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Heavy-quark potential at finite temperature from a string picture

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The heavy-quark potential at finite temperature is calculated from a string picture, where the string is regulated on the lattice. While the potential agrees exactly with those found previously in special cases such as the zero-temperature limit, the oscillational modes do add several interesting new terms at finite temperature.

The intention of this paper is to compute the heavy-quark potential at finite temperature predicted by the simple free-string model. The knowledge of the potential is important in understanding the dynamics of QCD and useful to experiments such as heavy-ion collisions. The potential can be used to guide the accurate measurement of the nonperturbative string tension by Monte Carlo simulation of lattice QCD.

The free energy of an infinitely heavy-quark–antiquark pair system can be calculated from the correlation function of two Polyakov loops on the lattice. Consider the case when a quark pair is introduced into the lattice at $\mathbf{0}$ and $\mathbf{R}$,

$$\Gamma(\mathbf{R},T)=\int \prod_x d[U_x]L(\mathbf{0})L^\dagger(\mathbf{R})e^{-S_p} \tag{1}$$

is the correlation function, where U_x is the gauge field variable, L is the ordered product of links along the loop in the T direction which represents a quark in the fundamental representation, and S_p is the plaquette action. $\Gamma(\mathbf{R},T)=e^{-\mathcal{F}(\mathbf{R},T)/T}=e^{-V(\mathbf{R},T)/T}$ determines the free energy and the free energy is the interaction potential in the static case. Inspired by the strong-coupling expansion of Eq. (1), a string model

$$\Gamma(\mathbf{R},T)\sim \sum_{\text{surfaces}} e^{-K(\text{area of the surface})}$$
$$\sim \int d[X_{\tau,\rho}]\exp\left[-\left[\frac{K}{2}\right]\int\int d\tau\, d\rho\, \nabla X\nabla X\right] \tag{2}$$

will be used as an approximation, where K determines the string tension and X will be defined on a two-dimensional lattice to regulate the operator ∇^2. This is the starting point for this paper and we will derive the free-string potential from it. The underlying physics is the belief that the strong self-interaction among gluons contracts the force field between quarks into an essentially one-dimensional flux tube and a large part of the dynamics of the QCD system is characterized by the motion of the flux tube. This is the basis for the string picture.

In the following, we will calculate Eq. (2) on the lattice in the continuum limit. This gives us the potential and it is the main result of this paper. For convenience, we will consider Polyakov loops of length $N_t=1/Ta$ with separation $N_r=R/a$ (both N_t and N_r are dimensionless integers), where R,T are the physical distance and temperature and a is the lattice unit. Then the size of the two-dimensional lattice is given by N_t and N_r and the continuum limit at finite temperature should be given by $N_t,N_r\to\infty$, $a\to 0$ with N_r/N_t fixed so that R and T are finite while the product RT is fixed.

By making a definite choice of parameters $X_4=\tau$ and $X_1=\rho$, Γ becomes

$$\Gamma(R,T)\sim \int \prod_{\tau,\rho} d^2X_\perp(\tau,\rho)\exp\left[-KN_tN_r-\tfrac{1}{2}K\int_0^{N_t}d\tau\int_0^{N_r}d\rho\,\nabla X_\perp\nabla X_\perp\right]. \tag{3}$$

Calculating the eigenvalues of $-\nabla^2$ on the lattice ($d^2/dx^2\to\delta_{x,x+\mu}+\delta_{x,x-\mu}-2\delta_{x,x}$, $x=\tau,\rho$) with periodic and fixed boundaries for N_t and N_r, respectively, we get[1]

$$\Gamma(R,T)=e^{-Ka^2N_tN_r}[\det(-\tfrac{1}{2}K\nabla^2)]^{-1}$$
$$\sim\exp\left[-Ka^2N_tN_r-\sum_{m,n}(\tfrac{1}{2}Ka^2\ln\lambda_{m,n})\right], \tag{4}$$

where

$$\lambda_{m,n}=4-2\cos(2\pi m/N_t)-2\cos(\pi n/N_r)\,, \tag{5}$$

with $-N_t/2<m\le N_t/2$, $0<n<N_r$. Let

$$F(N_t,N_r)=\sum_{m=1}^{N_t/2-1}\sum_{n=1}^{N_r-1}2f_2(m\pi/(N_t/2),n\pi/N_r)+\sum_{n=1}^{N_r-1}f_1(n\pi/N_r)\,, \tag{6}$$

where

$$f_2(x,y)=\ln(4-2\cos x-2\cos y)\ ,$$
$$f_1(y)=\ln(2-2\cos y)+\ln(6-2\cos y) \tag{7}$$

$[x=m\pi/(N_t/2),\quad y=n\pi/N_r,\quad$ and $\quad r=N_r/N_t=RT$ throughout this paper], then

$$\Gamma(R,T)=e^{-V(R,T)/T}$$
$$=\exp[-Ka^2N_tN_r-\ln(Ka^2/2)N_t(N_r-1)-F]\ . \tag{8}$$

Once we have calculated $F(N_t,N_r)$, we get $\Gamma(R,T)$ and hence $V(R,T)$. Since we are interested in the result near the continuum limit, we need only to find the asymptotic behavior of F. The function F is expected to have asymptotic leading terms involving $N_t\times N_r, N_t, N_r$ and these terms will carry all the regularization-scheme-dependent divergences (of the form of $1/a$ after changing back to physical T and R) in the continuum limit. Then there should be terms which are independent of the lattice cutoff. Besides a constant term, others can only depend on the ratio of N_r/N_t (no asymptotic terms with N_t^k/N_r^l or N_r^k/N_t^l for $k>l+1$ can exist since they would blow up quicker than the leading terms). We will call these terms (excluding the constant) the subleading asymptotic terms and they will not be influenced by the regularization scheme. The other terms in F will be the constant term and terms with at least one extra factor of $1/N_t\sim Ta$ or $1/N_r\sim a/R$, which will be of the order $O(a)$ near the continuum limit. These terms are not interesting and will be dropped in the continuum limit. All the physics is carried by the leading (after the renormalization of the string tension) and subleading asymptotic terms in the continuum limit. Finding these asymptotic terms in the right limit is the objective of following.

Before we go to the general case, let us discuss the results for two special cases. For the first case where $N_t\to\infty$ and $N_r(\text{finite})\gg 1$, by using Euler's sum formula[2] which must be applied first to sum over m and second to sum over n and

$$\sum_{i=1}^{I-1}\ln[2-2\cos(\pi i/I)]=\ln I\ , \tag{9}$$

we get the asymptotic terms of the two series summations

$$\sum_n\sum_m 2f_2(m\pi/(N_t/2),n\pi/N_r)$$
$$=-\sum_n f_1(n\pi/N_r)+C_1N_tN_r-C_2N_t-(\pi/12)N_t/N_r\ ,$$
$$\sum_n f_1(n\pi/N_r)=\ln N_r+N_r\ln(3+\sqrt{8})-\ln\sqrt{32}\ , \tag{10}$$

where

$$C_1=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi dx\,dy\,\ln(4-2\cos x-2\cos y)$$
$$=1.166\,243\,7\ ,$$
$$C_2=\tfrac{1}{2}\ln(3+2\sqrt{2})=0.881\,373\,6\ . \tag{11}$$

Then

$$F=C_1R/Ta^2-C_2/Ta-(\pi/12)/RT+O(a)\ , \tag{12}$$

near the continuum limit and V is of the form $(a\to 0)$

$$V(R,T)=\sigma R-(\pi/12)/R+\text{const}\ , \tag{13}$$

where the renormalized string tension is given by $\sigma=K+\ln(Ka^2/2)/a^2+C_1/a^2$. This is exactly the zero-temperature potential derived in Ref. 2. The $\pi/12$ factor can also be found from many other ways.[3] This is not surprising since the limit $(N_t\to\infty, N_ta=1/T$ finite and $N_r/N_t=RT\to 0)$ gives exactly $R\ll 1/T$, i.e., $T\to 0$ or the separation is much smaller than the thermodynamic scale.

For the other case where $N_r\to\infty$ and $N_t(\text{finite})\gg 1$, the order of the double summation must be reversed (the Euler sum formula cannot be used otherwise). However, the result for the double summation can be gotten by using the symmetry between N_r and $(N_t/2)$ and simply exchanging $N_r\Longrightarrow N_t/2$. The asymptotic behavior for the new limit is

$$F=C_1R/Ta^2-C_2/Ta-(\pi/3)RT+\ln(2RT)\ , \tag{14}$$

$$V(R,T)=\sigma R-(\pi/3)RT^2+T\ln(2RT)+\text{const}\ . \tag{15}$$

This is the result given by Ref. 1. It is the case of a very long string where the length is much longer than thermodynamic scale. This is because the requirement $N_r/N_t=RT\to\infty$ means $R\gg 1/T$, but T is bounded by T_c (hence $1/T$ is always larger than $1/T_c$) for the QCD string to be valid.

It is easy to see that the difference for the two results is due to the fact they represent the asymptotic behavior in two different extreme cases when the order of the summation is done differently. The problem comes from the Euler sum formula since both $f_1(y)$ and $f_2(x,y)$ are not continuously differentiable at $x=y=0$. For example, $(\partial f_2/\partial x)(0,0)=0/0$ and $(\partial f_2/\partial y)(0,0)=0/0$, both are indefinite depending on how fast $x\to 0$ and $y\to 0$. In the first case, $N_t\to\infty$ while N_r is held finite, i.e., $x\to 0$ is taken first while y is held finite. In the second case, it is just the opposite. Because of this, $\int dx\,[\lim_{y\to 0}(\partial f_2/\partial y)(x,y)]=0$ or $\pi/2$ for the two cases depending on the order of the limiting process $(y\to 0)$ and the integrating process $(x\to 0)$. This causes the difference in the two results. Note, in both cases, only the first summation has the difficulty, the second is well behaved.

However, it is perhaps more interesting to find the general solution and get the complete asymptotic behavior in the continuum limit. The proper continuum limit at finite temperature should be given by $N_t\to\infty$, and $N_r\to\infty$ with N_r/N_t fixed so that $RT=$finite. Since the actual value for x is limited to $[1,\ldots,(N_t/2-1)]\pi/(N_t/2)$ and y to $[1,\ldots,(N_r-1)]\pi/N_r$, the speed that x,y go to zero are exactly given by the relative size of $2/N_t$ and $1/N_r$. The above two cases of $x\to 0$ first (y held finite so that $N_r/N_t\to 0$) and $y\to 0$ first (x held finite so that $N_t/N_r\to 0$) reflect the ratio of $r=N_r/N_t$ and correspond to $r\to 0$ *or* ∞, respectively. What we need for the right asymptotic behavior is to keep $r=N_r/N_t$

$\sim x/2y$ fixed and finite while we take the limit $x\to 0$ *and* $y\to 0$.

To get the right subleading terms, we must take care of the indefinites properly. Let us consider a modified problem. For α positive and small,

$$F^{\alpha}(N_t,N_r)=\sum_{m=1}^{N_t/2-1}\sum_{n=1}^{N_r-1}2f_2((m+\alpha)\pi/(N_t/2),+(n+\alpha)\pi/N_r)+\sum_{n=1}^{N_r-1}f_1((n+\alpha)\pi/N_r)\ . \qquad (16)$$

The functions $f_i^{\alpha}(x,y)=f_i(x+2\alpha\pi/N_t,y+\alpha\pi/N_r)$, $i=1,2$ are clearly continuous and differentialable for $x,y\in[0,\pi]$ where $2m\pi/N_t\to x$ and $n\pi/N_r\to y$. Now we can use the Euler sum formula for the asymptotic terms and evaluate f_i^{α} and their derivatives at $(x=0,y=0)$. In this calculation, the two leading terms are not changed at all while the indefinites are regulated and correct control for the right continuum limit is guaranteed. Also, the symmetry between N_r and $N_t/2$ in the double summation is exactly preserve (it has been lost in the two special cases) and the summation order (hence the order of the limits) dependence is eliminated. Since all subleading terms involve only the ratio of $r=N_r/N_t$, the α dependence cancels out completely. The reason is that α/N_t and α/N_r coming from the functions $f_i(2\alpha\pi/N_t,\alpha\pi/N_r)$ and their derivatives contribute only through their ratio. One can check that the $O(a)$ terms are also finite as $\alpha\to 0$. The limit of $\alpha\to 0$ for the asymptotic expansion can be taken readily. This leads to the correct continuum limit

$$\begin{aligned}F=&C_1R/Ta^2-C_2/Ta+\tfrac{1}{2}\ln[1+(2r)^2]\\&-\left[\frac{\pi}{3}-\tfrac{2}{3}\arctan[1/(2r)]\right]RT\\&-\left[\frac{\pi}{12}-\tfrac{1}{6}\arctan(2r)\right]\Big/(RT)+\text{const}\\&+O(a)\ .\end{aligned} \qquad (17)$$

It is easy to check that the above result agrees exactly with the two special cases in the limits $r=0$ and ∞. Note, only the subleading terms are influenced for different r's and the leading terms are not changed at all. It is clear that as r increases or decreases, F changes from one of the special case to the other continuously.

Thus, we have found the static heavy-quark potential at finite temperature (in the continuum limit $a\to 0$, $T=1/N_ta$ and $R=N_ra$ finite) from a string picture. It is of the form

$$\begin{aligned}V(R,T)=&\sigma R-\tfrac{1}{2}T\ln[1+(2RT)^2]\\&-\left[\frac{\pi}{3}-\tfrac{2}{3}\arctan[1/(2RT)]\right]RT^2\\&-\left[\frac{\pi}{12}-\tfrac{1}{6}\arctan(2RT)\right]\Big/R+\text{const}\ .\end{aligned} \qquad (18)$$

This is the main result. Here σ is renormalized in the same way as given by Eq. (13).

Since F can be calculated accurately for a given size N_t,N_r by using a computer, we can perform a few numerical checks easily. Since the leading terms in F (i.e., $C_1N_tN_r-C_2N_t$) are dominant yet free from ambiguity, we will consider $\mathcal{F}\to F-C_1N_tN_r+C_2N_t$ only for the following to test the subleading terms.

One simple check is done by fixing N_t and changing N_r from $(0.1,0.2,\ldots,1.0)\times N_t$ to $(2,3,\ldots,10)\times N_t$. Figure 1 shows a comparison ($N_t=100$) between the calculated $\tilde{F}$'s (circles) and the (subleading) asymptotic function (curve). The agreement is very good. Another check is made by calculating $\tilde{F}(N_t,N_r)-\tilde{F}(N_t,N_t)$ for 30 points with $N_t,N_r\in[10,10\,000]$ and fitting to $\phi(r)-\phi(1)$ with

$$\begin{aligned}\phi(r)=&X_1\ln(1+4r^2)-[X_2-X_3\arctan(2r)]/r\\&-[X_4-X_5\arctan(1/2r)]r\end{aligned}$$

to eliminate the constant. The fitted results for $\{X_i: i=1,\ldots,5\}$ ($X_i=0.495\,87, 0.262\,63, 0.184\,11, 1.063\,54, 0.639\,73$) agree very well with their expected values. The conclusion is that Eq. (17) is the right asymptotic function, and more important, it agrees with the series very accurately beginning with $N_t,N_r>10$.

The biggest motivation is the hope that Eq. (18) might be a good potential model for QCD. Then the real check

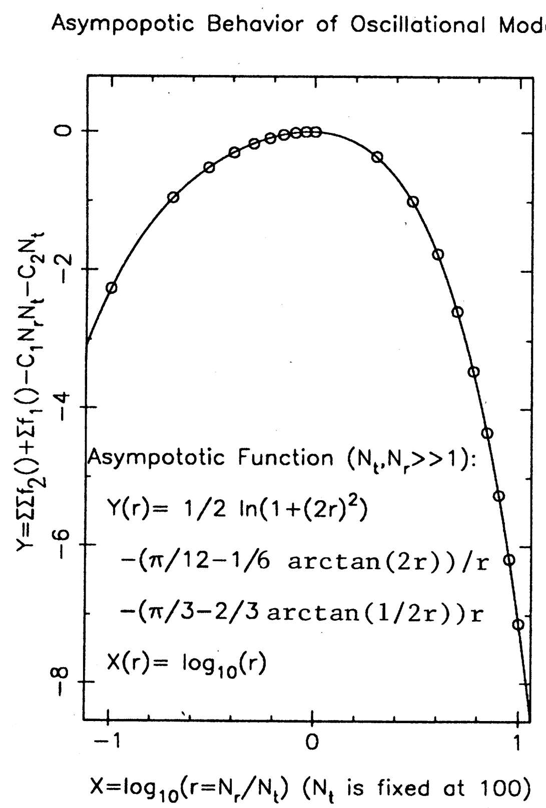

FIG. 1. Comparison between calculated $F(N_t,N_r)$ (circles) and its asymptotic function (curve). The leading terms have been eliminated for the Y coordinate in both cases and the X coordinate is the ratio of $r=N_r/N_t$ plotted on a logarithm scale.

is whether or not the result fits finite-temperature QCD. Since $\Gamma(\mathbf{R},T)$ for all the possible $\mathbf{R}$ on large lattices $(24^3 \times N_t)$ has been measured by Monte Carlo simulation of lattice QCD (Ref. 4) on supercomputers such as the Columbia parallel processor,[5] the new potential can be easily tested.[6] It is important to notice that most simulations of lattice QCD are done for $r = RT$ finite $(r \sim 1)$ so that a guidance for QCD at this range of R is very useful.

Now, we come to the conclusion. The heavy-quark potential is found based on a simple string model. It is the free-string potential at finite temperature. Just as the potential given by Ref. 3 has been used for guiding the zero-temperature physics, the potential here is expected to be useful for finite-temperature physics. To improve the simple model, for phenomenological purposes, we can add T dependence to the model by letting $K = K(T)$, or more directly by letting $\sigma = \sigma(T)$ for the potential which can be determined by lattice-gauge-theory simulation, for example (as is motivated by the simulation result[4]). This may provide a more suitable heavy-quark potential to compare with the dynamics of QCD.

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