Derivation of the three-body bound-state equation from the effective action

M. Komachiya, M. Ukita, and R. Fukuda
Department of Physics, Faculty of Science and Technology, Keio University, Yokohama 223, Japan
(Received 20 March 1989)

Using a recently developed method, a formal novel derivation of the three-body bound-state equation is presented. It is based on the second derivative of the effective action. The baryonic equation in the case of quantum chromodynamics is also derived.

I. INTRODUCTION

Recently a systematic method for deriving a bound-state equation has been proposed by one of the authors.[1] It starts from the effective action Γ which is obtained by the generating functional W of the connected Green's function through the Legendre transformation. The two-body bound-state equation in the field-theoretical case, i.e., the Nambu-Bethe-Salpeter (NBS) equation,[2] has been derived using this method.[1] By systematic we mean that the functional Γ is indeed a generating functional in the following sense. It determines the ground-state solution by the stationary requirement and the value of Γ at this stationary solution is related to the total energy of the ground state if the solution is time independent. Moreover, if we expand Γ around the stationary solution, the lowest term determines the particle spectra[1] and the higher orders determine the scattering of these particles;[3] we get the on-shell expansion of Γ where the expansion coefficients are the connected S-matrix elements.

The purpose of the present paper is to apply the same method for *deriving* the three-body bound-state equations. We do not claim that our method sheds new light on the way to solve the three-body bound-state equation but, since the method is formal and straightforward, it can be a firm basis of further systematic studies.

The three-body bound-state problem has a long history starting from the original work by Faddeev.[4,5] The same problem has been discussed[6] in the framework of quantum chromodynamics (QCD)—a bound-state equation corresponding to the baryon which is composed of three quarks.

Our approach has the following advantages.

(1) As is explained above, since it is a systematic approach, we can calculate various quantities related to the bound state. All the observable information comes from Γ.

(2) It can be applied to any quantum theory: quantum mechanics of few particles, field-theoretical system or the system with finite temperature.[1] *The formalism does not change according to the system considered.*

(3) There is a case where the ground state realizes the nonperturbative condensation. For example, we know the electron-pair condensation in superconductivity and the quark-antiquark condensation in massless QCD. We have to discuss the bound-state spectrum based on the above condensed ground state. Since we use the second derivative of Γ evaluated at the solution which satisfies the stationary condition of Γ, these condensation phenomena are automatically included in our formalism. It will be difficult to study the problem by a usual intuitive graphical approach. Our method is particularly useful in these situations.

We now summarize the results of Ref. 1 which are necessary for the following discussions. Let the dynamical coordinate be Φ_j, where j includes all the attributes of the coordinate: it includes the internal degrees of freedom and the indices specifying the particle species in the case of quantum mechanics or the component of the field variables as well as the space-time coordinates for the field-theoretical case. Consider the Lagrangian $L(\Phi) \equiv L(\Phi_1, \Phi_2, \ldots)$ and its action $I[\Phi] = \int dt\, L(\Phi)$ of the system and define $W[J]$ by

$$\exp(iW[J]) \equiv \int [d\Phi_j] \exp\{i(I[\Phi] + J_j O_j)\} \ , \qquad (1.1)$$

where O_j's are the arbitrary operators constructed out of Φ_j and $\int[d\Phi_j]$ denotes the path integral (or the functional integral in the field theory). The summation or the integration over the repeated indices is implied. The effective action $\Gamma[\phi]$ is defined by the Legendre transformation as

$$\Gamma[\phi] \equiv W[J] - J_j \frac{\delta W[J]}{\delta J_j} \ , \qquad \phi_j \equiv \frac{\delta W[J]}{\delta J_j} \ . \qquad (1.2)$$

The stationary condition

$$-J_j = \frac{\delta \Gamma[\phi]}{\delta \phi_j} = 0 \qquad (1.3)$$

determines the ground-state expectation value $\langle 0|O_j|0\rangle = \phi_j$. Let us denote one of the solutions to (1.3) by $\phi_j = \phi_j^{(0)}$ and look for another solution in the form $\phi_j = \phi_j^{(0)} + \Delta\phi_j$. Assuming $\Delta\phi_j$ is small we find the eigenvalue equation for $\Delta\phi_j$ which determines the particle spectrum:

$$\left[\frac{\delta^2 \Gamma[\phi]}{\delta\phi_j \delta\phi_k} \right]_0 \Delta\phi_k = 0 \ , \qquad (1.4)$$

where $(\ )_0$ denotes the value of $(\ )$ evaluated at $\phi_j = \phi_j^{(0)}$. Here we notice an identity

$$-\delta_{jk}=\left[\frac{\delta^2\Gamma[\phi]}{\delta\phi_j\delta\phi_l}\right]\left[\frac{\delta^2 W[J]}{\delta J_l\delta J_k}\right], \quad (1.5)$$

where

$$\left[\frac{\delta^2 W[J]}{\delta J_l\delta J_k}\right]_{J=0}=\langle 0|TO_lO_k|0\rangle_c \quad (1.6)$$

is the connected Green's function. Comparing (1.5) with (1.4), it is seen that we are looking for the pole of the Green's function. This is the reason why Eq. (1.4) determines the particle spectrum or the mode. Indeed for the time-independent solution $\phi_j^{(0)}$, we have shown how Eq. (1.4) looks and we demonstrated in particular that Eq. (1.4) coincides with the NBS equation if we choose a bilocal product of the field Φ_j as O_j; $O_j \propto \Phi_k\Phi_l$ (Ref. 1).

With these results at hand the three-body bound-state equation for the bosonic case is studied in Sec. II where $\lambda\Phi^4$ theory is used as an illustration. We derive the baryonic equation as a three-quark bound state in Sec. III. The underlying dynamics is QCD. We confirm the equivalence of our equation with the one derived by previous authors.[5,6] Section IV is devoted to several discussions. Our arguments below rely heavily on the results of De Dominicis and Martin.[7] Although the derivations of their formulas are rather involved, their final expressions are simple and elegant.

II. BOSON FIELD CASE

Following our formalism explained in the Introduction, the three-body NBS equations are derived if we take O_j in (1.1) as the nonlocal products of three fields. In this case, the effective action Γ can be calculated by the use of the general Legendre transformation rule given by De Dominicis and Martin.[7] Originally this rule was derived in the framework of the nonrelativistic quantum-statistical mechanics, but it can also be applied to our case with a small modification. We first summarize about this (modified) rule to derive the expression of Γ, which then allows us to get the three-body bound-state equation in the form of (1.4).

Let us examine the boson field case. We follow De Dominicis and Martin and start with the generating functional W defined in the form

$$\exp(iW[v_\nu])\equiv\int[d\Phi_j]\exp\left[i\left[v_1(j)\Phi_j+\frac{1}{2!}v_2(j,k)\Phi_j\Phi_k+\frac{1}{3!}v_3(j,k,l)\Phi_j\Phi_k\Phi_l+\frac{1}{4!}v_4(j,k,l,m)\Phi_j\Phi_k\Phi_l\Phi_m\right]\right], \quad (2.1)$$

where each v_ν ($\nu=1,\ldots,4$) is completely symmetrized with respect to its arguments. We have written (1.1) in the form (2.1) so that the external sources may be included in v_ν's. (Incidentally this starting point is different from the original one in Ref. 7, where W is defined through the grand partition function $e^W=\mathrm{Tr}e^{-H}$.) Using W, G_ν is then introduced,

$$G_\nu(j,k,\ldots)\equiv\nu!\frac{\delta W[v_\nu]}{\delta v_\nu(j,k,\ldots)}, \quad (2.2)$$

and its connected part is denoted as $\tilde{G}_\nu$. For example,

$$G_1(j)=\tilde{G}_1(j),\quad G_2(j,k)=\tilde{G}_2(j,k)+G_1(j)G_1(k),$$
$$G_3(j,k,l)=\tilde{G}_3(j,k,l)+G_1(j)\tilde{G}_2(k,l)+G_1(k)\tilde{G}_2(l,j)+G_1(l)\tilde{G}_2(j,k)+G_1(j)G_1(k)G_1(l). \quad (2.3)$$

In the following, instead of $\tilde{G}_3$, we use the amputated part C_3 defined as

$$\tilde{G}_3(j,k,l)\equiv C_3(j',k',l')\tilde{G}_2(j',j)\tilde{G}_2(k',k)\tilde{G}_2(l',l). \quad (2.4)$$

Now we can write down the De Dominicis–Martin Legendre transformation formula in the form[7]

$$F[G_1,\tilde{G}_2,C_3]\equiv W[v_\nu]-v_1(j)G_1(j)-\frac{1}{2!}v_2(j,k)G_2(j,k)-\frac{1}{3!}v_3(j,k,l)G_3(j,k,l) \quad (2.5a)$$

$$=-\frac{i}{2}\mathrm{Tr}\ln\tilde{G}_2+\frac{i}{2}\frac{1}{3!}C_3(j,k,l)\tilde{G}_2(j,j')\tilde{G}_2(k,k')\tilde{G}_2(l,l')C_3(j',k',l')-i\kappa. \quad (2.5b)$$

Here κ consists of one-, two-, and three-particle-irreducible vacuum graphs constructed out of $\tilde{G}_2$ (propagator), C_3 (three-point vertex), iv_4 (four-point vertex), and G_1 (vacuum expectation value of the field). Some of them are shown in Fig. 1 (graphically G_1 is written by the wavy line which directly connects to the vertices). The term "three-particle irreducible (3PI)" usually means the graphs which cannot be disconnected by cutting any three internal lines. But even when the graph is disconnected by this process, if one (and only one) of the disconnected parts is the C_3 vertex itself, we also call it the 3PI graph. Further we note that κ does not include the contribution of the vacuum graphs shown in Fig. 2. These are the conventions adopted in Ref. 7.

From (2.5), we can get the effective action Γ. For this purpose, let us take $v_\nu\equiv v_\nu^0+J_\nu$ where J_ν is the external source and v_ν^0 denotes the bare vertex. We notice $v_1^0=0$ and we set $J_4=0$. (J_4 is needed for the study of the four-body NBS equations.) Then, from (2.5a), (2.2) and (1.2), the relation between F and Γ is obtained:

$$F=W-J_1\frac{\delta W}{\delta J_1}-(v_2^0+J_2)\frac{\delta W}{\delta(v_2^0+J_2)}-(v_3^0+J_3)\frac{\delta W}{\delta(v_3^0+J_3)}$$
$$=W-J_1\frac{\delta W}{\delta J_1}-J_2\frac{\delta W}{\delta J_2}-J_3\frac{\delta W}{\delta J_3}-\frac{1}{2!}v_2^0G_2-\frac{1}{3!}v_3^0G_3=\Gamma-\frac{1}{2!}v_2^0G_2-\frac{1}{3!}v_3^0G_3 \; . \tag{2.6}$$

By using this relation, we next discuss an example and present the formal derivation of the three-body bound-state equation.

As the simplest model, let us examine the single-component $\lambda\Phi^4$ theory. Corresponding to (2.1), the generating functional W is defined as

$$\exp(iW)\equiv\int[d\Phi]\exp\left[i\left\{J_1(x)\Phi_x+\frac{1}{2!}[v_2^0(x,y)+J_2(x,y)]\Phi_x\Phi_y+\frac{1}{3!}J_3(x,y,z)\Phi_x\Phi_y\Phi_z\right.\right.$$
$$\left.\left.+\frac{1}{4!}v_4^0(x,y,z,w)\Phi_x\Phi_y\Phi_z\Phi_w\right\}\right] , \tag{2.7}$$

where $v_2^0(x,y)=iD_0^{-1}(x,y)$ (the inverse of the bare propagator), $v_3^0(x,y,z)=0$, and $v_4^0(x,y,z,w)=\lambda\delta(x-y)\delta(y-z)\delta(z-w)$. If x, y, etc., are repeated, the integration is understood. The effective action Γ is obtained from (2.5), (2.6), and (2.3) as

$$\Gamma=F+\frac{1}{2!}iD_0^{-1}(x,y)[\tilde{G}_2(x,y)+G_1(x)G_1(y)]$$
$$=-\frac{i}{2}\mathrm{Tr}\ln\tilde{G}_2+\frac{i}{2}\frac{1}{3!}C_3(x,x',x'')\tilde{G}_2(x,y)\tilde{G}_2(x',y')\tilde{G}_2(x'',y'')C_3(y,y',y'')$$
$$-i\kappa+\frac{i}{2}D_0^{-1}(x,y)\tilde{G}_2(y,x)+\frac{i}{2}D_0^{-1}(x,y)G_1(x)G_1(y) \; . \tag{2.8}$$

Then Eq. (1.3) becomes

$$\frac{\delta\Gamma}{\delta G_1(x)}=iD_0^{-1}(x,x')G_1(x')-i\frac{\delta\kappa}{\delta G_1(x)}=0 \; ,$$
$$\frac{\delta\Gamma}{\delta\tilde{G}_2(x,y)}=-\frac{i}{2}\tilde{G}_2^{-1}(y,x)+\frac{i}{2}D_0^{-1}(y,x)+\frac{i}{4}C_3(x,x',x'')\tilde{G}_2(x',y')\tilde{G}_2(x'',y'')C_3(y,y',y'')-i\frac{\delta\kappa}{\delta\tilde{G}_2(x,y)}=0 \; , \tag{2.9}$$
$$\frac{\delta\Gamma}{\delta C_3(x,y,z)}=\frac{i}{3!}\tilde{G}_2(x,x')\tilde{G}_2(y,y')\tilde{G}_2(z,z')C_3(x',y',z')-i\frac{\delta\kappa}{\delta C_3(x,y,z)}=0 \; .$$

These are the Schwinger-Dyson (SD) equations for G_1, $\tilde{G}_2$, and C_3. By using the solutions of the SD equations, $(G_1)_0$, $(\tilde{G}_2)_0$, and $(C_3)_0$, we can write the coupled NBS equations in the form of (1.4) as

$$\begin{bmatrix}\Gamma^{11} & \Gamma^{12} & \Gamma^{13}\\ \Gamma^{21} & \Gamma^{22} & \Gamma^{23}\\ \Gamma^{31} & \Gamma^{32} & \Gamma^{33}\end{bmatrix}_0\begin{bmatrix}\Delta G_1\\ \Delta\tilde{G}_2\\ \Delta C_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} , \tag{2.10}$$

where we have abbreviated

$$\frac{\delta^2\Gamma}{\delta G_1\delta G_1}=\Gamma^{11},\quad \frac{\delta^2\Gamma}{\delta G_1\delta\tilde{G}_2}=\Gamma^{12},\quad \frac{\delta^2\Gamma}{\delta G_1\delta C_3}=\Gamma^{13} ,$$

etc. They are given explicitly as

$$\Gamma^{11}_{x',x}=iD_0^{-1}(x,x')-i\frac{\delta^2\kappa}{\delta G_1(x')\delta G_1(x)} ,$$
$$\Gamma^{22}_{x'y',xy}=\frac{i}{4}[\tilde{G}_2^{-1}(y,x')\tilde{G}_2^{-1}(y',x)+\tilde{G}_2^{-1}(x,x')\tilde{G}_2^{-1}(y',y)]$$
$$+\frac{i}{4}[C_3(x,x',x'')\tilde{G}_2(x'',y'')C_3(y,y',y'')+C_3(y,x',x'')\tilde{G}_2(x'',y'')C_3(x,y',y'')]-i\frac{\delta^2\kappa}{\delta\tilde{G}_2(x',y')\delta\tilde{G}_2(x,y)} ,$$
$$\Gamma^{33}_{x'y'z',xyz}=\frac{i}{(3!)^2}[\tilde{G}_2(x,x')\tilde{G}_2(y,y')\tilde{G}_2(z,z')+(\text{permutations of } x',\ y',\ \text{and } z')]-i\frac{\delta^2\kappa}{\delta C_3(x',y',z')\delta C_3(x,y,z)} , \tag{2.11}$$

$$\Gamma^{12}_{x',xy}=\Gamma^{21}_{xy,x'}=-i\frac{\delta^2\kappa}{\delta G_1(x')\delta\widetilde{G}_2(x,y)}\ ,$$

$$\Gamma^{13}_{x',xyz}=\Gamma^{31}_{xyz,x'}=-i\frac{\delta^2\kappa}{\delta G_1(x')\delta C_3(x,y,z)}\ ,$$

$$\Gamma^{23}_{x'y',xyz}=\Gamma^{32}_{xyz,x'y'}=\frac{1}{2}\frac{i}{3!}[C_3(x',y'',z'')\widetilde{G}_2(y'',y)\widetilde{G}_2(z'',z)\delta(x-y')$$
$$+(\text{other five terms with cyclic permutations of } x,\ y,\ z,\ \text{and } x'\leftrightarrow y')]-i\frac{\delta^2\kappa}{\delta\widetilde{G}_2(x',y')\delta C_3(x,y,z)}\ .$$

Equation (2.10) is our three-body bound-state equation and is represented graphically in Fig. 3. Although it looks complicated, the interpretation of the graph is straightforward. Our equation has 3×3 form, because one-body and two-body channels couple to the three-body channel since they all have the same quantum number.

We can see the relation of our (2.10) with the conventional equation by eliminating the one-body and two-body channels from (2.10). Through the elimination of one-body channel, we get

$$\begin{bmatrix}\hat{\Gamma}_{22} & \hat{\Gamma}_{23}\\ \hat{\Gamma}_{32} & \hat{\Gamma}_{33}\end{bmatrix}_0\begin{bmatrix}\Delta\widetilde{G}_2\\ \Delta C_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\ , \tag{2.12}$$

where

$$\hat{\Gamma}_{jk}=\Gamma^{jk}-\Gamma^{j1}(\Gamma^{11})^{-1}\Gamma^{1k}\quad (j,k=2,3)\ . \tag{2.13}$$

The second term of (2.13) represents the one-particle-reducible contributions. In the same way, if the two-body channel is further eliminated from (2.12), we obtain

$$(\hat{\hat{\Gamma}}_{33})_0\Delta C_3=0\ , \tag{2.14}$$

where

$$\hat{\hat{\Gamma}}_{33}=\hat{\Gamma}_{33}-\hat{\Gamma}_{32}(\hat{\Gamma}_{22})^{-1}\hat{\Gamma}_{23}\ . \tag{2.15}$$

As we expect, the two-particle-reducible contributions appear in $\hat{\hat{\Gamma}}_{33}$.

Equations (2.12) and (2.14) can also be regarded as the three-body NBS equation. Here we notice that if the solutions $(G_1)_0$ and $(\widetilde{G}_2)_0$ are the perturbative ones, Eq. (2.14) is expected to coincide with the conventional NBS which is obtained without the source J_1 and J_2. But we stress again that if we do not introduce J_1 and J_2, we cannot include the nonperturbative aspects of $(G_1)_0$ and $(\widetilde{G}_2)_0$ systematically; we have advertised this fact in (3) of the Introduction.

In the above example we have taken the $\lambda\Phi^4$ theory, but the same argument can be applied to any type of bosonic interactions. For the systems which consists of only fermion fields, the rule of the Legendre transformation has also been given by De Dominicis and Martin[7] and its applications to the derivation of the bound-state equations are straightforward. In the next section, we will see that these De Dominicis–Martin rules can be generalized to the case where both boson and fermion fields exist. This is necessary for the discussion of QCD.

III. FERMION FIELD CASE—QCD THEORY

Now let us consider QCD as a realistic model and try to get the exact three-body bound-state equations for baryons. Equation (3.12) below is our result. In order to arrive at this final form, we have to go through several points.

Here we consider the boson and the fermion fields simultaneously. Let us take the quark field $q(x)$, the gluon field $A_\mu(x)$, and the ghost field $c(x)$. In order to treat these fields at the same time, we introduce $\Phi=(\bar{q},q,\bar{c},c,A)$ and use a notation $\Phi(j)$ where j denotes the species of the field as well as other degrees of freedom of the corresponding field. When we want to indicate the species of the field, we write it explicitly as $\Phi(\bar{q}_j)\equiv\bar{q}_j$, etc.

The action $I[\Phi,J]$ is written as

$$I[\Phi,J]\equiv J_1(j)\Phi(j)+\tfrac{1}{2}[J_2(j,k)+v_2^0(j,k)]\Phi(j)\Phi(k)$$
$$+\frac{1}{3!}[J_3(j,k,l)+v_3^0(j,k,l)]\Phi(j)\Phi(k)\Phi(l)$$
$$+\frac{1}{4!}v_4^0(j,k,l,m)\Phi(j)\Phi(k)\Phi(l)\Phi(m)\ , \tag{3.1}$$

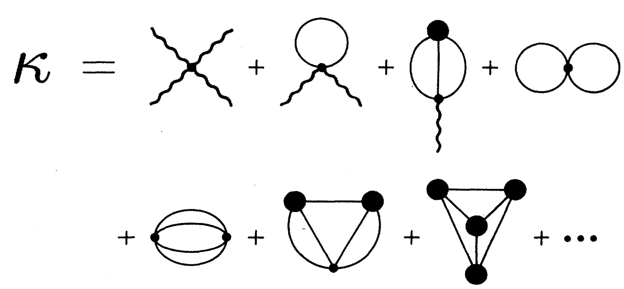

FIG. 1. The graphical representation of κ. We have adopted the same diagrammatical expressions as in Ref. 7; the solid line denotes $\widetilde{G}_2$ and the wavy line G_1. The three-point vertex and the four-point vertex represent C_3 and iv_4, respectively.

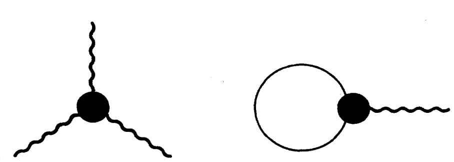

FIG. 2. The two graphs excluded from the definition of κ.

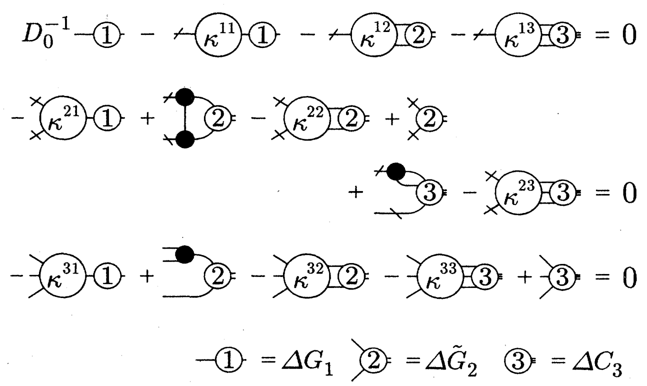

FIG. 3. The graphical representation of the coupled NBS equations in the boson field case. κ^{11}, κ^{12}, κ^{13}, etc., are defined in a similar way as Γ^{11}, Γ^{12}, Γ^{13}, etc., in (2.10). The slash across the line denotes the amputation of the propagator and G_1, $\tilde{G}_2$, and C_3 part of the graphs indicate the corresponding solution of the SD equations $(G_1)_0$, $(\tilde{G}_2)_0$, and $(C_3)_0$, respectively.

where the terms including v_ν^0 come from the usual QCD action. J_ν's are the external sources and each v_ν^0 or J_ν is properly symmetrized under the exchange of the indices. For example, $v_3^0(\bar{q}_j, q_k, A_l)$, the three-point vertex of quark-antiquark-gluon, has mixed symmetry, because under the exchange of the quark and the antiquark it changes the sign, but the exchange of the (anti)quark and the gluon does not. Moreover, since Φ contains both fermion and boson fields, v_ν^0 or J_ν includes the Grassmann number components in addition to the ordinary ones. Therefore, in (3.1), we have to treat the order of the variables in each term carefully.

We can follow the original derivation of the De Dominicis–Martin rule taking care of the sign due to the anticommutativity of these Grassman numbers. We summarize the result and write down the expression of Γ.

The generating functional $W[J]$ is defined as in the previous sections and the vacuum expectation values of the fields and the Green's functions are defined as

$$G_\nu(j,k,\ldots) \equiv \nu! \frac{\vec{\delta} W[J]}{\delta J_\nu(j,k,\ldots)} \tag{3.2}$$

for $\nu = 1,2,3$. In the above definition, we have used the (functional) left derivative. Here the left and right derivatives, denoted by $\vec{\delta}/\delta J$ and $\overleftarrow{\delta}/\delta J$, respectively, are given by the definition[8]

$$F[J+\delta J] - F[J] \equiv \delta J \frac{\vec{\delta} F[J]}{\delta J} \equiv \frac{\overleftarrow{\delta} F[J]}{\delta J} \delta J \ .$$

Of course, the left and right derivatives for the ordinary number are equivalent.

Next, as in the case of Sec. II, we define the two-point connected Green's function $\tilde{G}_2$ as

$$\tilde{G}_2(j,k) \equiv G_2(j,k) - G_1(j) G_1(k) \ , \tag{3.3}$$

the three-point connected Green's function $\tilde{G}_3$ and its amputated one $\tilde{C}_3$ as

$$\tilde{G}_3(j,k,l) \equiv G_3(j,k,l) - \sum_{P(j,k,l)} \epsilon^{P(j,k,l)} \left[\frac{1}{2!} G_1(j) \tilde{G}_2(k,l) + \frac{1}{3!} G_1(j) G_1(k) G_1(l) \right] , \tag{3.4}$$

$$\tilde{G}_3(j,k,l) \equiv \epsilon^{P(j',k',l';j,k,l)} C_3(j',k',l') \times \tilde{G}_2(j',j) \tilde{G}_2(k',k) \tilde{G}_2(l',l) \ . \tag{3.5}$$

Here the sum is taken over all the permutations of j,k,l and $\epsilon^{P(j,k,l)}$ is the sign coming from the permutation of $\Phi(j)\Phi(k)\Phi(l)$. The symbol $\epsilon^{P(j,k,l;j',k',l')}$ denotes the extra sign change which is caused by the rearrangement of the order of the field such as

$$\Phi(j)\Phi(k)\Phi(l)\Phi(j')\Phi(k')\Phi(l') \equiv \epsilon^{P(j,k,l;j',k',l')} \Phi(j)\Phi(j')\Phi(k)\Phi(k')\Phi(l)\Phi(l') \ .$$

In the usual approach[7] the three-point Green's functions such as $G_3(q_j, q_k, q_l)$ are not introduced but in our case it is convenient to have $G_3(q_j, q_k, q_l)$, etc., at hand although it will be set to zero at the end.

By the above preliminaries, we can derive the effective action Γ in the same way as in the case of the boson field. The differences arise because of the existence of the fermion fields in addition to the boson fields. We get

$$\begin{aligned}
\Gamma[G_1, \tilde{G}_2, C_3] &\equiv W - \sum_{\nu=1}^{3} \frac{1}{\nu!} J_\nu(j,k,\ldots) G_\nu(j,k,\ldots) \\
&= \tfrac{1}{2} v_2^0(j,k) G_2(j,k) + \frac{1}{3!} v_3^0(j,k,l) G_3(j,k,l) + \frac{i}{2} \mathrm{STr} \ln \tilde{G}_2 \\
&\quad + \frac{i}{2\times 3!} \sum_{\left[\substack{j,k,l \\ j',k',l'}\right]} \epsilon^{P(j,k,l;j',k',l')} C_3(j,k,l) \tilde{G}_2(j,j') \tilde{G}_2(k,k') \tilde{G}_2(l,l') \tilde{C}_3(j',k',l') - i\kappa \ ,
\end{aligned} \tag{3.6}$$

where κ is the sum of 1-, 2-, 3PI vacuum diagrams and $\tilde{C}_3$ is defined as

$$\Phi(j)\Phi(k)\Phi(l) \tilde{C}_3(j,k,l) \equiv C_3(j,k,l) \Phi(j)\Phi(k)\Phi(l) \ .$$

In writing Γ in the form (3.6), we have employed the notation *supertrace* (STr) or the related *superdeterminant*[8] (SDet). Let χ be the fermion field such as $\bar{q}$, q, $\bar{c}$, or c: then it is defined by

$$\mathrm{STr}\,\ln\widetilde{G}_2 \equiv \mathrm{Tr}\,\ln\widetilde{G}_2(\chi_j,\chi_k) - \mathrm{Tr}\,\ln[\widetilde{G}_2(A_j,A_k) - \widetilde{G}_2(A_j,\chi_{j'})\widetilde{G}_2^{-1}(\chi_{j'},\chi_{k'})\widetilde{G}_2(\chi_{k'},A_k)]$$
$$\equiv \ln \mathrm{SDet}\widetilde{G}_2\ ,$$
$$\mathrm{SDet}\widetilde{G}_2 \equiv \mathrm{Det}[\widetilde{G}_2(\chi_j,\chi_k)]\mathrm{Det}^{-1}[\widetilde{G}_2(A_j,A_k) - \widetilde{G}_2(A_j,\chi_{j'})\widetilde{G}_2^{-1}(\chi_{j'},\chi_{k'})\widetilde{G}_2(\chi_{k'},A_k)]\ .$$

The identities corresponding to (1.3) in this case are given by the replacement of the derivative with the *right* derivative and the SD equations are derived by setting $J=0$:

$$\nu!\frac{\overleftarrow{\delta}\Gamma[\widetilde{G}_\nu]}{\delta\widetilde{G}_\nu(j,k,\ldots)} \equiv -J_\nu(j,k,\ldots)=0 \quad (\nu=1,2,3)\ , \tag{3.7a}$$

where $\widetilde{G}_1\equiv G_1$. We can, of course, use the left derivative instead, but in that case the extra sign factor appears in front of J_ν. As for the variation with respect to C_3, however, it is useful to adopt the left derivative since C_3 is located at the left end in the definition (3.5). Then in terms of G_1, $\widetilde{G}_2$, and C_3, these equations take the expressions

$$\frac{\overleftarrow{\delta}\Gamma}{\delta G_1(j)}=0\ , \tag{3.7b}$$

$$2!\frac{\overleftarrow{\delta}\Gamma}{\delta\widetilde{G}_2(j,k)}=0\ , \tag{3.7c}$$

$$3!\frac{\overrightarrow{\delta}\Gamma}{\delta C_3(j,k,l)}=0\ . \tag{3.7d}$$

In (3.7), some of the Green's functions, which do not appear in the conventional QCD theory, can be set to zero, $C_3(q_j,q_k,q_l)=0$, for example. This is because they are nonzero when the external sources coupled to them are nonvanishing. We discuss here only the equations that have nontrivial solutions. There are six nontrivial SD equations up to the three-body operators. [We have set the trivial solutions such as $C_3(q_j,q_k,q_l)$ equal to zero.]

q-$\bar{q}$ propagator:

$$\widetilde{G}_2^{-1}(q_k,\bar{q}_j)+iv_2^0(q_k,\bar{q}_j)=[2iv_3^0(\bar{q}_{j'},q_k,A_{l'})-C_3(\bar{q}_{j'},q_k,A_{l'})]$$
$$\times C_3(\bar{q}_j,q_{j''},A_{l''})\widetilde{G}_2(q_{j''},\bar{q}_{j'})\widetilde{G}_2(A_{l''},\overline{A}_{l'})-2\frac{\overleftarrow{\delta}\kappa}{\delta\widetilde{G}_2(q_k,\bar{q}_j)}\ . \tag{3.8a}$$

c-$\bar{c}$ propagator:

$$\widetilde{G}_2^{-1}(c_k,\bar{c}_j)+iv_2^0(c_k,\bar{c}_j)=[2iv_3^0(\bar{c}_{j'},c_k,A_{l'})-C_3(\bar{c}_{j'},c_k,A_{l'})]$$
$$\times C_3(\bar{c}_j,c_{j''},A_{l''})\widetilde{G}_2(c_{j''},\bar{c}_{j'})\widetilde{G}_2(A_{l''},\overline{A}_{l'})-2\frac{\overleftarrow{\delta}\kappa}{\delta\widetilde{G}_2(c_k,\bar{c}_j)}\ . \tag{3.8b}$$

A-A propagator:

$$\widetilde{G}_2^{-1}(A_j,A_k)+iv_2^0(A_j,A_k)=[-iv_3^0(A_{j'},A_{k'},A_k)+\tfrac{1}{2}C_3(A_{j'},A_{k'},A_k)]C_3(A_{j''},A_{k''},A_j)\widetilde{G}_2(A_{j''},A_{j'})\widetilde{G}_2(A_{k''},A_{k'})$$
$$+[2iv_3^0(\bar{q}_{j'},q_{k'},A_k)-C_3(\bar{q}_{j'},q_{k'},A_k)]C_3(\bar{q}_{k''},q_{j''},A_j)\widetilde{G}_2(q_{j''},\bar{q}_{j'})\widetilde{G}_2(q_{k'},\bar{q}_{k''})$$
$$+[2iv_3^0(\bar{c}_{j'},c_{k'},A_k)-C_3(\bar{c}_{j'},c_{k'},A_k)]C_3(\bar{c}_{k''},c_{j''},A_j)\widetilde{G}_2(c_{j''},\bar{c}_{j'})\widetilde{G}_2(c_{k'},\bar{c}_{k''})$$
$$-2\frac{\overleftarrow{\delta}\kappa}{\delta\widetilde{G}_2(A_k,A_j)}\ . \tag{3.8c}$$

$\bar{q}$-q-A vertex:

$$C_3(\bar{q}_k,q_j,A_l)=iv_3^0(\bar{q}_k,q_j,A_l)-\widetilde{G}_2^{-1}(q_j,\bar{q}_{j'})\widetilde{G}_2^{-1}(q_{k'},\bar{q}_k)\widetilde{G}_2^{-1}(A_l,A_{l'})\frac{\overrightarrow{\delta}\kappa}{\delta C_3(\bar{q}_{j'}q_{k'}A_{l'})}\ . \tag{3.9a}$$

$\bar{c}$-c-A vertex:

$$C_3(\bar{c}_k,c_j,A_l)=iv_3^0(\bar{c}_k,c_j,A_l)-\widetilde{G}_2^{-1}(c_j,\bar{c}_{j'})\widetilde{G}_2^{-1}(c_{k'},\bar{c}_k)\widetilde{G}_2^{-1}(A_l,A_{l'})\frac{\overrightarrow{\delta}\kappa}{\delta C_3(\bar{c}_{j'}c_{k'}A_{l'})}\ . \tag{3.9b}$$

A-A-A vertex:

$$C_3(A_j,A_k,A_l)=iv_3^0(A_j,A_k,A_l)+\widetilde{G}_2^{-1}(A_j,A_{j'})\widetilde{G}_2^{-1}(A_k,A_{k'})\widetilde{G}_2^{-1}(A_l,A_{l'})\frac{\overrightarrow{\delta}\kappa}{\delta C_3(A_{j'}A_{k'}A_{l'})}\ . \tag{3.9c}$$

The solutions of these equations are denoted by $(G_\nu)_0$.

We now derive the NBS equations. We need only the second derivatives Γ'' of Γ in the case that all the sources vanish. Our NBS equation is written as

$$\nu!\left[\frac{\overleftarrow{\delta}}{\delta\tilde{G}_\mu(j',k',\ldots)}\left[\frac{\overleftarrow{\delta}\Gamma}{\delta\tilde{G}_\nu(j,k,\ldots)}\right]\right]_0 \times\Delta\tilde{G}_\mu(j',k',\ldots)=0\ , \qquad (3.10a)$$

where $[\ ]_0$ is $[\ ]$ evaluated at the solution $(G_\nu)_0$ (with the trivial solution set to zero). It is worth mentioning here that the spontaneous chiral-symmetry-breakdown solutions in QCD, for example, can be chosen as the solution $(G_\nu)_0$; we can pick up the solution corresponding to the nonperturbative condensation in this formulation as we have already noted.

We write (3.10a) explicitly in the forms of (3.7b)–(3.7d):

$$\Gamma^{11}(j';j)\Delta G_1(j')+\Gamma^{21}(j',k';j)\Delta\tilde{G}_2(j',k') +\Delta C_3(j',k',l')\Gamma^{31}(j',k',l';j)=0\ , \qquad (3.10b)$$

$$2!\Gamma^{12}(j';j,k)\Delta G_1(j')+2!\Gamma^{22}(j',k';j,k)\Delta\tilde{G}_2(j',k') +\Delta C_3(j',k',l')2!\Gamma^{32}(j',k',l';j,k)=0\ , \qquad (3.10c)$$

$$3!\Gamma^{13}(j';j,k,l)\Delta G_1(j')+3!\Gamma^{23}(j',k';j,k,l)\Delta\tilde{G}_2(j',k') +\Delta C_3(j',k',l')3!\Gamma^{33}(j',k',l';j,k,l)=0\ . \qquad (3.10d)$$

Here the notation $\Gamma^{\mu\nu}$ is the same as in Sec. II, except that the order of the derivatives is fixed and that we use the right derivative for G_1 and $\tilde{G}_2$ and the left for C_3. Note that some parts of the Γ'' vanish after the substitution of $(G_\nu)_0$. They are the parts which connect the sectors whose quantum numbers are different from each other, so that the $(\Gamma'')_0$ becomes block diagonal when we write them in the matrix form. For example, the NBS equations for the baryons are expected to be isolated from other parts of the NBS equations. In fact, (3.10) looks like

$$\begin{bmatrix} \text{Baryon part} & 0 & 0 \\ 0 & \text{Glueball and meson part} & 0 \\ 0 & 0 & \text{Other parts}\end{bmatrix}\begin{bmatrix}\Delta C_3(q,q,q)\\ \Delta C_3(\bar{q},\bar{q},\bar{q})\\ \Delta\tilde{G}_2(A,A)\\ \Delta\tilde{G}_2(q,\bar{q})\\ \Delta\tilde{G}_2(c,\bar{c})\\ \Delta C_3(A,A,A)\\ \Delta C_3(\bar{q},q,A)\\ \Delta C_3(\bar{c},c,A)\\ \vdots\end{bmatrix}=0\ . \qquad (3.11)$$

Let us concentrate on the baryonic part of (3.11). The diagonal elements

$$3!\frac{\vec{\delta}^2\Gamma}{\delta C_3(q,q,q)\delta C_3(q,q,q)}$$

and

$$3!\frac{\vec{\delta}^2\Gamma}{\delta C_3(\bar{q},\bar{q},\bar{q})\delta C_3(\bar{q},\bar{q},\bar{q})}$$

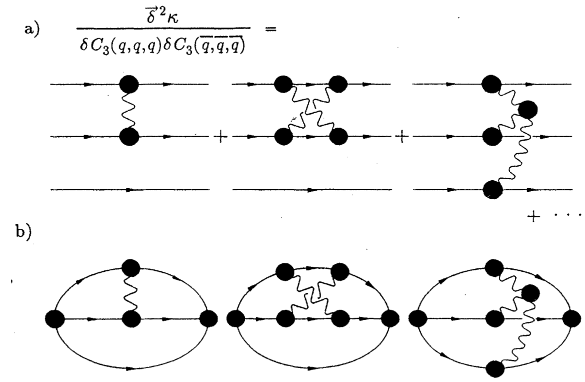

FIG. 4. (a) The graphical expression of the NBS kernel for the first several terms. (b) The several graphs appearing in κ. Each term corresponds to the graph in (a).

vanish, after we substitute $(G_\nu)_0$. We finally obtain the NBS equations for baryons in the simple form

$$\Delta C_3(q_{j'},q_{k'},q_{l'})\left[-(\tilde{G}_2)_0(q_{j'},\bar{q}_j)(\tilde{G}_2)_0(q_{k'},\bar{q}_k)(\tilde{G}_2)_0(q_{l'},\bar{q}_l) -3!\frac{\vec{\delta}^2\kappa}{\delta C_3(q_{j'},q_{k'},q_{l'})\delta C_3(\bar{q}_j,\bar{q}_k,\bar{q}_l)}\right]\equiv 0\ , \qquad (3.12a)$$

$$\Delta C_3(\bar{q}_{j'},\bar{q}_{k'},\bar{q}_{l'})\left[(\tilde{G}_2)_0(q_j,\bar{q}_{j'})(\tilde{G}_2)_0(q_k,\bar{q}_{k'})(\tilde{G}_2)_0(q_l,\bar{q}_{l'}) -3!\frac{\vec{\delta}^2\kappa}{\delta C_3(\bar{q}_{j'},\bar{q}_{k'},\bar{q}_{l'})\delta C_3(q_j,q_k,q_l)}\right]\equiv 0\ . \qquad (3.12b)$$

The two terms in the large parentheses come from the last two terms in (3.6). The solutions $\Delta C_3(q_j,q_k,q_l)$ and $\Delta C_3(\bar{q}_j,\bar{q}_k,\bar{q}_l)$ are the wave functions for the baryon or the antibaryon, considered as the bound states of three quarks or antiquarks, respectively. These two equations (3.12a) and (3.12b) are related by the charge conjugation so that they are not independent of each other. The graphical representations of the NBS kernel

$$3!\frac{\vec{\delta}^2\kappa}{\delta C_3(q,q,q)\delta C_3(\bar{q},\bar{q},\bar{q})}$$

are given in Fig. 4. Since κ contains only 1PI, 2PI, 3PI graphs [Fig. 4(b)], the kernel consists of 3PI six-point diagrams [3PI with respect to quark line, see Fig. 4(a)]. So they coincide with the results of the intuitive graphical expansion approach.[5,6]

IV. DISCUSSION

We have presented a formal scheme of deriving the three-body bound-state equation by the formalism of Ref. 1. Since our method is a systematic one, we can extend the method straightforwardly for any theory once the effective action is known. For example, the rule of evaluating the effective action of the four-body operator $O_j\propto\Phi_k\Phi_l\Phi_m\Phi_n$ has already been given in Ref. 7 so that we can easily write down the four-body bound-state equation. However, the bound state composed of such a high power of the field variables is interesting only when the bound states with the lower power do not exist. This is the case for three-body bound state of quarks, for which one- and two-body bound states of quarks are expected to be absent.

Our three-body bound-state equation takes into account the full quantum effects including the vacuum fluctuations of all the fields. This differs from the original Faddeev equations[2] where the equation is derived in the framework of the potential problem. In order to see the connection between our equation and the Faddeev equation, we have to make an approximation which correctly incorporates the above situations. However, we are mainly interested in the relativistic field theory so that we are not allowed to make such an approximation.

We have already developed an expansion scheme for the effective action Γ where the coefficients of the expansion are the connected S-matrix elements corresponding to the scattering among the particles found as the solution to the NBS equation (1.4). We can apply our method to find the S-matrix elements for the scattering among baryons in the framework of QCD. Before doing this task, we have to solve the SD equation to obtain $(G_\nu)_0$, which will show the nonperturbative nature in the gluonic sector, gluon condensation, for example, since it is expected to be crucial for the confinement of quarks.

In this paper no attempt has been made to solve explicitly our three-body bound-state equation. Some of the calculations for the QCD baryonic case have been performed in Ref. 6.

[1]R. Fukuda, Prog. Theor. Phys. **78**, 1487 (1987).

[2]Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950); E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

[3]R. Fukuda, M. Komachiya, and M. Ukita, Phys. Rev. D **38**, 3747 (1988).

[4]L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1961) [Sov. Phys. JETP **12**, 1014 (1961)]; Dok. Akad. Nauk SSSR **6**, 565 (1961) [Sov. Phys. Dokl. **6**, 384 (1961)]; **7**, 301 (1963) [**7**, 600 (1963)].

[5]K. Huang and H. A. Weldon, Phys. Rev. D **11**, 257 (1975). For recent developments, see the following review articles: *Modern Three-Hadron Physics*, edited by A. W. Thomas (Topics in Current Physics, Vol. 2) (Springer, Berlin, 1977); T. Sasakawa and T. Sawada, Prog. Theor. Phys. Suppl. **61**, 61 (1977).

[6]S. Machida and H. Nakkagawa, Prog. Theor. Phys. **54**, 243 (1975); **54**, 510 (1975); K. Tesima, Prog. Theor. Phys. Suppl. **77**, 285 (1987).

[7]C. De Dominicis and P. C. Martin, J. Math. Phys. **5**, 14 (1964); **5**, 31 (1964).

[8]K. B. Efetov, Adv. Phys. **32**, 53 (1983).