. The treatment of many problems given by the NRQM turns out to be a simplified version of the parametrization that can be derived from the underlying field theory. In particular we conclude the following. (a) The qualitative language of the NRQM is independent of the velocity of the quarks inside the hadrons or of the neglect of $q\bar{q}$ virtual pairs and virtual gluons; all that matters is the existence of a correspondence between the exact state of the hadrons and the nonrelativistic states with a fixed number of quarks of an appropriately constructed "model" Hamiltonian. (b) As to the quantitative results of the NRQM [such as, e.g., the fact that (1) provides a fair fit to the magnetic moments, or the fair estimate of the Σ-Λ mass difference and its sign using the nonrelativistic Breit-Fermi formula of the spin-dependent forces between quarks], this is another matter; this paper takes the first step: it shows that the parameters entering the NRQM description in the cases examined are in fact more important than all the other parameters that intervene in the general parametrization; but one still has to understand why the NRQM is such a reasonable approximation to the results of the basic theory or, in other words, why the simplified parametrization of the NRQM is successful.

All these conclusions emerge already from the treatment of the magnetic moments on which we now focus our attention.

II. AN OUTLINE OF THE PROCEDURE: THE MODEL STATES AND THE MODEL HAMILTONIAN

In this section we outline the procedure to derive Eq. (2); for some points see Appendix A. Call H the exact Hamiltonian of the relativistic theory of quarks and gluons to be taken as the starting point and let $|\psi_B\rangle$ be its eigenstates corresponding to the baryons B $(B=P,N,\Lambda,\Sigma^{\pm,0},\Xi^{-,0})$ at rest; we thus have $H|\psi_B\rangle=M_B|\psi_B\rangle$ where M_B is the mass of the baryon. The problem is to calculate the expectation value in $|\psi_B\rangle$ of the magnetic moment operator $\boldsymbol{\mathcal{M}}$:

$$\boldsymbol{\mathcal{M}}=\tfrac{1}{2}\int d^3\mathbf{r}\,\mathbf{j}(\mathbf{r})\times\mathbf{r}\ . \tag{3}$$

For the current $\mathbf{j}(\mathbf{r})$ in (3) we assume the canonical form

$$j_\mu(x)=e[\tfrac{2}{3}\bar{u}_R(x)\gamma_\mu u_R(x)-\tfrac{1}{3}\bar{d}_R(x)\gamma_\mu d_R(x) -\tfrac{1}{3}\bar{s}_R(x)\gamma_\mu s_R(x)] , \quad (4)$$

where the fields $u_R(x), d_R(x), s_R(x)$ are the renormalized fields of the respective quarks (we have suppressed a color index on the quark fields and a sum over colors in the current because this is unimportant for the following). In what follows the values of the renormalized (or "physical") masses of the quarks will not intervene explicitly, except to ensure that it is reasonable to treat only to first order flavor breaking [due to the $\lambda-\mathcal{P}$ (or $\mathcal{N}$) mass difference] and except when we will make contact with the NRQM. However, the mass renormalization implies that the "physical" (or better the "effective") masses of the quarks appearing in the quark propagator for calculations in the low-k^2 regime are not the bare masses (determined to be a few MeV by the algebra of currents); they must be identified with the constituent-quark masses, possibly in the range of some hundreds of MeV for the light quarks; in fact we identify the constituent quarks $\mathcal{P}$, $\mathcal{N}$, and λ [the symbols used in (1)] with the fields of the renormalized quarks:

$$\mathcal{P}\equiv u_R(x), \quad \mathcal{N}\equiv d_R(x), \quad \lambda\equiv s_R(x) .$$

The exact Hamiltonian H is assumed, as stated, to be flavor independent, except for the only breaking due to the mass difference between the λ quark renormalized mass $(m+\Delta m)$ and the common value (m) of the $\mathcal{P},\mathcal{N}$ renormalized masses; thus *the only flavor-breaking part of* H is

$$m\int d^3\mathbf{r}[\bar{u}_R(x)u_R(x)+\bar{d}_R(x)d_R(x)+\bar{s}_R(x)s_R(x)] +\Delta m\int d^3\mathbf{r}\,\bar{s}_R(x)s_R(x) . \quad (5)$$

The magnetic moment of a baryon is

$$\mathbf{M}_B=\langle\psi_B|\boldsymbol{\mathcal{M}}|\psi_B\rangle . \quad (6)$$

Below we suppress often the index B, restoring it when needed; to avoid confusion the index B will be used only to specify the baryon.

It is useful to read Eq. (6) as follows: $\mathbf{M}$ is the expectation value of a simple operator [the current $j_\mu(x)$ in (3) has the simple structure (4)] on the complicated state $|\psi\rangle$. In fact the state $|\psi\rangle$ appearing in (6), being the exact state of a strongly interacting system of quarks and gluons, is extremely complicated; it can be displayed as a superposition of an infinite number of Fock states, starting with three quarks; it includes four quarks and one antiquark, three quarks plus one gluon, and so on. Schematically,

$$|\psi\rangle=|qqq\rangle+|qqqq\bar{q}\rangle+|qqqG\rangle+\cdots , \quad (7)$$

where the ellipsis stands for the sum of an infinite number of additional states and the amplitudes that multiply each state (depending on the momenta, spins, flavors, colors of the intervening quarks, antiquarks, and gluons) have been left understood.

We introduce now an auxiliary Hamiltonian $\mathcal{H}$, nonrelativistic or semirelativistic (by this we mean that the "kinetic" energy of a particle with mass m can be written $p^2/(2m)$ or $[(p^2+m^2)^{1/2}-m]$, or also differently), that operates, by construction, only in the three-quark sector [compare Eq. (A4) of the Appendix]; the operator $\mathcal{H}$, called the "model Hamiltonian," has the sole purpose of providing a set of three-quark baryon states that we call the three-quark "model" states $|\phi_B\rangle$. In principle we can now write the exact state $|\psi_B\rangle$ in (6) as

$$|\psi_B\rangle=V|\phi_B\rangle , \quad (8)$$

where V is some very complicated unitary operator that, again in principle, can be constructed in terms of H and $\mathcal{H}$ using, for instance, the adiabatic construction of the bound states[3] (the question of a unitary V in the adiabatic construction is dealt with in the Appendix). Using (8) the expression (6) for $\mathbf{M}_B$ becomes

$$\mathbf{M}_B=\langle\phi_B|V^\dagger\boldsymbol{\mathcal{M}}V|\phi_B\rangle . \quad (9)$$

The difference between (6) and (9) is that in (6) $\boldsymbol{\mathcal{M}}$ is simple but the states $|\psi\rangle$ are very complicated; in (9), on the other hand, the states $|\phi\rangle$ are simple and all the complications are transferred into $V^\dagger\boldsymbol{\mathcal{M}}V$.

This has an advantage: $V^\dagger\boldsymbol{\mathcal{M}}V$ is indeed a complicated field operator, but since it has to act only on the coordinates (space, spin, flavor, color) of the *three* quarks present in the state $|\phi\rangle$, it must be (after contraction of all the field operators) necessarily a function of these coordinates only. In the following the three quarks in $|\phi\rangle$ will be numbered 1,2,3. Thus, after the elimination of all the creation and destruction operators, $V^\dagger\boldsymbol{\mathcal{M}}V$ in (9) behaves as a color-singlet three-body operator acting on 1,2,3.

In (6) $\boldsymbol{\mathcal{M}}$ transforms as an axial vector (since it is a magnetic moment) and the same is true for $V^\dagger\boldsymbol{\mathcal{M}}V$ in (9) because V is a rotationally invariant operator (it is expressed in terms of H and $\mathcal{H}$ which are both rotationally invariant). As to $|\phi\rangle$ in (9), it depends on how we select the model Hamiltonian $\mathcal{H}$. The main point now is that by choosing $\mathcal{H}$ as the *simplest, most naive, most unrefined* nonrelativistic quark model Hamiltonian, the parametrization of $\langle\phi|V^\dagger\boldsymbol{\mathcal{M}}V|\phi\rangle$ can be considerably simplified. We select $\mathcal{H}$ so that for the lowest octet (and decuplet) baryons the wave functions ϕ_B of the baryons states $|\phi_B\rangle$ have a nonrelativistic space-spin structure with the following properties: (1) ϕ_B is the product of a space part $X(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ symmetrical in $\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3$, times a spin-flavor part $W_B(1,2,3)$ times a color-singlet factor $S_c(1,2,3)$; (2) the space part has orbital angular momentum $\mathbf{L}=0$; thus $X\equiv X_{L=0}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$.

Altogether for the baryon B,

$$\phi_B=X_{L=0}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)W_B(1,2,3)S_c(1,2,3) . \quad (10)$$

The assumption $\mathbf{L}=0$ and the symmetry of the space wave function imply automatically that the spin factors $W_B(1,2,3)$ of the wave function have the SU(6) structure.

As to the selection of the model Hamiltonian $\mathcal{H}$, that is, of the model wave functions ϕ, the following remarks are in order.

(a) Because the operator V in (8) is written in terms of creation and destruction operators of Dirac particles, the spinors appearing in $W_B(1,2,3)$ must be four-component spinors; otherwise the operation of V on $|\phi_B\rangle$ would not be defined. This is achieved by simply completing the Pauli spinors of the NR wave function with two zeros in the lower components. This is entirely compatible with our nonrelativistic choice of the model Hamiltonian $\mathcal{H}$. In fact $\mathcal{H}$ operates on two-component Pauli spinors, but it is formally possible to enlarge the space of such spinors to that of the four-component spinors, provided that $\mathcal{H}$ is extended so that it does not connect the spaces of the upper and lower components and gives zero when operating on the latter.

(b) The $X_{L=0}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ part of the wave function (10) does not carry the baryon index B; this is because we have chosen $\mathcal{H}$ to be flavor independent [in particular the masses of the quarks in $\mathcal{H}$ are taken equal, all with the value $m+(\Delta m/2)$, in the notation of (5)]. All the flavor breaking is left to V: recall that V depends on the exact Hamiltonian H of the theory and H contains the flavor-breaking term (5). We might have made a different choice, assuming $\mathcal{H}$ to be flavor dependent, but, at least for the problem at hand, it is convenient to proceed as indicated.

The operator $V^\dagger \mathcal{M} V$ has a role only when inserted between three-quark states; this is because V in (8) contains $\mathcal{H}$ that was constructed specifically for the three-quark states. But, of course, one can introduce auxiliary Hamiltonians for states different from the three-quark ones, for instance, the $q\bar{q}$ states of the nonexotic mesons or the $qq\bar{q}\,\bar{q}$ states corresponding to exotic mesons (if these are bound); then V would operate also in sectors additional to the three-quarks one.

Calculating the expectation value of the field operator $V^\dagger \mathcal{M} V$ in the state $|\phi_B\rangle$ gives the same result as calculating the expectation value of a certain quantum-mechanical three-body operator $\tilde{\mathcal{M}}$ on the wave function ϕ_B in *ordinary three-body quantum mechanics;* the search for the most general parametrization of the magnetic moments is then reduced to writing the most general operator $\tilde{\mathcal{M}}$ transforming as an axial vector in ordinary nonrelativistic quantum mechanics. Some steps will be given in the next section.

III. THE EFFECTIVE THREE-BODY MAGNETIC-MOMENT OPERATOR

We first write explicitly the state $|\phi_B\rangle$, with the wave function ϕ_B given by (10), in the occupation number space. It is

$$|\phi_B\rangle=\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}\sum_{\rho_1\rho_2\rho_3}\varphi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\chi^B_{\rho_1\rho_2\rho_3}a^\dagger_{\mathbf{p}_1\rho_1}a^\dagger_{\mathbf{p}_2\rho_2}a^\dagger_{\mathbf{p}_3\rho_3}|0\rangle \ . \tag{11}$$

Here $|0\rangle$ is the vacuum of the noninteracting but mass-renormalized quark fields (with equal masses); $a^\dagger_{\mathbf{p}_k\rho_k}$ ($k=1,2,3$) is a creation operator of the kth quark with momentum $\mathbf{p}_k$ and with spin-flavor-color specified cumulatively by the index ρ_k; $\varphi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$, which includes a factor $\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)$ (baryon at rest), is the transform in momentum space of the rotation and translation-invariant function $X_{L=0}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$; $\chi^B_{\rho_1\rho_2\rho_3}$ is the transform to the space of the indices $\rho_1\rho_2\rho_3$ of the spin-flavor-color factor $W_B(1,2,3)S_c(1,2,3)$.

On inserting the above expression (11) into (9) we obtain

$$\mathbf{M}_B=\langle\phi_B|V^\dagger\mathcal{M}V|\phi_B\rangle =\sum_{\mathbf{p}'\rho'}\sum_{\mathbf{p}\rho}\varphi^*(\mathbf{p}')\chi^{B*}_{\rho'}\langle\mathbf{p}'\rho'|V^\dagger\mathcal{M}V|\mathbf{p}\rho\rangle\varphi(\mathbf{p})\chi^B_\rho \ , \tag{12}$$

where we have introduced the abbreviations

$$\mathbf{p}\equiv\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3,\quad \rho\equiv\rho_1\rho_2\rho_3 \ ,$$

and $|\mathbf{p}\rho\rangle$ stands for

$$|\mathbf{p}\rho\rangle=a^\dagger_{\mathbf{p}_1\rho_1}a^\dagger_{\mathbf{p}_2\rho_2}a^\dagger_{\mathbf{p}_3\rho_3}|0\rangle \ .$$

In (12) $\langle\mathbf{p}'\rho'|V^\dagger\mathcal{M}V|\mathbf{p}\rho\rangle$ is, of course,

$$\langle\mathbf{p}'\rho'|V^\dagger\mathcal{M}V|\mathbf{p}\rho\rangle=\langle 0|a_{\mathbf{p}_1'\rho_1'}a_{\mathbf{p}_2'\rho_2'}a_{\mathbf{p}_3'\rho_3'}V^\dagger\mathcal{M}Va^\dagger_{\mathbf{p}_1\rho_1}\times a^\dagger_{\mathbf{p}_2\rho_2}a^\dagger_{\mathbf{p}_3\rho_3}|0\rangle \ . \tag{13}$$

To calculate explicitly $\langle\mathbf{p}'\rho'|V^\dagger\mathcal{M}V|\mathbf{p}\rho\rangle$ from (13) one has to contract the creation and destruction operators present in $V^\dagger\mathcal{M}V$ with those acting on the vacuum on the right and on the left, and among themselves (the disconnected graphs disappear in the usual way). Once this contraction of the $a,a^\dagger$ is accomplished, the operator whose matrix elements are defined by (13) becomes an operator acting on the space of the three-quark states in ordinary nonrelativistic quantum mechanics. This operator, that we will name the "effective three-body magnetic moment operator" and indicate with $\tilde{\mathcal{M}}$, is thus

$$\tilde{\mathcal{M}}=\sum_{\mathbf{p}'\rho'}\sum_{\mathbf{p}\rho}|\mathbf{p}'\rho'\rangle\langle\mathbf{p}'\rho'|V^\dagger\mathcal{M}V|\mathbf{p}\rho\rangle\langle\mathbf{p}\rho| \ . \tag{14}$$

The expression (12) for the magnetic moment can be rewritten

$$\mathbf{M}_B=\langle\phi_B|\tilde{\mathcal{M}}|\phi_B\rangle =\sum_{\mathbf{p}'\rho'}\sum_{\mathbf{p}\rho}\langle\phi_B|\mathbf{p}'\rho'\rangle\langle\mathbf{p}'\rho'|\tilde{\mathcal{M}}|\mathbf{p}\rho\rangle\langle\mathbf{p}\sigma|\phi_B\rangle \tag{15}$$

since it is

$$\langle\mathbf{p}\rho|\phi_B\rangle\equiv\langle\mathbf{p}|X_{L=0}(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3)\rangle \times\langle\rho|W_B(1,2,3)S_c(1,2,3)\rangle\equiv\varphi(\mathbf{p})\chi^B_\rho \ . \tag{16}$$

It is evident that the effective three-body magnetic-moment operator $\tilde{\mathcal{M}}$ must transform as $\mathcal{M}$, that is, as an axial vector, under space rotations. In the next section we will parametrize the operator $\tilde{\mathcal{M}}$ (14).

IV. THE PARAMETRIZATION OF $\tilde{\mathcal{M}}$: ELIMINATION OF THE COLOR AND MOMENTUM DEPENDENCE

We now determine the most general form of the three-body axial vector (14). We are interested only in the expectation value of $\tilde{\mathcal{M}}$ in the states $|\phi_B\rangle$, chosen to have $L=0$. Decomposing,

$$\tilde{\mathcal{M}}=|L=0\rangle\langle L=0|\tilde{\mathcal{M}}|L=0\rangle\langle L=0| + |L\neq 0\rangle\langle L\neq 0|\tilde{\mathcal{M}}|L\neq 0\rangle\langle L\neq 0| \equiv \tilde{\mathcal{M}}'+\tilde{\mathcal{M}}'' \quad (17)$$

we call $\tilde{\mathcal{M}}'$, the part of $\tilde{\mathcal{M}}$ invariant under space rotations; $\tilde{\mathcal{M}}''$ includes all those three-body axial vectors that vanish after evaluating their expectation value on $|\phi_B\rangle$:

$$\langle\phi_B|\tilde{\mathcal{M}}''|\phi_B\rangle=0 \ . \quad (18)$$

We thus restrict to the parametrization of $\tilde{\mathcal{M}}'$. There is no need to pay attention to the color dependence of $\tilde{\mathcal{M}}'$ because the singlet factor $S_c(1,2,3)$ in ϕ_B always factorizes; because of this, the average of $\tilde{\mathcal{M}}'$ over the color variables can be performed independently from the average over the momentum, spin, and flavor. Because the color-singlet factor $S_c(1,2,3)$ is the same for all the baryons, the average over the color gives merely a common multiplicative factor that does not increase the number of final independent parameters. Therefore, from now on we forget the color and determine the most general dependence of $\tilde{\mathcal{M}}'$ on the momentum, spin, and flavor variables; we start with the momentum.

The expectation value in the state $|\phi_B\rangle$ (with $L=0$) of any axial vector constructed with $\mathbf{p}$ or $\mathbf{p}'$ (any "orbital angular momentum") vanishes. Therefore, the axial-vector property of $\tilde{\mathcal{M}}'$ has to be carried by the spin operators $\boldsymbol{\sigma}$ only; we thus write $\tilde{\mathcal{M}}'$ as

$$\tilde{\mathcal{M}}'=\sum_{\nu}\sum_{\mathbf{p},\mathbf{p}'} R_\nu(\mathbf{p},\mathbf{p}')\mathbf{G}_\nu(\boldsymbol{\sigma},f) \ . \quad (19)$$

In (19) the notation is as follows: $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ is the set (numbered by the index ν) of all the independent Hermitian axial-vector operators that can be formed as products of the spin operators $\boldsymbol{\sigma}_i$ of the three quarks with the flavor operators f. Each $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ is multiplied by a Hermitian $R_\nu(\mathbf{p},\mathbf{p}')$ (operating in the space of the vectors $\mathbf{p},\mathbf{p}'$), a scalar under space rotations for the reasons just stated. We now exploit once more the fact that the wave function ϕ_B is a product of a momentum and a spin-flavor factor. On evaluating first the expectation value of $\tilde{\mathcal{M}}'$ over the space part of the wave function we get

$$\langle X_{L=0}|\tilde{\mathcal{M}}'|X_{L=0}\rangle=\sum_\nu g_\nu \mathbf{G}_\nu(\boldsymbol{\sigma},f) \ , \quad (20)$$

where g_ν are numerical real coefficients [recall that $\varphi(p)$ is the Fourier transform of $X_{L=0}$] given by

$$g_\nu=\langle\varphi(p)|R_\nu(\mathbf{p},\mathbf{p}')|\varphi(p)\rangle \ . \quad (21)$$

We must determine, therefore, the set of spin-flavor operators $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$. Before doing this we shall show that the most general axial vector formed with three spins is much simpler than one would think, if we are only interested in its expectation value on a state described by a real spin-flavor function.

V. THE SPIN DEPENDENCE OF THE MOST GENERAL AXIAL VECTOR

The most general Hermitian axial vector formed with the spins $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3$ of three spin-$\frac{1}{2}$ particles and depending on the flavor operators f is given by the following expression (22) plus all the terms (possibly with different coefficients) obtained performing any permutation on 1,2,3:

$$\boldsymbol{\sigma}_1[a(f)+b(f)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)]+c(f)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2) + d(f)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_3) \ . \quad (22)$$

In (22) $a(f),b(f),c(f),d(f)$ are Hermitian operators acting on the flavor variables and having real matrix elements between real functions (we call such operators "real").

We will show that when calculating the expectation value of the operator (22) on a spin-flavor state with a real wave function and a given value of the total angular momentum J, (a) the cross-product terms in (22) give no contribution, and (b) the term $\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)$ can be rewritten purely in terms of $\boldsymbol{\sigma}_1$ and of the c number J. It follows that the most general Hermitian axial-vector operator $\mathbf{G}(\boldsymbol{\sigma},f)$ constructed in terms of the $\boldsymbol{\sigma}_i$'s of the three particles (and of the flavor operators f) is, when used for evaluating an expectation value as specified above, a combination of $\boldsymbol{\sigma}_1\Gamma_1^J(f)$, $\boldsymbol{\sigma}_2\Gamma_2^J(f)$, $\boldsymbol{\sigma}_3\Gamma_3^J(f)$ where $\Gamma_i^J(f)$ are three operators depending (for a given J) only on f.

Proof: Consider first the term $c(f)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)$ in (22); the same argument obviously holds for $d(f)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_3)$. We may restrict the attention to any component of $c(f)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)$, for instance, the z component; it is sufficient to prove the above statement for $c(f)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z$. Consider any spin-flavor state of 1,2,3 represented by a real function $R(1,2,3)$ of the spin-flavor variables; we will show that

$$\langle R|(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z c(f)|R\rangle=0 \ . \quad (23)$$

The most general form of a real function of the 1,2,3 spin-flavor variables is

$$R(1,2,3)=\alpha_1\beta_2R_a+\alpha_2\beta_1R_b+\alpha_1\alpha_2R_c+\beta_1\beta_2R_d \ , \quad (24)$$

where R_a,R_b,R_c,R_d are four real functions expressed in terms of α_3,β_3 (the spin basis for the third particle) and of the complete flavor basis for 1,2,3. Because it is $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z=2i(\sigma_1^+\sigma_2^- - \sigma_1^-\sigma_2^+)$ the operator $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z$ gives zero when operating on the $\alpha_1\alpha_2$ and on the $\beta_1\beta_2$ terms of R (24). When applied to $\alpha_1\beta_2$ and to $\alpha_2\beta_1$ it is $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z\alpha_1\beta_2=-2i\beta_1\alpha_2$ and $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z\alpha_2\beta_1=+2i\beta_2\alpha_1$.

It, therefore, follows that

$$\langle R|(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z c(f)|R\rangle=-2i\langle\alpha_1\beta_2R_a+\beta_1\alpha_2R_b|c(f)|\beta_1\alpha_2R_a-\alpha_1\beta_2R_b\rangle=-2i[\langle R_b|c(f)|R_a\rangle-\langle R_a|c(f)|R_b\rangle]=0 \ . \quad (25)$$

The vanishing of (25) is due to the assumed reality of $c(f)$ and to the fact that R_a and R_b are real functions:

$$\langle R_b|c(f)|R_a\rangle=\langle R_a|c(f)|R_b\rangle^*=\langle R_a|c(f)|R_b\rangle \ .$$

It must be stressed that the above property (25) holds for the *expectation value* of $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)c(f)$ over a real function, but not necessarily for a nondiagonal matrix element.

Note, incidentally (we will use this in Sec. XI), that also the scalar

$$(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{\sigma}_3 \tag{26}$$

(times any Hermitian real flavor-dependent operator) has a vanishing expectation value on any real spin-flavor state of three particles.

Proof:

$$\begin{aligned}(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\boldsymbol{\sigma}_3&=(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z\cdot\sigma_{3z}+(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_x\cdot\sigma_{3x}\\&\quad+(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_y\cdot\sigma_{3y}\\&\equiv(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z\cdot\sigma_{3z}+(\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_1)_z\cdot\sigma_{2z}\\&\quad+(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)_z\cdot\sigma_{1z} \ .\end{aligned}$$

The expectation value of $(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_k)_z\cdot\sigma_{jz}$ $(i\neq k\neq j)$ vanishes as it is seen using (25) in which $c(f)$ is replaced by $\sigma_{jz}c(f)$, a real operator.

Consider next the operator $\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)$ [multiplied in (22) with a real operator $b(f)$] and take its z component. It is

$$\sigma_{1z}(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)=\tfrac{1}{2}\sigma_{1z}[(\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)^2-6] \ .$$

On writing

$$2\mathbf{J}=\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3$$

we obtain

$$\sigma_{1z}(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)=\tfrac{1}{4}\sigma_{1z}[(2\mathbf{J}-\boldsymbol{\sigma}_1)^2-6]+\tfrac{1}{4}[(2\mathbf{J}-\boldsymbol{\sigma}_1)^2-6]\sigma_{1z}$$

and, with some algebra,

$$\sigma_{1z}(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)=\tfrac{1}{4}[(4|\mathbf{J}|^2-7)\sigma_{1z}+\sigma_{1z}(4|\mathbf{J}|^2-7)]-2J_z \ . \tag{27}$$

In calculating the expectation value of $\sigma_{1z}(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)$ on a state with a given J we can write $|\mathbf{J}|^2=J(J+1)$, a c number.

In conclusion the most general axial-vector operator (*as far as its expectation value on a real spin-flavor state with a given J is concerned*) is

$$\mathbf{G}(\boldsymbol{\sigma},f)=\boldsymbol{\sigma}_1\Gamma_1^J(f) \ \text{ or } =\boldsymbol{\sigma}_2\Gamma_2^J(f) \ \text{ or } =\boldsymbol{\sigma}_3\Gamma_3^J(f) \ , \tag{28}$$

where the $\Gamma_i^J(f)$ are real flavor operators. What we have proven, essentially, is that the most general axial vector formed with three spin-$\frac{1}{2}$ particles (under the italicized condition written above) is a combination of $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3$ and nothing else. It might appear strange that the only axial vectors are those listed in (28) since we can, for instance, multiply $\boldsymbol{\sigma}_1$ by $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ or by other scalar products of the spin matrices, and remain with an axial vector. The answer appears from Eq. (27), and Eqs. (29) and (30) below, where we have restricted to the z components (the x and y behave similarly)

$$\sigma_{1z}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)=\sigma_{2z}+i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_z \ , \tag{29}$$

$$\begin{aligned}\sigma_{1z}[\boldsymbol{\sigma}_1\cdot(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)]&=(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)_z+i[\boldsymbol{\sigma}_1\times(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)]_z\\&=(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)_z+i(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\sigma_{3z}\\&\quad-i(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)\sigma_{2z} \ .\end{aligned} \tag{30}$$

On recalling that the expectation value of $(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_k)$ gives zero, as shown above, Eqs. (29) and (30) [in conjunction with (27)] exemplify how the multiplication of $\boldsymbol{\sigma}_i$ by scalar products of $\boldsymbol{\sigma}$'s does not create new axial vectors with a nonzero expectation value in addition to (28).

VI. THE PARAMETRIZATION OF $\tilde{\mathcal{M}}$: THE SPIN-FLAVOR DEPENDENCE

We now list the independent $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ that appear in (20); having clarified the spin dependence, we must determine the most general flavor operator $\Gamma^J(f)$ in (28).

Call $P^{\mathcal{P}},P^{\mathcal{N}},P^{\lambda}$ the projection operators for the $\mathcal{P},\mathcal{N},\lambda$ quarks; in terms of the Gell-Mann matrices λ_3,λ_8 with $\lambda_3=\text{Diag}[1,-1,0]$ and $\lambda_8=\text{Diag}[1,1,-2]$ the projection operators are, of course,

$$\begin{aligned}&P^{\mathcal{P}}=\tfrac{1}{6}(2+3\lambda_3+\lambda_8), \quad P^{\mathcal{N}}=\tfrac{1}{6}(2-3\lambda_3+\lambda_8) \ ,\\&P^{\lambda}=\tfrac{1}{3}(1-\lambda_8) \ .\end{aligned} \tag{31}$$

In terms of the above projection operators and calling ψ the column symbol u_R,d_R,s_R the current density (4) takes the form

$$j_\mu(x)=e\bar{\psi}(x)\gamma_\mu P^q\psi(x) \ , \tag{32}$$

where P^q stands for the combination of projection operators:

$$P^q=\tfrac{2}{3}P^{\mathcal{P}}-\tfrac{1}{3}P^{\mathcal{N}}-\tfrac{1}{3}P^{\lambda} \ . \tag{33}$$

The mass Hamiltonian (5), breaking the flavor, is rewritten

$$m\int d^3\mathbf{r}\,\bar{\psi}(\mathbf{r})\psi(\mathbf{r})+\Delta m\int d^3\mathbf{r}\,\bar{\psi}(\mathbf{r})P^{\lambda}\psi(\mathbf{r}) \ . \tag{34}$$

Consider now the operator $\tilde{\mathcal{M}}$ given by (14). No matter how complicated is the calculation leading, after the elimination of the creation and destruction operators of the fields, to the expression of $\tilde{\mathcal{M}}$ in terms of the variables of the three quarks, the result of this calculation must be linear in P_i^q [where P^q is the combination (33) of the projection operator and i is a quark index, $i=1,2,3$].

Indeed, if in the course of the calculation of $\tilde{\mathcal{M}}$, we keep all the P^{λ}'s without simplifying P_i^q with P_i^{λ} [compare Eq. (35) below], the P^q appearing in the current (32) remain in the final result for $\tilde{\mathcal{M}}$; this is due to the fact that the only operators in flavor space present in the Hamiltonian H are the P^{λ}'s [appearing in the mass-breaking part of (34)] and the P_i^{λ} commute with the P_k^q for any choice of the indices i,k. Of course, in simplifying the end result one will note that the product of P_i^q and P_i^{λ} with the same index i is

$$P_i^q\cdot P_i^\lambda=-\tfrac{1}{3}P_i^\lambda \tag{35}$$

due to the fact that $P_i^{\mathcal{P}}\cdot P_i^\lambda=P_i^{\mathcal{N}}\cdot P_i^\lambda=0$ and $P_i^\lambda\cdot P_i^\lambda=P_i^\lambda$; then, because of Eq. (35), P^q can disappear if multiplied by a P^λ with the same quark index. But below we will list the possible flavor factors in the final expression of $\tilde{\mathcal{M}}$ (more precisely of $\tilde{\mathcal{M}}'$) before the use of (35).

Thus for all the baryons of the octet and decuplet, except the Ω^-, the operator $\Gamma(f)$ appearing in (28) must have necessarily one of the following forms (the indices i,k,j have the values 1,2,3):

$$P_i^q,\quad P_i^q\cdot P_k^\lambda,\quad P_i^q\cdot P_k^\lambda\cdot P_j^\lambda\ . \tag{36}$$

To include the Ω^- one must add $P_i^q\cdot P_1^\lambda\cdot P_2^\lambda\cdot P_3^\lambda$.

It follows that the most general operator $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ in (28) for the octet [to be then inserted in (20)] is the product of an operator chosen from the basis $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3$ in the spin space and an operator from the basis (36) (with all the choices of i,k,j) in the flavor space.

An additional property of the $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$'s is due to the spin-flavor functions $W_B(1,2,3)$ being symmetric in 1,2,3 for the lowest baryon octet and decuplet. Hence, each $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ in (20) must be symmetric in 1,2,3. Thus all the $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ are obtained by taking one of the operators (36) (with the index i chosen to be, say, 1 and the other indices k,j different or equal among them and from 1 in all the possible ways), multiplying this operator by $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}_2$ or $\boldsymbol{\sigma}_3$ in all the possible ways and symmetrizing. By doing so the following 11 different $\mathbf{G}_\nu(\boldsymbol{\sigma},f)$ are obtained [we list them before the simplification (35)] (we have included also $\mathbf{G}_{11}$ that is nonzero only for the Ω^-):

$$\mathbf{G}_1=\sum_i P_i^q\boldsymbol{\sigma}_i,\quad \mathbf{G}_2=\sum_i P_i^qP_i^\lambda\boldsymbol{\sigma}_i\ ; \tag{37}$$

$$\begin{aligned}&\mathbf{G}_3=\sum_{i\neq k}P_i^q\boldsymbol{\sigma}_k,\quad \mathbf{G}_4=\sum_{i\neq k}P_i^qP_i^\lambda\boldsymbol{\sigma}_k,\quad \mathbf{G}_5=\sum_{i\neq k}P_i^qP_k^\lambda\boldsymbol{\sigma}_k\ ,\\ &\mathbf{G}_6=\sum_{i\neq k}P_i^q\boldsymbol{\sigma}_iP_k^\lambda,\quad \mathbf{G}_7=\sum_{i\neq k\neq j}P_i^qP_k^\lambda\boldsymbol{\sigma}_j\ ;\end{aligned} \tag{38}$$

$$\begin{aligned}&\mathbf{G}_8=\sum_{i\neq k\neq j}P_i^qP_k^\lambda P_j^\lambda\boldsymbol{\sigma}_i,\quad \mathbf{G}_9=\sum_{i\neq k\neq j}P_i^q\boldsymbol{\sigma}_kP_k^\lambda P_j^\lambda\ ,\\ &\mathbf{G}_{10}=\sum_{i\neq k\neq j}P_i^qP_i^\lambda\boldsymbol{\sigma}_kP_j^\lambda,\quad \mathbf{G}_{11}=\sum_{i\neq k\neq j}P_i^qP_i^\lambda\boldsymbol{\sigma}_iP_k^\lambda P_j^\lambda\ .\end{aligned} \tag{39}$$

Nothing can be done, without additional assumptions, with 11 parameters; below we restrict ourselves to considering the flavor violation to first order. In this case only the G_ν's from G_1 to G_7 (the vector symbol on the G_ν's is often suppressed from now on) intervene: from the experimental values of the seven magnetic moments of the octet baryons we then determine their coefficients and compare the result with that given by the NRQM description. Moreover, from the knowledge of the above coefficients, it will be possible to predict (only to first order in flavor breaking) the $\Sigma^0\to\Lambda\gamma$ rate.

First order in flavor breaking means that among the above 11 terms we keep only those linear, at most, in the P^λ's. However, $(P_i^\lambda)^n=P_i^\lambda$ for any (integer positive) n, so that, in fact, we keep to all orders n the flavor-breaking contributions additive in the quarks; we exclude flavor-breaking effects of second order or more coming from terms carrying the indices of two different quarks.

The structure of (37)–(39) simplifies by introducing the abbreviations

$$\boldsymbol{\Sigma}^q\equiv\sum_i P_i^q\boldsymbol{\sigma}_i,\quad \boldsymbol{\Sigma}^\lambda\equiv\tfrac{1}{3}\sum_i P_i^\lambda\boldsymbol{\sigma}_i \tag{40}$$

as well as the charge Q, the strangeness S, and the total spin $2\mathbf{J}$:

$$Q\equiv\sum_i P_i^q,\quad S\equiv-\sum_i P_i^\lambda,\quad 2\mathbf{J}\equiv\sum_i\boldsymbol{\sigma}_i\ . \tag{41}$$

The first seven G_ν's ($\nu=1,\ldots,7$) in terms of the quantities (40) and (41) are reported below; we have used the simplification (35):

$$\begin{aligned}&G_1=\boldsymbol{\Sigma}^q,\quad G_2=-\boldsymbol{\Sigma}^\lambda,\quad G_3=Q\cdot(2\mathbf{J})-\boldsymbol{\Sigma}^q\ ,\\ &G_4=\tfrac{1}{3}S\cdot(2\mathbf{J})+\boldsymbol{\Sigma}^\lambda,\quad G_5=3Q\cdot\boldsymbol{\Sigma}^\lambda+\boldsymbol{\Sigma}^\lambda\ ,\\ &G_6=-S\cdot\boldsymbol{\Sigma}^q+\boldsymbol{\Sigma}^\lambda\ ,\\ &G_7=-Q\cdot S\cdot(2\mathbf{J})-\tfrac{1}{3}S\cdot(2\mathbf{J})-3Q\cdot\boldsymbol{\Sigma}^\lambda+S\cdot\boldsymbol{\Sigma}^q-2\boldsymbol{\Sigma}^\lambda\ .\end{aligned}$$

G_8 to G_{11} can be expressed similarly; they contain additional powers of S, up to third power.

Each of the seven G_ν's linear in the flavor perturbation is a linear combination of products of the quantities (40) and (41) containing at most the first power of S *or* of $\boldsymbol{\Sigma}^\lambda$. Therefore, the most general form for the magnetic moment $\mathbf{M}$ to first order in flavor perturbation is

$$\begin{aligned}\mathbf{M}=\sum_{1\nu}^{7}g_\nu\mathbf{G}_\nu&\equiv\mu\boldsymbol{\Sigma}^q+A\boldsymbol{\Sigma}^\lambda+FQ\cdot(2\mathbf{J})+HS\cdot(2\mathbf{J})\\ &\quad+LQ\cdot\boldsymbol{\Sigma}^\lambda+KS\cdot\boldsymbol{\Sigma}^q+GQ\cdot S\cdot(2\mathbf{J})\ ,\end{aligned} \tag{42}$$

where $g_1,g_2,g_3,\ldots,g_7$ or, equivalently, μ,A,F,H,L,K,G are seven parameters. Equation (42) is Eq. (2), written with different symbols.

We make a final remark. The effective magnetic moment operators G_1 to G_7 can be grouped in various ways: one is to form a class with only the "additive" operators, those that consist of sum of terms over one quark at a time, and another class with all the others; in this way G_1 and G_2 belong to a class and G_3,G_4,G_5,G_6,G_7 to another one. [In the NRQM (in its simplest form) only the additive G_1 and G_2 intervene.] Another way is to group together those G_ν's where the same quark carries the spin *and* the charge producing the magnetic moment and assign to another class the G_ν's related to the spins of two different quarks; this division puts together G_1,G_2,G_6 and in another class G_3,G_4,G_5,G_7 that imply the intervention of the spins of two different quarks. Once the coefficients of the various G_ν's have been determined from the data, one can check if their magnitude reflects somehow the above groupings.

VII. DETERMINATION OF THE SEVEN PARAMETERS

The expectation values of $\Sigma_z^q,\Sigma_z^\lambda$ for the seven baryons $P,N,\Lambda,\Sigma^{+,-},\Xi^{0,-}$ are given in Table I. The last line gives the matrix elements of Σ_z^q and Σ_z^λ for the transition

$\Sigma^0 \to \Lambda\gamma$ (to be used in the next section).

Equation (42) and the values of $\Sigma_z^q, \Sigma_z^\lambda$ of Table I lead to the following expressions for the seven magnetic moments (we indicate by the baryon symbol the magnetic moment of the baryon in proton magnetons):

$$\begin{aligned} &P=\mu+F,\quad N=-\frac{2\mu}{3},\quad \Lambda=-\frac{\mu}{3}+\frac{A}{3}-H+\frac{K}{3}\ ,\\ &\Sigma^+=\mu-\frac{A}{9}+F-H-\frac{L}{9}-K-G\ ,\\ &\Sigma^-=-\frac{\mu}{3}-\frac{A}{9}-F-H+\frac{L}{9}+\frac{K}{3}+G\ ,\\ &\Xi^0=-\frac{2\mu}{3}+\frac{4A}{9}-2H+\frac{4K}{3}\ ,\\ &\Xi^-=-\frac{\mu}{3}+\frac{4A}{9}-F-2H-\frac{4L}{9}+\frac{2K}{3}+2G\ . \end{aligned} \tag{43}$$

The P and N magnetic moments depend only on the two parameters μ and F. Note that this remains true also if all flavor-breaking terms are included. It follows (in proton magnetons) that

$$\mu=2.869,\quad F=-0.076\ . \tag{44}$$

The $\Lambda, \Sigma^{+,-}, \Xi^{0,-}$ determine the remaining five parameters; they are

$$\begin{aligned} &A=-\tfrac{3}{2}(\Xi^0+N)+\tfrac{9}{2}\Lambda-\tfrac{3}{4}(\Sigma^++\Sigma^-)\ ,\\ &K=+\Xi^0-\tfrac{1}{2}N-\tfrac{3}{2}\Lambda-\tfrac{1}{4}(\Sigma^++\Sigma^-)\ ,\\ &H=-\tfrac{1}{6}(\Xi^0+N)-\tfrac{1}{3}(\Sigma^++\Sigma^-)\ ,\\ &G=\tfrac{1}{4}\Lambda+\tfrac{5}{12}\Sigma^--\tfrac{1}{4}\Sigma^+-\tfrac{1}{2}\Xi^0+\tfrac{1}{6}\Xi^-+\tfrac{5}{6}(P+N)\ ,\\ &L=-\tfrac{3}{2}(\Xi^-+\Xi^0)+\tfrac{3}{2}P+\tfrac{9}{4}\Sigma^--\tfrac{3}{4}\Sigma^++\tfrac{9}{2}\Lambda\ . \end{aligned} \tag{45}$$

Inserting in (45) the values $P=2.793$, $N=-1.913$, and

$$\Sigma^+=2.48,\quad \Sigma^-=-1.16,\quad \Lambda=-0.61\ ,$$
$$\Xi^0=-1.25,\quad \Xi^-=-0.65\ ,$$

TABLE I. The expectation values of $\Sigma_z^q \equiv \sum_i P_i^q \sigma_{iz}$ and $\Sigma_z^\lambda \equiv \frac{1}{3}\sum_i P_i^\lambda \sigma_{iz}$ for the baryon-octet states; in the last line the transition matrix elements of the same quantities between Σ^0 and Λ.

	$\langle \Sigma_z^q \rangle$	$\langle \Sigma_z^\lambda \rangle$
P	$+1$	0
N	$-\frac{2}{3}$	0
Λ	$-\frac{1}{3}$	$+\frac{1}{3}$
Σ^+	$+1$	$-\frac{1}{9}$
Σ^-	$-\frac{1}{3}$	$-\frac{1}{9}$
Ξ^0	$-\frac{2}{3}$	$+\frac{4}{9}$
Ξ^-	$-\frac{1}{3}$	$+\frac{4}{9}$
	$\langle \Sigma^0\uparrow\vert\Sigma_z^q\vert\Lambda\uparrow\rangle=-1/\sqrt{3}$	$\langle \Sigma^0\uparrow\vert\Sigma_z^\lambda\vert\Lambda\uparrow\rangle=0$

we obtain

$$\begin{aligned} &A=+1.005,\quad K=+0.289,\quad H=+0.086\ ,\\ &G=-0.155,\quad L=-0.175\ . \end{aligned} \tag{46}$$

We stress that these are, to first order in flavor breaking, the exact values of the parameters A,K,H,G,L; the values that the exact relativistic theory, if it satisfies the two conditions stated in the Introduction, must produce.[4] Alternatively, in terms of the g_ν's used in the first expression of **M** in Eq. (42) one has

$$\begin{aligned} &g_1=2.79,\quad g_2=-0.94,\quad g_3=-0.076\ ,\\ &g_4=0.41,\quad g_5=0.097,\quad g_6=-0.134,\quad g_7=0.155\ . \end{aligned} \tag{47}$$

We have listed the values of the g_ν's because the G_ν's, of which they are the coefficients, are normalized, by construction, more homogeneously than Σ^q, Σ^λ, $Q\cdot(2\mathbf{J})\cdots Q\cdot S\cdot(2\mathbf{J})$ that multiply μ, A, F, H, K, G, L in the second form of (42); however, below we will continue to refer mainly to $\mu, A, F\ldots G, L$. On writing

$$A=\mu(1-a)\ , \tag{48}$$

where a is the usual parameter $a=m_{\mathcal{P}}/m_\lambda$ appearing in the NRQM, we get

$$a=0.649\ . \tag{49}$$

Thus the value of a ($\simeq 0.65$) used to fit the NRQM description remains valid in the complete description, in spite of the many additional degrees of freedom. The next largest parameter in (42) is K; this is not introduced usually in the NRQM description; it represents a correction to the magnetic moment of the individual quark linear in the strangeness, that is, essentially linear in the mass of the hadron in which the quarks are.

Each of the other parameters is rather smaller than A: $|G/A|\simeq 0.15$, $|L/A|\simeq 0.17$, $|H/A|\simeq 0.085$. In some cases, however, G,L,H have an appreciable effect; for instance, the value of the combination

$$\Lambda+2\Sigma^-+\Sigma^+=G-4H+\tfrac{1}{9}L-F=-0.44 \tag{50}$$

depends essentially on the small value of $H=0.086$.

In conclusion it remains true that, though affected by many corrections, the NRQM parametrization is verified in an exact description, in the following sense: μ and A, the only parameters that the NRQM description introduces, are indeed the largest ones; all the other parameters turn out to be at most 15% of μ. Were it not for the fact that it is not easy to define normalized invariants, we might even be tempted to generalize the above conclusion into a criterion assessing the reliability of the NRQM: that in the parametrization of a process the coefficients of terms different from those introduced in the NRQM are at most 15%; and that in cases in which the NRQM implies that the "same" values of a coefficient should be used in calculating two different processes; the word "same" is correct to 15%.

But, of course, the real question is not this, it is to understand why the correct relativistic theory (say, QCD) leads to predictions confirming the assumption of additivity, the main assumption of the NRQM. Take, for instance, the P and N magnetic moments; in this case, independently of any flavor-breaking approximation, only the terms μ and F in (42) intervene, as we have stated. Why does the correct theory predict a value of F as small as it is?

Finally it does not seem possible, at least if the experimental values of the magnetic moments will stabilize near to those used above, to classify the parameters additional to μ and A (or to g_1 and g_2) in a hierarchy of decreasing importance, in particular the distinction between different types of G_v's corresponding to the second subdivision given at the end of Sec. VII is not reflected in the dominance of K over the remaining parameters F, HG, L.

VIII. THE $\Sigma^0 \to \Lambda\gamma$ TRANSITION RATE

The parametrization of Sec. V for an axial vector holds only for an expectation value, as stated. For a nondiagonal matrix element additional terms can intervene. In fact $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)c(f)$, where $c(f)$ is a real operator acting on the flavor variables only, can have a nonzero nondiagonal matrix element. In calculating the $\Sigma^0 \to \Lambda\gamma$ rate operators of this form can thus give a contribution and thus the number of parameters can increase. We want to show, however, that in calculating the $\Sigma^0 \to \Lambda\gamma$ $M1$ *rate*, correct to first order in flavor breaking, no new parameter, in addition to those present in (42) appears. This is seen as follows: by the Wigner-Eckart theorem we can consider only the transition matrix element between the $\Lambda\uparrow$ and $\Sigma^0\uparrow$ states, that is the matrix element

$$\langle \Sigma^0\uparrow|(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z c(f)|\Lambda^0\uparrow\rangle \ . \tag{51}$$

This is purely imaginary: in fact, $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z = 2i(\sigma_1^+\sigma_2^- - \sigma_2^-\sigma_1^+)$ and $c(f)$ is real; because $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z c(f)$ is Hermitian, its coefficient (call it D) must be real if it has to be a Hermitian operator. Therefore, the rate $\Sigma^0 \to \Lambda\gamma$ turns out to be proportional to

$$|\langle \Sigma^0\uparrow|M_z|\Lambda^0\uparrow\rangle|^2 + D^2|\langle \Sigma^0\uparrow|(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z c(f)|\Lambda^0\uparrow\rangle|^2 \ , \tag{52}$$

where M_z is the operator (42). In the rate there are no interference terms between the M_z matrix element and $D\langle \Sigma^0\uparrow|(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z c(f)|\Lambda^0\uparrow\rangle$. Now consider the terms of order zero in flavor breaking; this means to select $c(f)$ in (51) simply as P^q. Then the transition operator is either

$$(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_k)_z P_j^q \quad (i \neq k \neq j) \ ,$$

or (53)

$$(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_k)_z (P_i^q - P_k^q) \quad (i \neq k) \ .$$

But it is

$$\langle \Sigma^0\uparrow|(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z P_3^q|\Lambda\uparrow\rangle = 0$$

and (54)

$$\langle \Sigma^0\uparrow|(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z (P_1^q - P_2^q)|\Lambda\uparrow\rangle = 0$$

as can be checked directly.

It must be added that there exist flavor-breaking matrix elements with the factor $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)_z$ that do not vanish, as those obtained, for instance, choosing $c(f) = P_1^q \cdot P_1^\lambda$ $(= -P_1^\lambda/3)$; however, in view of (52) they contribute to the *rate* at second order in flavor breaking. We conclude that in a calculation of the rate $\Sigma^0 \to \Lambda\gamma$ correct to first order in flavor breaking, only the part with the operator M_z in (42) contributes; moreover the coefficients $\mu, A, K, \ldots, L$ appearing in M_z have the same values deduced in Sec. VII from the (diagonal) magnetic moments, because Σ^0 and Λ have the same $J = \frac{1}{2}$ [if we had to calculate a matrix element between an octet state $(J = \frac{1}{2})$ and a decuplet state $(J = \frac{3}{2})$ the coefficients, in view of (27), would not be expected to be the same]. Therefore, it is

$$\langle \Sigma^0\uparrow|M_z|\Lambda^0\uparrow\rangle = (\mu + KS)\langle \Sigma^0\uparrow|\Sigma_z^q|\Lambda^0\uparrow\rangle + A\langle \Sigma^0\uparrow|\Sigma_z^\lambda|\Lambda^0\uparrow\rangle = -(\mu - K)\frac{1}{\sqrt{3}} = -\frac{2.869 - 0.289}{\sqrt{3}} = -1.49 \quad (|\text{expt}| = 1.61 \pm 0.09) \ , \tag{55}$$

where use has been made of the values in the last line of Table I.

Before ending this section we consider the following point. Both for the magnetic moments and for the $\Sigma^0 \to \Lambda\gamma$ matrix element we have written the most general expression (correct to first order in flavor breaking) of an axial-vector (and flavor) operator and parametrized the appropriate expectation value or transition matrix element. In some presentation of the adiabatic procedure the operator V (transforming the model state into the exact state) is not unitary. Then one might object that one has to parametrize not the matrix element of an axial vector but rather the product of the matrix element of an axial vector and of a scalar—the normalization factor of $V|\phi_B\rangle$—depending on the baryon. If this were true, the resulting parametrization would be more complicated; to first order in flavor breaking, this would introduce an additional type of term (and thus an additional parameter): namely,

$$\langle A|\boldsymbol{\Sigma}^\lambda \cdot \mathbf{J}|B\rangle \cdot \langle A|\Sigma_z^q|B\rangle \ , \tag{56}$$

where A and B are the baryons involved ($A \equiv B$ for the magnetic moments, $A = \Sigma^0, B = \Lambda$ for the transition matrix element); the connection between the $\Sigma^0 \to \Lambda\gamma$ matrix element and the magnetic moments would fail, even to first order in flavor breaking.

However, there is no problem here. Clearly our procedure is more general than the adiabatic method; but even if we stick to it, V can be written as a unitary operator multiplied by a singular baryon-dependent phase factor (compare the Appendix). Thus the quantity to be calculated is purely the expectation value (or transition matrix element) of an axial vector.

IX. THE $\Delta\to P\gamma$ $M1$ TRANSITION

The parametrization of Secs. VI and VII cannot be used to calculate the $\Delta\to P\gamma$ $M1$ matrix element. A discussion of this and the related question of the quasiforbiddenness of the $E2$ $\Delta\to P\gamma$ transition illustrates, however, the relationship between the exact results and those of the NRQM; in particular we may ask if the presence of nonadditive contributions to the $M1$ transition operator with coefficients not larger than $\approx 15\%$ of those of the additive contributions can account for the deviation of the experimental $M1$ transition matrix element from that predicted by the NRQM. The deviation in this case is rather large:[5] the matrix element of the NRQM is a factor of ≈ 1.45 below the truth; the rate a factor of ≈ 2.1 [this includes in the calculated *matrix element* a factor of $1-\frac{1}{6}k^2\langle r^2\rangle=0.82$ obtained with $(\langle r^2\rangle)^{1/2}=0.8$ F; a smaller effective $(\langle r^2\rangle)^{1/2}$ of the three quarks would imply a smaller deviation].

The first question is why we cannot expect to calculate exactly the $M1$ transition matrix element for $\Delta\to P\gamma$ using the magnetic moment operator of Eq. (42) (or better the part of it relevant to strangeness zero, the terms μ and F); we recall that, indeed, in the NRQM the $M1$ matrix element for the transition is calculated using just the term μ in (42). There are two reasons.

(1) The calculation of the $\Delta\to P\gamma$ transition is not that of a diagonal matrix element; therefore, terms of the form (with η real)

$$\mathbf{M}^{\eta}=\frac{\eta}{2}\sum_{i,k}(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_k)(P_i^q-P_k^q) \tag{57}$$

can contribute, in addition to the terms of the form

$$\hat{\mu}\sum_i \boldsymbol{\sigma}_i P_i^q\ . \tag{58}$$

(2) It is not true that $\hat{\mu}$ to be used in (58) for calculating the transition matrix element is precisely equal to μ in (42).

Note that (57) and (58) are the only terms that can contribute; a term $Q\cdot(2\mathbf{J})$ [present in (42) with coefficient F] does not intervene in $\Delta\to P\gamma$ because $\mathbf{J}$ has no matrix elements between different eigenstates.

The rate can, therefore, be expressed in terms of the following matrix elements:

$$\left\langle\Delta\frac{1}{2}\left|\hat{\mu}\sum_i\sigma_{iz}P_i^q\right|P\frac{1}{2}\right\rangle=\frac{2}{3}\sqrt{2}\hat{\mu}\ , \tag{59}$$

$$\left\langle\Delta\frac{1}{2}\left|\frac{\eta}{2}\sum_{i,k}(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_k)_z(P_i^q-P_k^q)\right|P\frac{1}{2}\right\rangle=i2\sqrt{2}\eta\ . \tag{60}$$

Because of the imaginary unit in the right-hand side of (60) the two matrix elements add in quadrature in the rate; it is

$$\left|\left\langle\Delta\frac{1}{2}\left|\hat{\mu}\sum_i\sigma_{iz}P_i^q+\frac{\eta}{2}\sum_{i,k}(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_k)_z(P_i^q-P_k^q)\right|P\frac{1}{2}\right\rangle\right|^2=\frac{8}{9}(\hat{\mu}^2+9\eta^2)\ . \tag{61}$$

As stated above, the rate calculated with $\eta=0$ and with $\hat{\mu}=2.79$ (the value suggested by the NRQM) is ≈ 2.1 times smaller than the experimental value. The term $9\eta^2$ on the right-hand side of (61) acts in the right direction to correct this difference, but is largely insufficient if η is, say, $\approx 15\%$ of μ. We, therefore, turn to $\hat{\mu}$ in (58).

Why, in spite of the fact that the operator $V^{\dagger}\mathcal{M}V$ is the same operator both for the octet ($J=\frac{1}{2}$) and the decuplet ($J=\frac{3}{2}$), are $\hat{\mu}$ and μ different? To answer this note that (for how it has been derived) the expression (42) of the effective magnetic moment can only be used for the expectation value of $\mathbf{M}$ in a state of given J [compare the remark before (28)]. To get an expression for $V^{\dagger}\mathcal{M}V$ that applies also to the transition matrix element for $J=\frac{3}{2}$ to $J=\frac{1}{2}$, it is convenient to start again from (22), that gives the most general spin-flavor operator before the simplifications brought by (27). We now omit from (22) the $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)$ and $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_3)$ terms because they have been just discussed, and write for $a(f)$ and $b(f)$ the most general expression appropriate to deal, as we do here, with the sector of strangeness zero; because this expression must be linear in P^q and taking into account obvious symmetry requirements we write

$$a(f)=\alpha P_1^q+\delta(P_2^q+P_3^q),\quad b(f)=\beta P_1^q+\gamma(P_2^q+P_3^q)\ ,$$

where $\alpha,\beta,\gamma,\delta$ are real coefficients. The most general effective magnetic moment operator to be used (in the zero strangeness sector) for calculating both the diagonal and the transition matrix elements is then

$$\mathbf{M}=\sum_{\text{perm}}[\alpha P_1^q+\delta(P_2^q+P_3^q)]\boldsymbol{\sigma}_1+[\beta P_1^q+\gamma(P_2^q+P_3^q)]\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)\ , \tag{62}$$

where the sum over the perm(utations) means that one has to add to the term (123) indicated in (62) the terms (312) and (231). With some algebra, using in particular Eq. (27), the expression (62) can be rewritten

$$\begin{aligned}\mathbf{M}=&(\alpha-\delta)\boldsymbol{\Sigma}^q\\&+(\beta-\gamma)[\tfrac{1}{4}(4|\mathbf{J}|^2-7)\boldsymbol{\Sigma}^q+\tfrac{1}{4}\boldsymbol{\Sigma}^q(4|\mathbf{J}|^2-7)]\\&+[\delta-\beta+\tfrac{1}{2}\gamma(4|\mathbf{J}|^2-7)]Q\cdot(2\mathbf{J})\ .\end{aligned} \tag{63}$$

In calculating the expectation value over an octet state with $J=\frac{1}{2}$, (63) reduces to

$$\mathbf{M}(J=\tfrac{1}{2})=[\alpha-\delta-2(\beta-\gamma)]\boldsymbol{\Sigma}^q+(\delta-\beta+2\gamma)Q(2\mathbf{J})\ . \tag{64}$$

For a $J=\frac{3}{2}\to J=\frac{1}{2}$ transition (63) becomes

$$\mathbf{M}(\tfrac{3}{2}\to\tfrac{1}{2})=(\alpha-\delta+\beta-\gamma)\boldsymbol{\Sigma}^q\ . \tag{65}$$

Comparing (64) with (42) we have

$$\mu=\alpha-\delta-2(\beta-\gamma), \quad F=\delta-\beta+2\gamma \tag{66}$$

and comparing with (58) it is

$$\hat{\mu}=\alpha-\delta+\beta-\gamma . \tag{67}$$

Therefore,

$$\hat{\mu}=\mu+3(\beta-\gamma) . \tag{68}$$

Unfortunately we have no way to determine $\beta-\gamma$; nevertheless, (68) is of some interest; with the factor 3 in front of $(\beta-\gamma)$, it is sufficient in (68) to take $(\beta-\gamma)=0.35$ $(\equiv 0.12\mu)$ and to assume the same order of magnitude for η in front of (57), to have $(\hat{\mu}^2+9\eta^2)/\mu^2\simeq 2.1$; once more the parameters not included in the NRQM are smaller than 15% of those included.

X. THE $\Delta\to P\gamma$ $E2$ MATRIX ELEMENT

The usual $E2$ operator in the NRQM has the structure

$$Q_M=\sum_i P_i^q\Omega_M(\mathbf{r}_i)+\sum_i P_i^q[\boldsymbol{\sigma}_i * \mathbf{V}(\mathbf{r}_i)]_M , \tag{69}$$

where the index M specifies the component of the spherical tensor of order 2. The terms on the right-hand side of (69), characterized by an additive structure in the quark indices, originate in the NRQM as follows: the first is due to the "conduction" current; it consists of the charge P_i^q times a quadrupole operator $\Omega_M(\mathbf{r}_i)$ of the quark space coordinates. The second (due to the "magnetic" current) is the product of a spin $\boldsymbol{\sigma}_i$ and an axial-vector operator of the space-momentum coordinates $\mathbf{V}(\mathbf{r}_i)$. Because of the above structure of the $E2$ operator and of the property (10) of the baryon wave function, the matrix element of Q_M (69) between the Δ and P states vanishes:[6] the first [second] term of (69) vanishes because the matrix element of $\Omega_M(\mathbf{r}_i)$ $[\mathbf{V}(\mathbf{r}_i)]$ between two $L=0$ states is zero. The experiment essentially confirms this selection rule; a recent analysis[7] shows that the ratio of the amplitudes quadrupole/$M1$ is not exactly zero but is small (different estimates range from 0.009 to 0.024); that is 5–13 % of the value ($\simeq 0.18$) expected[6] if the transition were not inhibited.

It is natural to ask which possible structures, in addition the additive ones that we have written, can be added to the right-hand side of (69) if the operator in (69) has in general to be interpreted as the quadrupole effective operator, the quadrupole part of $V^\dagger j_\mu(x)V$. Because the model functions have both $L=0$, the effective quadrupole operator has to be a tensor formed with the spins of the intervening quarks. We do not discuss the most general structure but limit to write a possible form for such an effective quadrupole operator:

$$\chi\sum_{i>k}(P_i^q+P_k^q)[\sigma_{iz}\sigma_{kz}-(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)/3]S(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) , \tag{70}$$

where $S(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ is a scalar function of the coordinates (and momenta) and the zz component only has been written. This term has a nonzero matrix element between Δ and P and the small value of the amplitude mentioned above corresponds to a small value (compared to $\hat{\boldsymbol{\mu}}$) of $\chi\langle X_{L=0}|S|X_{L=0}\rangle$ in (70). Thus although in principle the effective operator (70) might have led to a large $E2$ transition, it turns out again that its contribution is depressed, as suggested by the NRQM.

I will not make this more quantitative here, but add the following remark: in this description the space part of the wave functions of the Δ and P continue to be taken as perfectly spherical symmetric; all the asphericity, due, in the NRQM language, to configuration mixing,[8] is transferred into the effective operator (70). At the same time the operator (70) also includes other effects, for instance, exchange of $q\bar{q}$ pairs and relativistic effects.

As already stated we are dealing here with a correspondence between exact and model states characterized by the operator V; depending on the case at hand V can be thought to be applied to the model states, thus producing configuration mixing and all that; or it can leave unaltered the simple model states and transform an operator O into $V^\dagger OV$; both choices are equally good, but, of course, one should avoid doing both things together.

XI. PARAMETRIZATION OF THE MASSES

If we replace in Eq. (6) the magnetic moment operator $\mathcal{M}$ by the exact Hamiltonian H of the system, the same argument used to parametrize the magnetic moments leads to a parametrization of the masses of the baryons for the octet and decuplet. The difference is that instead of an axial vector we must now parametrize a scalar (under space rotations). Again the factorization property (10) of the model state reduces the expectation value of the Hamiltonian to a combination of spin-flavor operators multiplied by some coefficients, the same for all the states of the octet and decuplet. As to the spin-flavor invariants that can intervene, we recall the following.

(1) As shown in Sec. V the expectation value of $\boldsymbol{\sigma}_1\cdot(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)$ on a real spin-flavor wave function vanishes; the same is true for $\boldsymbol{\sigma}_1\cdot(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)\Gamma(f)$ where $\Gamma(f)$ is any real flavor operator. The only spin invariants that intervene in parametrizing a scalar are thus

$$1, \quad (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k) . \tag{71}$$

(2) As to the structures in the flavor space, the charge operator P^q will now be absent, of course (we will not consider the electromagnetic mass differences); because the only flavor operator intervening in the theory is P^λ, only products of P^λ's can intervene, containing, of course, a maximum of three P^λ's.

Because each spin-flavor operator has to be symmetrical in the coordinates of the three quarks (again because the spin-flavor structure of our model states is symmetrical) the following nine independent operators can intervene in the expression of the masses:

$$\begin{gathered} 1, \quad \sum_i P_i^\lambda, \quad \sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k), \quad \sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(P_i^\lambda+P_k^\lambda) , \\ \sum_{\substack{i\neq k\neq j \\ (i>k)}}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)P_j^\lambda, \quad \sum_{i>k}P_i^\lambda P_k^\lambda, \quad \sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)P_i^\lambda P_k^\lambda , \\ \sum_{\substack{i\neq k\neq j \\ (i>k)}}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)P_j^\lambda(P_i^\lambda+P_k^\lambda), \quad P_1^\lambda P_2^\lambda P_3^\lambda . \end{gathered} \tag{72}$$

Again, as for the magnetic moments, we will limit ourselves to consider terms of first order in flavor breaking [with the meaning of this clarified in Sec. VI, the statement before Eq. (40)]. Excluding all the above terms with two or more P^λ's, the most general expression for the mass, to first order in flavor breaking is

$$M=M_0+B\sum_i P_i^\lambda+C\sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)+D\sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(P_i^\lambda+P_k^\lambda)+E\sum_{\substack{i\neq k\neq j\\(i>k)}}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)P_j^\lambda\ . \tag{73}$$

Expression (73) contains five parameters; because we must fit four octet masses and four decuplet ones, we will get three (well-known) mass relations.

Before this a remark is appropriate: the argument just given applies to the expectation value of any quantity independent of the charge P^q, scalar under rotations; it uses, in fact, only the factorizability of the states and the list (72) of invariants in the spin-flavor space. Thus it applies not only to the Hamiltonian H, but also to its square H^2, or to H^3 or more generally to any function $F(H)$ of H. This has two consequences.

(1) It shows in general (not only for the case of the masses) that the number of independent invariants [for the masses those in the list (72)] has to be always larger than or equal to the number of the states; if it were not so we would have rigorous relations between the masses of the states which would be true simultaneously for the masses, for their squares or for any function $F(M)$ of the masses; this is clearly impossible except for the uninteresting case of complete mass degeneracy.

(2) If we introduce an approximation, that of neglecting invariants that correspond to flavor breaking of second order or more, then one can obtain relations between the masses (the Gell-Mann–Okubo mass formula will be seen to be one of them); but these relations can equally be derived for any power or function of the masses. Of course, these relations do not have all the same accuracy: if it is, say, a fair approximation to treat flavor breaking to first order when one is dealing with H, it is certainly a less fair approximation to do the same in dealing with $H^{10}=(H_0+H_1)^{10}$. What is in this sense the best function of H, such that the effect of the neglected flavor-breaking terms is minimum is hard to say, *a priori* (but it is hard to see a reason why the quadratic mass formulas should be better than the linear ones, a long-debated question; this is some justification for the linear mass formulas). Of course, all the above remarks apply also to the conventional group theoretical derivation of, say, the Gell-Mann–Okubo mass formula.

Similar considerations hold also for the magnetic moments; we might have parametrized $H\mathcal{M}H$ instead of $\mathcal{M}$, for instance, in a way formally identical to that used for $\mathcal{M}$. We did not raise this point for the magnetic moments only because the situation for the masses is much more familiar.

XII. THE MASS FORMULAS AND THEIR RELATIONSHIP WITH THOSE OBTAINED WITH THE NRQM

We now write the masses of the octet and decuplet baryons in terms of the coefficients $M_0,\ldots,E$ of the parametrization (73). Writing again S=strangeness $=-\sum_i P_i^\lambda$ and noting that

$$\sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)=\tfrac{1}{2}[4J(J+1)-9]\ , \tag{74}$$

$$\sum_{\substack{i\neq k\neq j\\(i>k)}}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)P_j^\lambda=(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)P_3^\lambda+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)P_2^\lambda+(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)P_1^\lambda=-\tfrac{1}{2}[4J(J+1)-9]S-\sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(P_i^\lambda+P_k^\lambda)\ , \tag{75}$$

$$\sum_{i>k}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(P_i^\lambda+P_k^\lambda)=\sum_i(\boldsymbol{\sigma}_iP_i^\lambda\cdot 2\mathbf{J})+3S \tag{76}$$

we get

$$M=(M_0-\tfrac{9}{2}C)+S(3D+\tfrac{3}{2}E-B)+\frac{C}{2}[4J(J+1)]+(D-E)\sum_i(\boldsymbol{\sigma}_iP_i^\lambda\cdot 2\mathbf{J})-\frac{E}{2}S[4J(J+1)]\ . \tag{77}$$

Defining

$$\tilde{M}_0\equiv M_0-\tfrac{9}{2}C,\quad \beta=3D+\tfrac{3}{2}E-B,\quad \gamma=\frac{C}{2}\ ,\quad \delta=D-E,\quad \epsilon=-\frac{E}{2} \tag{78}$$

expression (77) can be rewritten

$$M=\tilde{M}_0+\beta S+\gamma[4J(J+1)]+\delta\sum_i(\boldsymbol{\sigma}_iP_i^\lambda\cdot 2\mathbf{J})+\epsilon S[4J(J+1)]\ . \tag{79}$$

Note now that

$$\left\langle\uparrow\left|\sum_i(\boldsymbol{\sigma}_iP_i^\lambda\cdot 2\mathbf{J})\right|\uparrow\right\rangle_{J=1/2}=3\left\langle\uparrow\left|\sum_i\sigma_{iz}P_i^\lambda\right|\uparrow\right\rangle \tag{80}$$

and

$$\left\langle\uparrow\left|\sum_i(\boldsymbol{\sigma}_iP_i^\lambda\cdot 2\mathbf{J})\right|\uparrow\right\rangle_{J=3/2}=5\left\langle\uparrow\left|\sum_i\sigma_{iz}P_i^\lambda\right|\uparrow\right\rangle\ , \tag{81}$$

where $|\uparrow\rangle$ means a state with J_z, respectively, $\frac{1}{2}$ and $\frac{3}{2}$. The values of $\langle\uparrow|\sum_i\sigma_{iz}P_i^\lambda|\uparrow\rangle$ are listed below for the **8** and **10** baryons:

N	Λ	Σ	Ξ	Δ	Σ^*	Ξ^*	Ω
0	1	$-\frac{1}{3}$	$\frac{4}{3}$	0	1	2	3

From (79), we get, therefore (baryon symbols stand for baryon masses),

$$N=\tilde{M}_0+3\gamma,\quad \Lambda=\tilde{M}_0-\beta+3\gamma+3\delta-3\epsilon\ ,\quad \Sigma=\tilde{M}_0-\beta+3\gamma-\delta-3\epsilon,\quad \Xi=\tilde{M}_0-2\beta+3\gamma+4\delta-6\epsilon \tag{82}$$

and

$$\Delta=\tilde{M}_0+15\gamma,\quad \Sigma^*=\tilde{M}_0-\beta+15\gamma+5\delta-15\epsilon\;,$$
$$\Xi^*=\tilde{M}_0-2\beta+15\gamma+10\delta-30\epsilon\;, \tag{83}$$
$$\Omega=\tilde{M}_0-3\beta+15\gamma+15\delta-45\epsilon\;.$$

With five parameters and eight masses we get three mass relations: the Gell-Mann–Okubo formula for the octet

$$\underset{(1129)}{\frac{N+\Xi}{2}}=\underset{(1135)}{\frac{3\Lambda+\Sigma}{4}} \tag{84}$$

and the equal-spacing formula for the decuplet:

$$\underset{139}{\Omega-\Xi^*}=\underset{149}{\Xi^*-\Sigma^*}=\underset{152}{\Sigma^*-\Delta}\;. \tag{85}$$

In preparation for the comparison with the results of the NRQM we give below the values (in MeV) of the parameters $\tilde{M}_0,\beta,\gamma,\delta,\epsilon$ in (79) obtained from expressions (82) and (83); we also display M_0,B,C,D,E, the parameters of (77). They are

$$\tilde{M}_0=865.2,\quad \beta=-228.6,\quad \gamma=24.5\;,$$
$$\delta=-19.4,\quad \epsilon=-1.5\;,$$

or, alternatively,

$$M_0=1086,\quad B=188.4,\quad C=49.2\;,$$
$$D=-15.4,\quad E=3.0\;. \tag{87}$$

One can check [as already seen from the mass formulas (84) and (85)] that formula (79) [or (77)] represents the data to a few percent, in spite of the fact that flavor breaking has been taken into account only to first order. Note, incidentally, that the values of the parameters listed in (86) are not a best fit; for instance, to mention only the parameters that will play a role in the following discussion, δ comes simply from $\Sigma-\Lambda=-4\delta$; γ is obtained from $\Delta-N=12\gamma$; $\epsilon=-1.5$ is taken as the average of two determinations; (a) $\Sigma^*-\Sigma-(\Xi^*-\Xi)=12\epsilon$, giving $\epsilon=-2$ and (b) $\Sigma^*-\Sigma-6\delta-12\gamma=-12\epsilon$, giving $\epsilon=-1$.

We can compare these results with those obtained in a nonrelativistic quark model; we will do this for the version of the NRQM due to De Rújula, Georgi, and Glashow[9] where the interactions between the quarks (and, in particular the spin-dependent interactions) are given by the Fermi-Breit approximation to the QCD one-gluon-exchange potential. The mass formula for the baryons, to first order in flavor breaking, is given by Eq. (6) of Ref. 9. Comparing that formula with our mass formula in its first form (77), we see that it becomes equal to ours [to order $(\Delta m/m_\lambda)^2$] if we suppress in (77) the parameter E,

$$E=0\;, \tag{88}$$

and if D and C are related by

$$\frac{D}{C}=-\frac{m_\lambda-m_{\mathcal{P}}}{m_\lambda}\equiv-\frac{\Delta m}{m_\lambda}\;, \tag{89}$$

where the m's are the quark masses. [In particular, using (78), the value (89) of D/C and (88) one can verify that our formulas for the masses (82) and (83) satisfy Eqs. (5) and (11) of Ref. 9.]

We see from (87) that the value of the parameter E, the only one outside of the NRQM description (with two-body potentials), is indeed small, with respect to the other parameters; moreover the ratio $(-D/C)$ is $(15.4/49.2)=0.31$ from (87); and this number is in good agreement with the value $(1-a)=0.35$ obtained [Eq. (49)] from the analysis of the magnetic moments, which value, in a NRQM description, can also be interpreted as $\Delta m/m_\lambda$.

XIII. CONCLUSION

We summarize below the main points.

(1) We have given general parametrizations of the baryon magnetic moments and masses. By "general" we mean that they can be deduced from the exact field theory, whatever it is, provided that the only flavor breaking in the theory is due to the quark mass difference and that the electromagnetic current is carried only by the quarks (in the future we hope to examine how necessary these restrictions are). The general parametrizations contain more parameters than data; this is always true, as has been remarked in Sec. XI. However, if we keep only those terms that break the flavor to first order, then the number of parameters reduces and several consequences emerge. One is that the resulting parametrizations are approximated well by a NRQM; this is true both for the magnetic moments and for the masses. Suppose, for instance, that from the data on the magnetic moments we had found that the parameters F,K,H,G,L in (42) were as important as μ and A. Clearly the NRQM description could not have been maintained. Similarly suppose that comparing the parametrization of the masses with reality we had found that the parameter E in (73) were as important as M_0,B,C,D; or we found that $(-C/D)$ in (73) was very different from $A/\mu(=1-a)$ in (42). Again the NRQM description could not have been maintained; the success of the model would then have been a chance.

Another comment on the same theme is appropriate: by the transformation V that relates the model states with the true states, we have established a connection between the description in a relativistic field theory and the "nuclear physics" language of the NRQM. This connection is general, but to find that the values of the parameters in nature are near to those suggested by the NRQM is an additional fact, in a sense it is the really interesting fact. To add another example take, for instance, the circumstance, stressed in Ref. 9, that the Σ is heavier than the Λ; it depends on the sign of $(-C/D)$ being equal to that of $\Delta m/m_\lambda$; again this can be deduced in a NRQM, with the potential used in Ref. 9; of course, one would like to know how it depends on the underlying field theory, but this then becomes equivalent to asking which features of a field theory are capable of producing a description which is well approximated by the NRQM—an old question and a question for the future.

(2) Always in the approximation of disregarding flavor breaking to orders higher than the first, the $\Sigma^0\rightarrow\Lambda\gamma$ rate has been related to the magnetic moments of $P,N,\Lambda,\Sigma^\pm,\Xi^{0,-}$. This relation [Eq. (55)] is a general one, independent of any model, depending only on the above approximation. It is in a way the equivalent for the mag-

netic moments of the Gell-Mann–Okubo relation for the masses.

(3) The $M1$ transition $\Delta \to P\gamma$ has always been a case with a discrepancy larger than usual (by a factor of 1.45 in the matrix element) from the NRQM. We have shown that a comparatively small percentage ($\approx 0.12\mu$) of a term not coming from the simple NRQM is multiplied (in this case) by a factor of 3 and can explain the discrepancy. We hope to reexamine in the future along the same line the vector-meson-pseudoscalar-meson$+\gamma$ $M1$ transitions and establish how near is the μ intervening in the quark spin-flip transitions of these mesons to the μ of a quark inside the proton.

(4) In the usual description the small violation of the $E2$ $\Delta \to P\gamma$ selection rule arises from a mixing of D states in the S states of the intervening baryons. Here we have parametrized the same effect in terms of a two-quark spin-tensor operator. This is nothing new, simply a different way of representing the same effect; in this way the same term includes not only the configuration mixing but also the relativistic and $q\bar{q}$-exchange effects. The important point is that we know this new term to be small: the $E2$ amplitude is from 0.9% to 2.5% the $M1$ amplitude.

(5) Coming to the masses, we have already summarized part of the results at point (1) above. The Gell-Mann–Okubo formula for the octet and the equal-spacing rule for the decuplet are given from our parametrization, after keeping only first-order flavor breaking. We recall that first-order flavor breaking means neglecting all the terms with products of two or more P_i^λ coming from different quarks; flavor-breaking effects that depend additively on each quark are kept to all orders; it is possible for this to be a reason why the Gell-Mann–Okubo formula is so unexpectedly accurate.

(6) A final point to be mentioned very briefly is this: the correspondence between model states and exact states produced by the operator V allows to clarify, qualitatively, the question of the existence or nonexistence of hadrons with exotic quantum numbers. The exact exotic hadron states are obtained from the model states with exotic quantum numbers by application of the "adiabatic" operator V. We may assume that exotic meson states arising as, say, $V|qq\bar{q}\,\bar{q}\rangle$ or glueballs $V|GG\rangle$ are not seen often because they are high in mass and decay very rapidly into nonexotic states $V|q\bar{q}\rangle$. In fact two situations can occur: (a) it is possible that a few of these states are not so high in mass, while the majority are; (b) all those states are high in mass and therefore, in practice, nonexistent. This indicates that, besides the obvious interest in looking experimentally for states that are unambiguously of exotic nature, it might help to examine on theoretical models if a situation of type (a) (a few states relatively low in mass, the majority high) can occur without too artificial assumptions. In this frame the finding of exotic states appears to be rather natural and not in contrast with the NRQM; it would simply indicate that a situation of type (a) occurs.

We hope to discuss in the future along the same lines other cases of quantitative predictions of the NRQM, including the $V \to P\gamma$ decays, the semileptonic decays, and other weak processes.

Note added in proof. On expressing μ and K in terms of the baryon magnetic moments, one can check that Eq. (55) $[\langle \Sigma^0\uparrow|M_z|\Lambda^0\uparrow\rangle = -(\mu-K)/\sqrt{3}]$ coincides, as it must, with Eq. (8b) of S. Okubo [Phys. Lett. **4**, 14 (1963)] derived using only first-order T_3^3 breaking of flavor. Note that, because of Eq. (28), the number of parameters in our full spin-flavor parametrization of the magnetic moments is not higher than that of Okubo's analysis, that refers only to the flavor space [compare Okubo's Eqs. (7) and (8) and the footnote (†) on p. 15 of his paper].

APPENDIX

Let H be the exact Hamiltonian of the quarks and gluons; to be specific we may have in mind the H of QCD. We will indicate by $q_R(x)$ a mass renormalized quark field, with (renormalized) mass m; as stated in the text m is taken to be the mass that appears in the quark propagator $(\not{k} - im)^{-1}$ for low values of the momentum transfer k^2 ($|k| \approx R^{-1}$, with R the size of the hadron; here it is unnecessary to specify $|k|$ more precisely); we assume to identify m with the mass of a constituent quark. We introduce the Fock states of the independent quarks and gluons with renormalized mass and, suppressing the index R, call $|qqq\rangle$ a state of three quarks and no gluon. Call $\eta = \sum |qqq\rangle\langle qqq|$ the projection operator into the states of three quarks and no gluons:

$$\eta|qqq\rangle = |qqq\rangle \ , \tag{A1}$$

$$\eta|\neq qqq\rangle = 0 \ . \tag{A2}$$

We rewrite H identically as

$$H = \eta H \eta + (1-\eta)H(1-\eta) + \eta H(1-\eta) + (1-\eta)H\eta \ . \tag{A3}$$

Introduce now the model Hamiltonian $\mathcal{H}$ which is a typical nonrelativistic quark-model Hamiltonian acting only on the Fock space of the states of three quarks and no gluons. We decompose H as

$$H = K_0 + K_1 \tag{A4}$$

with

$$K_0 = \eta\mathcal{H}\eta + (1-\eta)H(1-\eta) \ ,$$

$$K_1 = \eta H(1-\eta) + (1-\eta)H\eta + \eta H\eta - \eta\mathcal{H}\eta$$

having added (to K_0) and subtracted (from K_1) the model Hamiltonian $\eta\mathcal{H}\eta$. Referring to the baryons we assume that $\eta\mathcal{H}\eta$ has degenerate eigenvalues M_0^0 for all the octet and decuplet baryon states:

$$\eta\mathcal{H}\eta\phi_B = M_0^0\phi_B \quad (B = N, \Lambda, \Sigma, \Xi, \Delta, \Sigma^*, \Xi^*, \Omega) \ , \tag{A5}$$

where $|\phi_B\rangle|0 \text{ gluons}\rangle$ are the $L=0$ model states, called simply $|\phi\rangle$ in the text. Because in the three-quark sector K_0 and $\mathcal{H}$ coincide, $|\phi_B\rangle$ are the degenerate eigenstates of K_0:

$$K_0|\phi_B\rangle = M_0^0|\phi_B\rangle \ . \tag{A6}$$

In the part $\eta\mathcal{H}\eta$ of K_0 the masses of the $\mathcal{P}$, $\mathcal{N}$, and λ quarks are taken as equal [as implied by (A5)]; the flavor-breaking mass term [Eq. (5) of the text] appears in the term $(1-\eta)H(1-\eta)$ of K_0 and in the term $\eta H\eta$ of

K_1. The term $(1-\eta)H(1-\eta)$ of K_0 includes, in particular, the Hamiltonian of the noninteracting gluons; K_1 contains the interaction terms $\eta H(1-\eta)$ and $(1-\eta)H\eta$ of the quark-gluon Hamiltonian.

We now regard K_0 as the unperturbed Hamiltonian, K_1 as the perturbation; imagine inserting K_1 adiabatically, and construct the true states $|\psi\rangle$ with the procedure of Gell-Mann and Low[3] (this procedure of construction is of course, not compulsory, but we imagine using it to show that at least one method of construction exists). Writing $K_1(t)=\exp(+iK_0t)K_1\exp(-iK_0t)$ the adiabatic $U(t,t_0)$ satisfies $i\dot{U}_\alpha(t,t_0)=\exp(-\alpha|t|)\cdot K_1(t)U_\alpha(t,t_0)$ [with $\alpha>0$, $U(t_0,t_0)=1$] and the $|\psi\rangle$'s for the exact bound states corresponding to the lowest $|\phi\rangle$'s are

$$|\psi_B\rangle=\lim_{\alpha\to 0}\exp(-w_B/\alpha)\cdot U_\alpha(0,-\infty)|\phi_B\rangle\ , \tag{A7}$$

where w_B is purely imaginary ($w_B+w_B^*=0$) [so that the factor of $\exp(-w_B/\alpha)$ in front of (A7) is a pure phase factor that eliminates the singularity coming from the $\lim_{\alpha\to 0}U_\alpha(0,-\infty)$; w_B in (A7) is related to the $S=U(+\infty,-\infty)$ matrix element of the $\phi_B\to\phi_B$ transition by $\lim_{\alpha\to 0}\langle\phi_B|S|\phi_B\rangle=\exp(2w_B/\alpha)$]. The operator V introduced in the text can be, therefore, written explicitly as

$$V=\lim_{\alpha\to 0}\exp(-w_B/\alpha)\cdot U_\alpha(0,-\infty)\ . \tag{A8}$$

The formula for the magnetic moment has the form used in the text: namely,

$$\mathbf{M}=\langle\phi|V^\dagger\mathcal{M}V|\phi\rangle\ . \tag{A9}$$

As can be seen in a few passages, this is the same as the formula used frequently for practical calculations:

$$\mathbf{M}=\frac{\langle\phi|T(\mathcal{M}(0)S)|\phi\rangle}{\langle\phi|S|\phi\rangle}\equiv\langle\phi|T(\mathcal{M}(0)S)|\phi\rangle_C\ , \tag{A10}$$

where the index C means "connected." However, formula (A8) for V is not that written most frequently:

$$|\psi\rangle=\lim_{\alpha\to 0}\frac{U_\alpha(0,-\infty)|\phi\rangle}{\langle\phi|U_\alpha(0,-\infty)|\phi\rangle}\ . \tag{A11}$$

Although the final formulas for the physical quantities are always the same, in (A11) the denominator is not a pure phase factor.

We have insisted a little on this to make it clear that the magnetic moment can be written, in the adiabatic procedure, simply as the expectation value of an axial vector, not as the expectation value of an axial vector divided by a baryon-dependent scalar as one might think at first sight having in mind a V defined on the basis of (A11) [compare the remarks at the end of Sec. VIII]; for the mass formulas we have to parametrize just a scalar, the expectation value of the Hamiltonian, not the quotient (or product) of two scalars. A similar conclusion holds for the $\Sigma^0\to\Lambda\gamma$ rate; there one has to parametrize the square modulus of the matrix element $\langle\phi_\Sigma|V^\dagger\mathcal{M}V|\phi_\Lambda\rangle$; in forming the square modulus the singular phase factors disappear.

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[1](a) G. Morpurgo, Physics (N.Y.) **2**, 95 (1965) [also reproduced in J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969), p. 132]; (b) G. Morpurgo, in *XIV International Conference on High Energy Physics,* Vienna, 1968, edited by J. Prentki and J. Steinberger (CERN, Germany, 1968), p. 225. In Sec. 5.1 of (b) the idea of the V correspondence between model and exact states was introduced and a parametrization of the magnetic moments was discussed [Eq. (28)]; however, at that time we did not realize the simplifications described here in Sec. V nor those coming from flavor breaking being due only to the mass terms.

[2]For the status of the magnetic moments compare the surveys of J. Franklin: (A) experimental and theoretical status of baryon magnetic moments, (B) theoretical status of baryon magnetic moments in *High Energy Spin Physics,* proceedings of the Eighth International Symposium, Minneapolis, Minnesota, 1988, edited by K. Heller (AIP Conf. Proc. No. 187) (AIP, New York, 1988). They contain a wide set of references; however, Refs. 4 and 5 of (A) do not give a correct account of the historical evolution: that the static quark model implies automatically the SU(6) wave functions was first underlined strongly in Ref. 1(a) (p. 101); that a measurement of the Λ magnetic moment might determine the magnetic moment of the strange quark was noted already in C. Becchi and G. Morpurgo, Phys. Rev. **140**, B687 (1965). I thank Professor J. Franklin for a copy of his papers prior to publication.

[3]M. Gell-Mann and F. Low, Phys. Rev. **84**, 181 (1951); for detailed presentations of the adiabatic procedure see, also, (a) S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row and Peterson, New York, 1961); (b) P. Nozières, *Le Probleme à N corps* (Dunod, Paris, 1963), p. 147.

[4]For $\Sigma^+=2.38$ (and the other magnetic moments as above) one obtains $A=1.08$, $K=0.31$, $H=0.12$, $G=-0.13$, $L=-0.10$. The most recent values of the magnetic moments are given in Ref. 2(A).

[5]R. H. Dalitz and D. G. Sutherland, Phys. Rev. **154**, 1608 (1967) (in estimating the rate we used a Δ width of 110 MeV).

[6]C. Becchi and G. Morpurgo, Phys. Lett. **17**, 352 (1965).

[7]R. Davidson, N. C. Mukhopadhyay, and R. Wittman, Phys. Rev. Lett. **56**, 804 (1986).

[8]A. Le Yaouanc, L. Oliver, O. Pène, and J. Raynal, Phys. Rev. D **15**, 844 (1977); N. Isgur, G. Karl, and R. Koniuk, Phys. Rev. Lett. **19**, 1269 (1978).

[9]A. De Rújula, H. Georgi, and S. Glashow, Phys. Rev. D **12**, 147 (1975). For convenience we transcribe Eqs. (6), (5), and (11) of this paper quoted in the text: Eq. (6),

$$M=A+B\sum_i\frac{\Delta m_i}{m_{\mathcal{P}}}+C\sum_{i>j}\mathbf{s}_i\cdot\mathbf{s}_j[1-(\Delta m_i+\Delta m_j)/m_{\mathcal{P}}]$$

(here A,B,C are the coefficients of De Rújula, Georgi, and Glashow different from ours); Eq. (5), $m_{\mathcal{P}}/m_\lambda=2(\Sigma^*-\Sigma)/(2\Sigma^*+\Sigma-3\Lambda)$; Eq. (11), $\Sigma-\Lambda=\frac{2}{3}[1-(m_{\mathcal{P}}/m_\lambda)](\Delta-N)$.