# String distributions above the Hagedorn energy density

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The formalism for discussing energy and charge distribution functions in the microcanonical ensemble is presented and applied to strings. This yields information of direct physical significance in string statistical mechanics. Above the Hagedorn energy density the free string gas exhibits a number of interesting features. For toroidal compactification, the distribution of the total energy among strings of various energies depends upon the number of noncompact spatial dimensions $d$. For $d=0$ the energy is distributed uniformly among strings at all energy scales, while for $d\geq 3$ a single string captures most of the energy. The imposition of conservation laws does not alter this qualitative picture. We show that the $d=1$ and $d=2$ cases are qualitatively different from the others.

## I. INTRODUCTION

An issue of fundamental importance in string theory is its behavior in the Planck and higher-energy regime. This regime is where the theory is likely to be most interesting and where might lurk new insights into its fundamental degrees of freedom, into the structure of spacetime, and a framework for the unification of spacetime with matter. At the very least we expect significant departures from conventional theory. This is borne out from our at-present-meager knowledge of this regime through studies of string statistical mechanics at high energies[1–18] and high-energy-scattering thought experiments.[19,20] An immediate objective is to explore diverse properties of the theory in this regime as it stands and to try to achieve a unified understanding of these phenomena. In this paper we discuss how useful information can be extracted about a gas of strings above a Hagedorn energy density, $\sigma_0$ (Ref. 15). This could be relevant for applications to the very early Universe in the context of superstring theory, to cosmic strings close to their time of creation, and to hadronic matter close to the deconfinement temperature.

Consider the following thought experiment: a gas of charged particles is enclosed in a large isolated box. This box has a small opening, outside which sits a detector which measures the energy, momentum, and all the charges of all particles that emanate from it. After a long period of time the accumulated data give us the distribution of the total energy, momentum, and charge of the gas in the box among individual particles of specified energy, momentum, charge, etc. A similar thought experiment can also be carried out in the case of a gas of strings with an energy density much higher than the Hagedorn energy density (which is characterized by the string length scale $\sqrt{\alpha'}$). For instance, assuming that the whole system has a total energy $E$ and a total charge $Q$ ($Q$ corresponds to a generic conserved quantity including momentum and is taken to be discrete for the moment for ease of writing), one can ask: what is the average number of strings in the box in a given energy range $\epsilon$ to $\epsilon+d\epsilon$ and having a fixed charge $q$? Denote this number by $\mathcal{D}(\epsilon,q;E,Q)d\epsilon$. Then

$$\mathcal{D}(\epsilon;E,Q)d\epsilon\equiv\sum_{q=-\infty}^{\infty}\mathcal{D}(\epsilon,q;E,Q)d\epsilon \tag{1}$$

equals the average number of strings in the energy range $\epsilon$ to $\epsilon+d\epsilon$, irrespective of their charge, given that the whole system has total energy $E$ and total charge $Q$. Similarly, for the same system

$$\mathcal{D}(q;E,Q)\equiv\int_0^E d\epsilon\,\mathcal{D}(\epsilon,q;E,Q) \tag{2}$$

equals the average number of strings of charge $q$, irrespective of their energy. We would like to determine in this paper such distribution functions from the theory in a microcanonical treatment.

The distributions we are studying are traditionally referred to as "inclusive" distributions in high-energy multiparticle production.[21] This terminology originates from the fact that, for example, $\mathcal{D}(\epsilon;E,Q)$ measures the relative frequency of occurrence of a specific substate $|\psi\rangle$ contained in a set of states that includes all possible states for the remaining gas, subject to an overall constraint of total energy and charges. We ask for the distribution as a function of variables specifying a particular substate $|\psi\rangle$, without caring about the rest of the system. For $\mathcal{D}(\epsilon;E,Q)$, the substate is a single string with energy $\epsilon$. [The formalism to be discussed can also be used for multistring substates, e.g., when a pair of strings is chosen at random from the gas, one can ask for the relative frequency that their energies are $\epsilon_1$ and $\epsilon_2$. The analogous "multipoint inclusive energy distributions," $\mathcal{D}(\epsilon_1,\ldots,\epsilon_k;E,Q)$, that provide a measure of frequencies of such occurrences are of special interest when one includes interactions.]

 

We consider first an even simpler distribution function, the single-string inclusive energy distribution, denoted $\mathcal{D}(\epsilon;E)$ and defined such that $\mathcal{D}(\epsilon;E)d\epsilon$ equals the average number of strings of energy between $\epsilon$ and $\epsilon+d\epsilon$ in a gas whose total energy is fixed to be $E$ but whose total charge is arbitrary. We can imagine that the box is in equilibrium with a neutral "charge-bath" with which it can exchange charge, but not energy, which is fixed to be $E$. (This picture will be made more precise later.) Once determined, $\mathcal{D}(\epsilon;E)$ can be used to estimate the average number of strings in a given range of energy. For example, $\mathcal{N}(\epsilon_1,\epsilon_2;E) \equiv \int_{\epsilon_1}^{\epsilon_2} d\epsilon\, \mathcal{D}(\epsilon;E)$ equals the average number of strings in the energy range $[\epsilon_1,\epsilon_2]$ given that the total system has energy $E$. Similarly $\mathcal{E}(\epsilon_1,\epsilon_2;E) \equiv \int_{\epsilon_1}^{\epsilon_2} d\epsilon\, \epsilon \mathcal{D}(\epsilon;E)$ is the average total energy carried by all the strings in the energy range $[\epsilon_1,\epsilon_2]$ given that the total system has energy $E$. This can be used to estimate the average number of high- or low-energy strings in the gas. [When integrated from 0 to $E$, $\mathcal{N}(0,E;E)$ and $\mathcal{E}(0,E;E)$ by definition give the total average number $\mathcal{N}_{\text{total}}$ of strings in the gas and the total energy $E$, respectively.]

For an ordinary gas of particles, the $\epsilon$ dependence of $\mathcal{D}(\epsilon;E)$ is typically $\mathcal{D}(\epsilon;E) \propto \epsilon^{\alpha} e^{-\beta\epsilon}$, which is schematically plotted in Fig. 1(a). [Figure 1(b) shows $\epsilon\mathcal{D}(\epsilon;E)$.] There is a power-law growth (due to the density of states), a single peak, and an exponential decay (the Boltzmann factor). The region of $\epsilon$ where the peak occurs represents the average energy of a particle in the box of total energy $E$. We will argue that the results are strikingly different for a gas of free strings, and in fact depend upon the compactification. In this paper we consider strings in a space that is toroidally compactified. The total spatial dimension is $D$ ($D=25$ or 9); there are $d$ uncompactified spatial dimensions and $D-d$ toroidally compactified dimensions with radii $R_i$ ($i=1,\ldots,D-d$), which for purposes of this paper are taken to be of the order $\sqrt{\alpha'}$. It turns out that the results depend significantly on the number of uncompactified dimensions. At energy densities much larger than the Hagedorn energy density $\sigma_0$, the picture is summarized in Figs. 2(a) and 2(b) for $d \geq 3$, and in Figs. 3(a) and 3(b) for $d=0$.

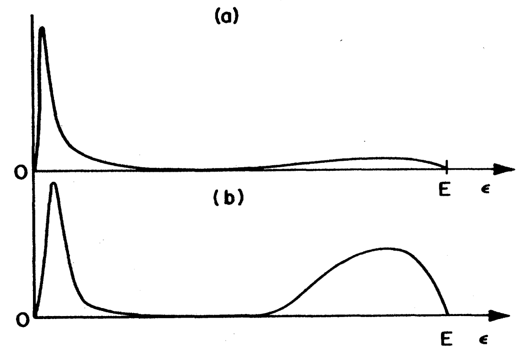


FIG. 2. (a) Schematic plot of $\mathcal{D}(\epsilon,E)$ as a function of $\epsilon$ for a string gas with $d \geq 3$. (b) Schematic plot of $\epsilon\mathcal{D}(\epsilon,E)$ as a function of $\epsilon$ for a string gas with $d \geq 3$.

The area under curves 2(a) and 3(a) between values of $\epsilon$ in the range $[\epsilon_1,\epsilon_2]$ is by definition $\mathcal{N}(\epsilon_1,\epsilon_2;E)$ and under 2(b) and 3(b) is $\mathcal{E}(\epsilon_1,\epsilon_2;E)$. Figures 2(a) and 2(b) together with the discussion in Sec. III give quantitative substance to the Frautschi-Carlitz picture[2,8,9,15–17] that a single string contains most of the energy of the gas. In Fig. 2(a), the area under the first peak is much larger than the second peak (under which the area is essentially unity) implying that there is essentially one very energetic string and many more low-energy strings. In Fig. 2(b), the area under the second peak is much larger than the first (provided the energy density is much larger than the Hagedorn energy density), implying that the single energetic string captures most of the energy. These results hold only for $d \geq 3$.

For $d=0$, Fig. 3(b) tells us that we have a kind of a scale-invariant distribution. The total energy is distributed equally among strings of all energies over most of the energy domain. (The energy deposited near the end points $\epsilon=0$ and $\epsilon=E$ is small.) Both these results are in sharp contrast to the usual case and to each other. For $d=1$ and 2, the general analysis is more complicated and the results under specific assumptions are described later.

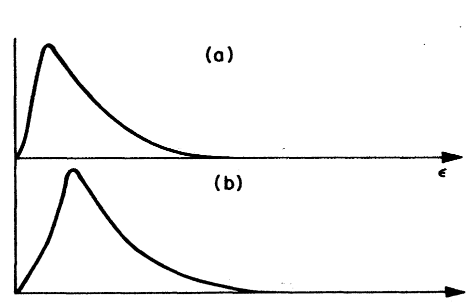


FIG. 1. (a) Schematic plot of $\mathcal{D}(\epsilon,E)$ as a function of $\epsilon$ for an ordinary gas. (b) Schematic plot of $\epsilon\mathcal{D}(\epsilon,E)$ as a function of $\epsilon$ for an ordinary gas.

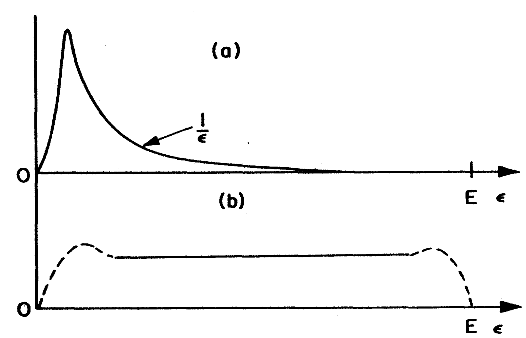


FIG. 3. (a) Schematic plot of $\mathcal{D}(\epsilon,E)$ as a function of $\epsilon$ for a string gas with $d=0$. (b) Schematic plot of $\epsilon\mathcal{D}(\epsilon,E)$ as a function of $\epsilon$ for a string gas with $d=0$.

In a toroidal compactification of $D-d$ spatial dimensions, the conserved quantities are the momenta in the noncompact spatial directions, and the discrete momenta and windings in the compact directions. Taking their conservation into account restricts the allowed states of the system, but we find that $\mathcal{D}(\epsilon;E,Q)$ as a function of $\epsilon$ possesses the same behavior as described above for $\mathcal{D}(\epsilon;E)$ for any finite $Q$, provided $E$ is large. Thus the distribution of the total energy among individual strings of various energies is unaltered. For $d\geq 3$, one very energetic string still captures most of the energy and for $d=0$ the distribution is again flat.

In Sec. II we discuss the general formalism for obtaining the above density functions that characterize physical quantities in the microcanonical ensemble. The discussion is valid for any statistical system. In Sec. III we apply the formalism to strings and quantify the results mentioned above. Section IV discusses the incorporation of conserved quantities other than the energy by the use of chemical potentials in general and also obtains the results when applied to strings. A short summary and some questions are presented in Sec. V.

## II. FORMALISM FOR INCLUSIVE ENERGY DISTRIBUTIONS

In this section we discuss how to obtain $\mathcal{D}(\epsilon;E)$ and other quantities of interest. The basic quantity to be used in the discussion that follows is the single-particle or single-string density of states

$$f(\epsilon)=\sum_a \delta(\epsilon-E_a)\ , \tag{3}$$

where $a$ labels all single-particle (-string) states so that $f(\epsilon)$ counts the number of single-particle (-string) states at a given energy $\epsilon$. For relativistic particles $f(\epsilon)\sim\epsilon^{d+1/2}$, and for strings its asymptotic behavior is given by

$$f(\epsilon)=CV\frac{e^{\beta_0\epsilon}}{\epsilon^{d/2+1}}[1+O(\epsilon^{-1})]\ . \tag{4}$$

$V$ is the volume of the uncompactified spatial dimensions, and it is absent for $d=0$. (However, for notational simplification, we adopt the convention in this paper that $V=1$ for $d=0$.) $V$ is to be taken to infinity (the thermodynamic limit) at the end of the calculation. $\beta_0$, the inverse of the Hagedorn temperature, is given by $(2\pi^2\alpha')^{1/2}(\sqrt{\omega_l}+\sqrt{\omega_r})$, where $(\omega_l,\omega_r)$ is (2,2), (2,1) and (1,1) respectively, for the closed bosonic string, heterotic string, and type-II superstring; and $C=(2\pi^2\alpha'\beta_0)^{-d/2}(\omega_l\omega_r)^{d/4}$. Here the "box" is all of space; the difference between the compact and noncompact dimensions of the box is that in the former string-winding modes are included and in the latter they are excluded.

The basic form of (4) is well known in the literature (Refs. 1–18 for $d\geq 1$ and Ref. 14 for $d=0$). A recent rigorous proof including the precise form of the correction term and the prefactor $C$ from first principles is given in Ref. 15. We recall in particular that, when all dimensions are compactified, $d=0$, the leading correction is suppressed exponentially, $O(e^{-\eta\epsilon})$, rather than by powers of $\epsilon^{-1}$, where $\eta\sim R_i^2/\sqrt{\alpha'}$ or $\alpha'^{3/2}/R_i^2$, whichever is smaller. Equation (4) is valid only at $\epsilon$ sufficiently large, say $\epsilon\geq m_0$, where $m_0$ is of order $\alpha'^{-1/2}$. A complete knowledge of $f(\epsilon)$ would enable us to compute, in principle, any thermodynamic quantity for a gas of free strings. However, as we shall see, even our knowledge of the expression in the restricted domain $\epsilon\geq m_0$ is sufficient to draw a number of significant conclusions.

To begin, we obtain a general expression for $\mathcal{D}(\epsilon;E)$ in terms of $f(\epsilon)$ which is valid for any noninteracting system. The microcanonical distribution function $\Omega(E)$ or the total density of states of the ensemble is given by

$$\Omega(E)=\sum_\alpha \delta(E-E_\alpha)\ , \tag{5}$$

where $\alpha$ sums over all multiparticle (-string) states so that $\Omega(E)$ counts the total number of states at given energy $E$. Write $\Omega(E)=\sum_{N=1}^\infty \Omega_N$, where $\Omega_N(E)=\sum_{\alpha_N}\delta(E-E_{\alpha_N})$ is the number density of states $\alpha_N$ possessing $N$ particles (strings). If we insert in the sum over $\alpha_N$ the quantity $\sum_{i=1}^N \delta(\epsilon-E_i)d\epsilon$ where $E_i$ is the energy of the $i$th particle (string) in the state $\alpha_N$, the resulting expression $\sum_{\alpha_N}\delta(E-E_{\alpha_N})\sum_{i=1}^N\delta(\epsilon-E_i)d\epsilon$ clicks only for those states $\alpha_N$ which have at least one particle (string) of energy between $\epsilon$ and $\epsilon+d\epsilon$. Indeed for each state it clicks as many times as the number of such particles (strings) it possesses. Thus the above expression counts the number of $N$-particle (-string) states that possess a particle (string) of energy between $\epsilon$ and $\epsilon+d\epsilon$ and weights each state by the number of such particles (strings) in it. To get the average number of such particles (strings) in the gas we must sum over $N$ and divide by $\Omega(E)$. Therefore

$$\mathcal{D}(\epsilon;E)=\frac{1}{\Omega(E)}\sum_{N=1}^\infty\sum_{\alpha_N}\delta(E-E_{\alpha_N})\sum_{i=1}^N\delta(\epsilon-E_i)\ . \tag{6}$$

In the Maxwell-Boltzmann statistics one can obtain a simple explicit expression for $\mathcal{D}(\epsilon;E)$ in terms of $f(\epsilon)$ and $\Omega(E)$. (The validity of the Maxwell-Boltzmann statistics in the context of a noninteracting high-energy string gas was discussed in Ref. 15.) For MB statistics we have

$$\begin{aligned}\Omega_N(E)&=\sum_{\alpha_N}\delta(E-E_{\alpha_N})\\&=\frac{1}{N!}\int_0^E\prod_{i=1}^N dE_i f(E_i)\delta\left[E-\sum_{i=1}^N E_i\right]\ .\end{aligned} \tag{7}$$

Inserting $\sum_{i=1}^N\delta(\epsilon-E_i)$ in (7) and doing the $E_i$ integration for the $i$th term in the sum gives $f(\epsilon)\Omega_{N-1}(E-\epsilon)$. Using this result in (6) gives

$$\mathcal{D}(\epsilon;E)=\frac{1}{\Omega(E)}f(\epsilon)[\Omega(E-\epsilon)+\delta(E-\epsilon)]\ . \tag{8}$$

This is the desired expression for the number density of particles (strings) of a given energy $\epsilon$. The $\delta(E-\epsilon)$ which comes from the $N=1$ term is insignificant compared to $\Omega(E-\epsilon)$; it is the contribution of states that correspond to having only one particle (string) in the gas occupying all the energy $E$. [A similar analysis also allows us to express multiparticle (-string) inclusive distributions in terms of $f(\epsilon)$ and $\Omega(E)$ (Ref. 22).]

Before proceeding further it is perhaps worthwhile contrasting a string gas with an ordinary classical gas involving $N$ weakly interacting identical particles in a large volume $V$. Let the total energy be $E$, and consider the large-$V$ limit with $E/V$ and $N/V$ fixed. Let $|\psi\rangle$ be an arbitrary $n$-particle state with a total energy $\epsilon$; it is intuitively clear that the desired inclusive distribution is proportional to $f_\psi(\epsilon)\Omega_{N-n}(E-\epsilon)$, where $f_\psi(\epsilon)$ is the $n$-particle density of states and $\Omega_N(E)$ is the microcanonical distribution function. In the large-$V$ limit, $\ln\Omega_{N-n}(E)\sim\ln\Omega_N(E)\equiv S(E,N,V)=O(N\ln E)$; an expansion of $\Omega_{N-n}(E-\epsilon)$ in $\epsilon$ then leads to the familiar Boltzmann distribution

$$\mathcal{D}_\psi(\epsilon;E)\propto f_\psi(\epsilon)e^{-\beta\epsilon}\ , \tag{9}$$

where $\beta$ is the inverse canonical temperature and is related to $E$ by $\beta\equiv\partial S(E,N,V)/\partial E$. In particular, the distribution is exponentially small for $\epsilon$ large since $f_\psi(\epsilon)$ increases only as a power of $\epsilon$ [Figs. 1(a) and 1(b)].

For a string gas above the Hagedorn energy density, Eq. (9) no longer follows from (8). The notion of a canonical temperature now breaks down. Furthermore, since the corresponding density of states $f_\psi(\epsilon)$ now increases exponentially, nontrivial structure can emerge when $\epsilon$ is of the order $E$. In Sec. III we shall concentrate on single-string inclusive distributions and discuss their resulting novel features at high energy densities.

If one knows $f(\epsilon)$ and $\Omega(E)$ completely, one would know $\mathcal{D}(\epsilon;E)$ for all $\epsilon$ for a fixed $E$. But if only asymptotic expressions are known we can only find $\mathcal{D}(\epsilon;E)$ quantitatively in the "central" region where $\epsilon/E$ is finite, nonzero, and less than unity. To discuss end-point regions, $\epsilon\sim 0$ or $\epsilon\sim E$, it will be useful to introduce a complex-$\beta$ representation for $\mathcal{D}(\epsilon;E)$.

Expressing the delta function $\delta(E-E_\alpha)$ in (5) as a Fourier integral:

$$\delta(E-E_\alpha)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(E-E_\alpha)}$$
$$=\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta(E-E_{\alpha_N})}\ ,\quad \beta=L+ik\ ,$$

this gives

$$\Omega(E)=\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E}Z(\beta)\ , \tag{10}$$

where $Z(\beta)\equiv\sum_\alpha e^{-\beta E_\alpha}$ is the canonical partition function. For free particles (strings), carrying out an analysis analogous to (7) and summing over $N$, one finds $Z(\beta)=e^{\tilde f(\beta)}-1$, where

$$\tilde f(\beta)\equiv\int_0^\infty d\epsilon\, e^{-\beta\epsilon}f(\epsilon)\ . \tag{11}$$

From (4) we note that for strings we need $L>\beta_0$, so that (11) converges. In Ref. 15 we have shown how the complex-$\beta$ representation for $f(\epsilon)$, i.e.,

$$f(\epsilon)=\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta\epsilon}\tilde f(\beta)\ , \tag{12}$$

and for $\Omega(E)$, i.e., Eq. (10), can be used to extract the asymptotic behavior of $f(\epsilon)$ and $\Omega(E)$. We now discuss how other quantities of interest are obtained in the complex-$\beta$ representation.

Consider first

$$\mathcal{N}_{\rm total}=\Omega(E)^{-1}\int_0^\infty d\epsilon\, f(\epsilon)[\Omega(E-\epsilon)+\delta(E-\epsilon)]\ .$$

Write $\Omega(E-\epsilon)=\int_0^\infty dE'\Omega(E')\delta(E-(\epsilon+E'))$ and express the $\delta(E-(\epsilon+E'))$ as a Fourier integral:

$$\delta(E-(\epsilon+E'))=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik[E-(\epsilon+E')]}$$
$$=\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta[E-(\epsilon+E')]}\ ,$$

with $L>\beta_0$. After interchanging the order of the $\epsilon$ and $\beta$ integrals, this gives

$$\mathcal{N}_{\rm total}=\frac{1}{\Omega(E)}\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E+\tilde f(\beta)}\tilde f(\beta)\ . \tag{13}$$

Using the same steps we can obtain analogous expressions for $\mathcal{N}(\epsilon_1,\epsilon_2;E)$ and $\mathcal{E}(\epsilon_1,\epsilon_2;E)$. The result is

$$\mathcal{N}(\epsilon_1,\epsilon_2;E)=\frac{1}{\Omega(E)}\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E+\tilde f(\beta)}\tilde f(\epsilon_1,\epsilon_2;\beta)\ , \tag{14}$$

$$\mathcal{E}(\epsilon_1,\epsilon_2;E)=\frac{1}{\Omega(E)}\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E+\tilde f(\beta)}\times\left[-\frac{\partial}{\partial\beta}\tilde f(\epsilon_1,\epsilon_2;\beta)\right]\ , \tag{15}$$

where

$$\tilde f(\epsilon_1,\epsilon_2;\beta)\equiv\int_{\epsilon_1}^{\epsilon_2}d\epsilon\, e^{-\beta\epsilon}f(\epsilon)\ . \tag{16}$$

Equation (14) differs from (13) in that $\tilde f(\beta)$ is replaced by $\tilde f(\epsilon_1,\epsilon_2;\beta)$. By narrowing the interval, $\epsilon_2\to\epsilon_1+\delta\epsilon_1$, (14) simply reproduces (8).

Equation (14) will be useful in discussing the average number of low- and high-energy strings in the ensemble. Choose an intermediate energy scale $m$, $m_0<m<E$, and call all strings with energy $\epsilon>m$ as energetic strings and all those with $\epsilon<m$ as low-energy strings. Denote their average numbers in the ensemble by $N_>(m;E)$ and $N_<(m;E)$, respectively. Then, from (14),

$$\mathcal{N}_>=\mathcal{N}(m,E;E)$$
$$=\frac{1}{\Omega(E)}\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E+\tilde f(\beta)}\tilde f(m,\infty;\beta)\ , \tag{17}$$

$$\mathcal{N}_<=\mathcal{N}(0,m;E)$$
$$=\frac{1}{\Omega(E)}\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E+\tilde f(\beta)}\tilde f(0,m;\beta)\ . \tag{18}$$

Note that in $\mathcal{N}_>$, we can use $\tilde f(m,\infty;\beta)$ in place of $\tilde f(m,E;\beta)$ because $\mathcal{D}(\epsilon;E)$ vanishes if $\epsilon>E$. Thus all quantities of physical interest, i.e., $\Omega$, $\mathcal{D}(\epsilon;E)$, $\mathcal{N}_{\rm total}$, $\mathcal{N}_>$, etc., are expressed in terms of $\tilde f(\beta)$, $\tilde f(\epsilon_1,\epsilon_2;\beta)$, etc., which in turn are completely determined from $f(\epsilon)$.

## III. SINGLE-STRING INCLUSIVE ENERGY DISTRIBUTION

In this section we apply the results of Sec. II to the specific example of strings, starting from expression (4).

To begin we discuss how our knowledge of the large-$\epsilon$ behavior of $f(\epsilon)$, Eq. (4), determines $\tilde{f}(\beta)$, etc., and hence the other physical quantities. From (11) we have $\tilde{f}(\beta)=\tilde{f}(0,m_0;\beta)+\tilde{f}(m_0,\infty;\beta)$. Note that $\tilde{f}(0,m_0;\beta)$ admits a power-series expansion at $\beta_0$:

$$\tilde{f}(0,m_0;\beta)=V\sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n(\beta-\beta_0)^n b_n \ ,$$

where $Vb_n=\int_0^{m_0}d\epsilon\, e^{-\beta\epsilon}\epsilon^n f(\epsilon)$ are all positive coefficients. On the other hand, (4) implies that $\tilde{f}(m_0,\infty;\beta)$ is singular at $\beta_0$:

$$\tilde{f}(m_0,\infty;\beta)=CV\int_{m_0}^{\infty}d\epsilon\, e^{-(\beta-\beta_0)\epsilon}\epsilon^{-d/2-1}\left[1+O\left[\frac{1}{\epsilon}\right]\right]$$

$$\simeq CV(\beta-\beta_0)^{d/2}\Gamma\left[-\frac{d}{2},m_0(\beta-\beta_0)\right] , \quad (19)$$

where $\Gamma(\alpha,x)$ is the incomplete gamma function. Since the analytic structure of $\Gamma(\alpha,x)$ is known, this allows us to separate out the singular and regular parts of $\tilde{f}(m_0,\infty;\beta)$ at $\beta_0$. Further, one can combine with $\tilde{f}(0,m_0;\beta)$ to get $\tilde{f}(\beta)$. The result is

$$\tilde{f}(m_0,\infty;\beta)\simeq h(\beta)(\beta-\beta_0)^{d/2}+a(\beta) \ , \quad (20)$$

$$\tilde{f}(\beta)\simeq h(\beta)(\beta-\beta_0)^{d/2}+\lambda(\beta) \ . \quad (21)$$

Here

$$h(\beta)=(-1)^{[(d+1)/2]}\pi CV\Gamma\left[\frac{d}{2}+1\right]^{-1}$$

and

$$h(\beta)=(-1)^{[(d/2)+1]}CV\Gamma\left[\frac{d}{2}+1\right]^{-1}\ln[m_0(\beta-\beta_0)]$$

for $d$ odd and even, respectively. The regular parts can be expanded in a power series:

$$a(\beta)=V\sum_{n=0}^{\infty}\frac{1}{n!}a_n(m_0)(\beta-\beta_0)^n$$

and

$$\lambda(\beta)=V\sum_{n=0}^{\infty}\frac{1}{n!}\lambda_n(\beta-\beta_0)^n \ ,$$

where $\lambda_n=a_n(m_0)+(-1)^n b_n$. The coefficients $a_n$ can be determined from various known expansions for the incomplete gamma function, and they are given by $a_n(m_0)=(-1)^{n+1}Cm_0^{n-d/2}/(n-d/2)$ for all $n$ when $d$ is odd and for $n\geq d/2+1$ when $d$ is even. For $n\leq d/2$ ($d$ even), the expressions are $a_0(d=0)=-\gamma$, $a_0(d=2)=CV/m_0$, $a_1(d=2)=CV(\gamma-1)$, etc. ($\gamma$ is the Euler number.) The corrections to Eq. (20) arise from the $O(1/\epsilon)$ corrections in Eq. (4) for $d\neq 0$. They give rise to terms that are less singular $[\sim(\beta-\beta_0)^{d/2+1}]$ than the one shown in Eq. (20) and also modify the regular part $a(\beta)$.

This completely expresses $\tilde{f}(\beta)$ in terms of $f(\epsilon)$. Note that $\tilde{f}(m,\infty;\beta)$ required in $N_>$ is given by (20) with $m_0$ replaced by $m$ and $\tilde{f}(0,m;\beta)$ required in $N_<$ is simply $\tilde{f}(\beta)-\tilde{f}(m,\infty;\beta)$. Our ignorance about $f(\epsilon)$ is mainly in not knowing the precise value of $m_0$ and the coefficients $b_n$ which reflect $f(\epsilon)$ for small $\epsilon$. Here we have given formulas for $\lambda_n$ in terms of the cutoff $m_0$ and $b_n$. However, they depend ultimately only on $\alpha'$ and the compactification radii and are independent of $m_0$. For low $n$, the $\lambda_n$ have simple physical meanings in a canonical approach. That is seen from $Z(\beta)=e^{\tilde{f}(\beta)}-1$ which implies that $\tilde{f}(\beta)$ is essentially the free energy, $-\partial\tilde{f}(\beta)/\partial\beta$ is the average energy $\langle E\rangle$ in the canonical ensemble, and $\partial^2\tilde{f}(\beta)/\partial\beta^2$ equals the energy fluctuations $\langle E^2\rangle-\langle E\rangle^2=(1/\beta^2)C_V$. From (21) we see that at $\beta=\beta_0$ the free energy diverges for $d=0$ and equals $\lambda_0 V$ for $d\geq 1$. The average canonical energy density at $\beta_0$ diverges for $d\leq 2$ and equals $-\lambda_1$ for $d\geq 3$. (Note that $\sigma_0\equiv-\lambda_1$ is positive for $d\geq 3$.) The canonical specific heat at $\beta_0$ diverges for $d\leq 4$ and equals $\beta_0^2\lambda_2 V$ for $d\geq 5$.

In Ref. 15 we obtained (21) from first principles and deduced (4). Here we have deduced (21) from (4) thereby establishing their equivalence. It now remains to perform the $\beta$ integration in (10), (13), (14), and (15) to obtain $\Omega(E)$, $N_{\text{total}}$, etc. This can be done by the method described in Ref. 15. We find qualitatively different results for $d\geq 3$ and $d=0,1,2$. We now present these results in turn.

### A. $d\geq 3$

When $d\neq 0$ the volume $V$ of the uncompactified spatial dimensions is another large parameter in addition to $E$. We discuss the situation when the energy density $\sigma\equiv E/V$ is finite but ranging from small to large values while $V\to\infty$. An intrinsic quantity in the theory that has the dimension of an energy density is $\sigma_0\equiv-V^{-1}\partial\lambda/\partial\beta|_{\beta_0}=-\lambda_1$, where $\lambda(\beta)$ is the nonsingular part of $\tilde{f}(\beta)$. Henceforth we shall refer to $\sigma_0$ as the Hagedorn energy density. All relevant energy densities will be compared with $\sigma_0$, and we use the notation $\bar{\sigma}\equiv\sigma-\sigma_0$ and $\bar{E}\equiv E-\sigma_0 V$.

Consider first the case $d\geq 3$. Equations (10) and (21) give

$$\Omega(E)=CV\frac{e^{\beta_0 E+\lambda_0 V}}{\bar{E}^{\,d/2+1}}\left[1+O\left[\frac{1}{\bar{E}}\right]+O\left[\frac{V}{\bar{E}^{\,\alpha}}\right]\right] , \quad (22)$$

where $\alpha=\frac{3}{2}$ for $d=3$ and $\alpha=2$ for $d\geq 4$. Thus the corrections are small for any finite value of $\bar{\sigma}$ since $V$ is large and the leading term gives a good approximation to the full expression as long as $\sigma>\sigma_0$. (A reason for results at $d=1,2$ being different from $d\geq 3$ is the fact that for $d=1,2$ these corrections are no longer small and an infinite number of terms have to be taken into account.)

We can now use (22) and (4) in (8) to discuss $\mathcal{D}(\epsilon;E)$. Let us first concentrate on two "end regions": $0\leq\epsilon\leq m_0$ and $\bar{E}-\Delta\leq\epsilon\leq E$ with $\Delta=O(\sigma_0 V)$. In the low-$\epsilon$ domain we do not know the precise form of $f(\epsilon)$, but we may use (22) for $\Omega(E-\epsilon)$. Similarly, in the high-$\epsilon$ domain we do not know the form of $\Omega(E-\epsilon)$, but we may use (4) for $f(\epsilon)$. Thus in these two domains we have, respectively,

$$\mathcal{D}_L(\epsilon;E)\simeq e^{-\beta_0\epsilon}f(\epsilon)(1-\epsilon/\bar{E})^{-(d/2+1)} , \quad (23)$$

$$\mathcal{D}_H(\epsilon;E)\simeq e^{-\beta_0\tilde{\epsilon}-\lambda_0 V}\Omega(\tilde{\epsilon})[(E/\bar{E})(1-\tilde{\epsilon}/E)]^{-(d/2+1)} , \quad (24)$$

where $\tilde{\epsilon} \equiv E-\epsilon$. The leading term in (23) is independent of $E$ (depends only on $\epsilon$), since $m_0 \ll E$. The leading term in (24) is independent of $E$ (depends only on $\tilde{\epsilon}$), if $\sigma_0 V \ll E$. Thus at energy densities much above the Hagedorn energy density, the behavior of $\mathcal{D}(\epsilon;E)$ close to both the end points $\epsilon=0,E$ becomes an $E$-independent "limiting" distribution: it depends only on the distance from the end point. At the end points, of course $\mathcal{D}(\epsilon;E)$ goes to zero because at low energies $f(\epsilon)$ and $\Omega(E)$ go as a positive power due to the excitation of the low-mass modes only.

In the central region, $m_0 < \epsilon < \bar{E}-\Delta$, we can use both (4) and (22) to obtain the explicit form

$$\mathcal{D}_C(\epsilon;E) \simeq \frac{CV}{\bar{E}^{\,d/2+1}}[x(1-x)]^{-(d/2+1)} , \qquad (25)$$

where $x \equiv \epsilon/\bar{E}$. This expression is interesting for the fact that $\mathcal{D}(\epsilon;E)$ does not continue to decrease for large $\epsilon$; in fact it rises when $\epsilon > \bar{E}/2$. It also implies that at the lower end point of the central region, $\epsilon=m_0$, its value $CVm_0^{-(d/2+1)}$ is much greater than its value at the upper end point $\epsilon=\bar{E}-\Delta$, which is $CV\Delta^{-d/2}$. This is because the low-$\epsilon$ domain is much narrower than the high-$\epsilon$ domain.

The above information implies a schematic plot of $\mathcal{D}(\epsilon;E)$ vs $\epsilon$ as shown in Fig. 2(a). The significant feature is that there are two peaks, a sharp and high peak between zero and $m_0$ and a broad and low peak between $\bar{E}-\Delta$ and $E$. The value in the central region is small [suppressed relative to the value at the beginning of the central region by a factor of approximately $(m_0/E)^{d/2+1}$]. The tail of the first peak starts off as a power law $\sim\epsilon^{-(d/2+1)}$. Another instructive plot is the graph $\epsilon\mathcal{D}(\epsilon;E)$ [Fig. 2(b)]. The only thing lacking is a quantitative knowledge of how high these peaks are.

To obtain that, let us compute $\mathcal{N}_{\text{total}}$, which is the area under the whole curve in Fig. 2(a), and next $\mathcal{N}_>$, which is defined as the area under the curve from a value $m$ (chosen to be much greater than $m_0$ but less than $\bar{E}$) up to $E$. Using (21) and (20) in (13) and (17) we get

$$\mathcal{N}_{\text{total}} \simeq 1+\left[\frac{d}{2}+1\right]\left[\frac{\sigma}{\sigma_0}-1\right]^{-1}+\lambda_0 V , \qquad (26)$$

$$\mathcal{N}_> = 1+\frac{2CV}{dm^{d/2}}\left[1+O\left[\frac{m}{\bar{E}}\right]\right] . \qquad (27)$$

From (27) one finds that there are always many massive strings present, $\mathcal{N}_> \sim E^{1-\xi}$, when $m\sim E^{\xi}$, $0<\xi \le 2/d < 1$. As $m$ approaches $\bar{E}$, the second term in (27) is suppressed since $d \ge 3$, with a single energetic string surviving in this limit. The number of low-energy strings is given by $\mathcal{N}_< = \mathcal{N}_{\text{total}} - \mathcal{N}_>$ which grows with $V$ linearly. This is the statement that the area under the low-energy peak is much greater than under the high-energy peak.

The situation is reversed for the energy-distribution plot, $\epsilon\mathcal{D}(\epsilon;E)$ [Fig. 2(b)]. In this case, the second peak has more area under it than the first when $\sigma$ is much higher than the Hagedorn energy density $\sigma_0$. To see that we compute $\mathcal{E}_>$ from (15) with the result

$$\mathcal{E}_> \simeq E - V\sigma_0\left[1-\frac{2C}{\sigma_0(d-2)}m^{1-d/2}\right] . \qquad (28)$$

The second term in the large parentheses is again a small correction as $m$ approaches $\bar{E}$. Thus, when $\sigma \gg \sigma_0$, we see that the second peak takes up most of the total area which equals $E$. While the height of the first peak, $O(\lambda_0 V)$, is much higher than that of the second, $O(\sigma/\sigma_0)$, most of the energy is nevertheless in the single energetic string, which is precisely the Frautschi-Carlitz scenario.[2]

### B. $d=0$

We now turn to the $d=0$ case. Here the singular term in $\tilde{f}(\beta)$ is $-\ln[(\beta-\beta_0)]$, which makes $\beta_0$ a simple pole of $Z(\beta)$ (Ref. 15). Equation (10) then gives

$$\Omega(E) = \frac{1}{m_0}e^{\beta_0 E+\lambda_0}[1+O(e^{-\eta E})] . \qquad (29)$$

Note that, unlike in $\Omega(E)$ for $d \ge 3$, and, as in $f(\epsilon)$ for $d=0$, the leading correction is suppressed exponentially and not by powers. The detailed form of the correction and its consequences, e.g., of yielding a positive specific heat, are discussed in Ref. 15, and $\eta$ is the same as for $f(\epsilon)$. Thus, at $R\sim\alpha'^{1/2}$, for the leading term to be the dominant contribution, we must have $E \gtrsim m_0$. In this paper we discuss the number densities in this regime where one can drop the correction term. The contribution of the corrections, which become important when the Universe expands, will be reported elsewhere.[18]

To discuss the string number distribution in $\epsilon$ we again divide the domain into three regions: $0 \le \epsilon \le m_0$, $m_0 < \epsilon < E-m_0$, and $E-m_0 \le \epsilon \le E$. The results are depicted in Figs. 3(a) and 3(b). Using (29) and the same argument as before, in the low-energy domain,

$$\mathcal{D}_L(\epsilon;E) \simeq e^{-\beta_0\epsilon}f(\epsilon) . \qquad (30)$$

Again this is independent of $E$. It follows that the total number of strings in this domain is a constant, $\mathcal{N}_L \simeq b_0$, and the energy deposited in these strings is also a constant, $\mathcal{E}_L \simeq b_1$. In the central domain, using (4) and (29) in (8), we find

$$\mathcal{D}_C(\epsilon;E) \simeq \frac{1}{\epsilon} . \qquad (31)$$

Integrating this, the number of strings in the central domain is $\mathcal{N}_C \simeq \ln[(E-m_0)/m_0]$. In this domain, the energy distribution function $\epsilon\mathcal{D}(\epsilon;E)$ is constant and unity, thus the energy deposited in this domain is $\mathcal{E}_C \simeq E-2m_0$. From this it follows that the energy deposited in the high-energy domain is again a constant, $\mathcal{E}_H \simeq 2m_0 - b_1$, independent of $E$. In this domain, from (4), we have

$$\mathcal{D}_H(\epsilon;E) \simeq \frac{m_0}{\epsilon}e^{-\beta_0\tilde{\epsilon}-\lambda_0}[\Omega(\tilde{\epsilon})+\delta(\tilde{\epsilon})] . \qquad (32)$$

When multiplied by $\epsilon$ we see that the resulting expression depends only on $\tilde{\epsilon}$ and not on $E$. $\mathcal{E}_C \simeq E-2m_0$ is the bulk of the energy and the above analysis substantiates the result mentioned in Ref. 15: namely, that the energy is distributed equally over most of the spectrum. This is reflected in the flat portion of the curve in Fig. 3(b). The dotted end regions reflect our ignorance of $f(\epsilon)$ and $\Omega(E)$ for $\epsilon$, $E<m_0$. Since $m_0$ is of the order $\eta^{-1}$, (30) and (32) together with the condition $\beta_0>\eta$ suggest the existence of the two small peaks shown in Fig. 3(b). We do know that the area under them is $\mathcal{E}_L \simeq b_1$ and $\mathcal{E}_H \simeq 2m_0 - b_1$, which is small compared to the area under the central region.

To compute the number of strings in the high-energy domain, it is convenient to first determine $\mathcal{N}_{\text{total}}$ from (13):

$$\mathcal{N}_{\text{total}} = \ln\frac{\bar{E}}{m_0} + b_0 + O\left[\frac{m_0^2}{\bar{E}^2}\right] . \tag{33}$$

Subtracting $\mathcal{N}_L + \mathcal{N}_C$ from this one finds that $\mathcal{N}_H \simeq \mathcal{E}_H/E$ vanishes asymptotically.

Similarly, for $m_0 < m < E - m_0$, we have

$$\mathcal{N}_> = \ln\frac{\bar{E}}{m} + O\left[\frac{m_0^2}{\bar{E}^2}\right] , \tag{34}$$

which implies that the number of energetic strings, each with an energy more than $m$, decreases logarithmically with $m$, so as to accommodate the flat distribution in $\epsilon$.

It is remarkable that the $d=0$ case exhibits behavior opposite to the $d \geq 3$ case. Here most of the energy is deposited in the central region, whereas for $d \geq 3$ this region contained almost nothing and most of the energy was concentrated in the two end regions. Further, here the energy deposited in the upper end region is a constant independent of total energy, whereas for $d \geq 3$ it was the number of strings in the upper end region that was a constant. In the $d=0$ case the energy is distributed equally on all length scales, whereas for $d \geq 3$ one energetic string dominates at high-energy densities. Both these cases are strikingly different from the usual free gas of a finite number of light particle species, e.g., Eq. (9), which is shown schematically in Fig. 1 for a contrast.

### C. $d=1$

The cases $d=1$ and $d=2$ turn out to be more complicated than the rest. A principal reason is that the corrections to the leading term in (22) are of the order $V/\sqrt{E}$ for $d=1$ and order $(V/E)\ln E$ for $d=2$ and the infinite series of corrections can no longer be ignored for finite energy densities. However, at finite energy densities, the integration over $\beta$ in (10) has a saddle point. For $d=1$ the saddle-point integration gives (up to corrections to be discussed shortly)

$$\Omega(E) \simeq \frac{CV}{\bar{E}^{3/2}} e^{\beta_0 E + \lambda_0 V - \pi C^2 V^2/\bar{E}} . \tag{35}$$

Substituting in (8) for the central region we obtain

$$\mathcal{D}(\epsilon;E) \simeq CV \left[\frac{\bar{E}}{\epsilon(\bar{E}-\epsilon)}\right]^{3/2} e^{-\pi C^2 V^2 \epsilon/\bar{E}(\bar{E}-\epsilon)} . \tag{36}$$

Here we see that $\mathcal{D}(\epsilon;E)$ is damped out exponentially in the region where $\epsilon$ is a significant fraction of $\bar{E}$. This is unlike the situation in $d=0$ or $d \geq 3$ where $\epsilon\mathcal{D}(\epsilon;E)$ was constant or rising. Thus for this case we have energetic strings exponentially suppressed.

In the saddle-point method the corrections to (35) come from the $\lambda_2, \lambda_3, \ldots$ terms in $\lambda(\beta)$. For instance, keeping $\lambda_2$ would change the exponential factor $e^{-\pi C^2 V^2/\bar{E}}$ in (35) to $\exp\{-(\pi C^2 V^2/\bar{E})[1+O(\lambda_2 C^2 V^3/\bar{E}^3)]\}$. (The $\bar{E}$ in the denominator is also modified by a similar factor.) These corrections are small if $\lambda_2 C^4 V^5/\bar{E}^4 \ll 1$. Thus Eqs. (35) and (36) are valid for the situation $\sigma \gg O(C(\lambda_2 V)^{1/4})$. At finite energy densities this requires $\lambda_2$, etc., to be close to zero.

Notice that if we have $E > V^2$ rather than $E \sim V$ (in the thermodynamic limit $V \to \infty$ this corresponds to the situation of strictly infinite energy density) the exponential factor $e^{-\pi C^2 V^2/\bar{E}}$ becomes essentially unity and the expression reduces to the one analogous to $d \geq 3$. This is to be expected because the corrections $VE^{-1/2}$ mentioned above now become small. In this limit we recover the picture that holds for $d \geq 3$ also for the $d=1$ case.

### D. $d=2$

For $d=2$ we consider two cases: one when $\sigma_0 < \bar{\sigma} < C\ln(CV/m_0)$ and the second when $\bar{\sigma} > C\ln(CV/m_0)$. In the former, the saddle-point analysis gives

$$\Omega(E) = \frac{e^{\lambda_0 V + \beta_0 E}}{(2\pi CVm_0)^{1/2}} \exp\left[-\frac{CV}{m_0}\exp(-1-\bar{\sigma}/C) - \tfrac{1}{2}(1+\bar{\sigma}/C)\right]\left[1 + O\left[\frac{V\lambda_2}{m_0^2}e^{-2(1+\bar{\sigma}/C)}\right]\right] . \tag{37}$$

We study this in the regime where $\lambda_2$, etc., are essentially zero so that the corrections can be ignored. In this regime, energetic strings are exponentially suppressed as can be seen from

$$\mathcal{D}(\epsilon;E) \simeq CV\epsilon^{-2} e^{-a\epsilon/m_0} . \tag{38}$$

Here $a = e^{-(1+\bar{\sigma}/C)} - m_0(2CV)^{-1}$ is a positive number. In the second case (of strictly infinite energy densities) we find

$$\Omega(E) \simeq CV\left[\bar{E} - CV\ln\left[\frac{CV}{m_0}\right]\right]^{-2} e^{\lambda_0 V + \beta_0 E} .$$

This is again similar to the $d \geq 3$ case. Therefore, above the Hagedorn energy density, so long as $\sigma$ is finite, a single-energetic-string-dominant picture holds only for $d \geq 3$ (the lower limit happens to coincide with the number of physical uncompactified dimensions). This completes our discussion for single-string number densities when all string states are included in the ensemble.

## IV. CONSERVATION LAWS AND INCLUSIVE DISTRIBUTIONS

We now turn to what happens when we take into account other conservation laws in string theory which was proposed in Ref. 15. In the case of toroidal compactifications that we have been considering, the conserved quantities are the total winding number $W_i$ and the discrete momenta $M_i$ in each compact spatial dimension $(i=1,2,\ldots,D-d)$, and the momenta $P_j$ in the noncompact dimensions. For the heterotic string, the U(1) charges for the 16 internal directions are also conserved.

The analog of $\mathcal{D}(\epsilon;E)$ when the total charge is fixed to be $Q$ is $\mathcal{D}(\epsilon;E,Q)$ defined in Eq. (2), and we would like to compare the two. Defining

$$f(\epsilon,q)\equiv \sum_{\text{all single-string states}} \delta(\epsilon-\epsilon_a)\delta_{q,q_a}$$

and

$$\begin{aligned}\Omega(E,Q)&\equiv \sum_{\text{all multistring states}} \delta(E-E_\alpha)\delta_{Q,Q_\alpha}\\ &=\sum_{N=1}^{\infty}\Omega_N(E,Q)\\ &=\sum_{N=1}^{\infty}\frac{1}{N!}\sum_{q_1,q_2,\ldots,q_N=-\infty}^{\infty}\int_0^E\prod_{i=1}^{N}[d\epsilon_i f(\epsilon_i,q_i)]\delta\left[E-\sum_{i=1}^{N}E_i\right]\delta_{Q,\sum q_i}\end{aligned}$$

in analogy with (3) and (7) and following the same steps as for Eqs. (6)–(8) we find

$$\mathcal{D}(\epsilon,q;E,Q)=\frac{1}{\Omega(E,Q)}f(\epsilon,q)[\Omega(E-\epsilon,Q-q)+\delta(E-\epsilon)\delta_{Q,q}]\ . \qquad (39)$$

Thus to compute (1) we need $f(\epsilon,q)$ and $\Omega(E,Q)$. These are obtained by introducing a chemical potential $\mu$ for the conserved charge in addition to $\beta$ which was introduced for obtaining the energy dependence. Using steps similar to those above (13), one can see that

$$f(\epsilon,q)=\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}\int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}e^{\beta\epsilon+\mu q}\tilde{f}(\beta,\mu)\ , \qquad (40)$$

$$\Omega(E,Q)=\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}\int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}e^{\beta E+\mu Q}Z(\beta,\mu)\ , \qquad (41)$$

where $Z(\beta,\mu)\equiv e^{\tilde{f}(\beta,\mu)}-1$ and

$$\begin{aligned}\tilde{f}(\beta,\mu)&\equiv\sum_{q=-\infty}^{\infty}\int_0^E d\epsilon\, e^{-\beta\epsilon-\mu q}f(\epsilon,q)\\ &=\sum_a e^{-\beta\epsilon_a-\mu q_a}\ . \end{aligned} \qquad (42)$$

A chemical potential is to be introduced for each conserved charge. Thus for toroidally compactified strings, $\mu q\to\mu_i m_i+\nu_i w_i+\rho_j p_j$, where $m_i$ and $w_i$ are the momenta and winding in the compact directions and $p_j$ are the momenta in the noncompact directions that characterize a single-string state $a$. Since $p_j$ are continuous, $\rho_j$ will be integrated from $-i\infty$ to $i\infty$, instead of $-i\pi$ to $i\pi$.

We have computed (42) from first principles using the string spectrum. We find that the leading behavior of the singular term in $\beta$ depends in a very simple way on the chemical potentials. The chemical potentials simply change the location of the singularity in $\beta$ from $\beta_0$ to $\beta_0(\mu,\nu,\rho)$. That is, the singular term of (21) is modified to

$$\tilde{f}(\beta,\mu,\nu,\rho)\simeq h(\beta)(\beta-\beta_0(\mu,\nu,\rho))^{d/2}\ . \qquad (43)$$

In $h(\beta)$ also $\beta_0$ is replaced by $\beta_0(\mu,\nu,\rho)$. The latter is found to be of the form

$$\beta_0(\mu,\nu,\rho)=\left[\left\{\left[\omega_l+\frac{1}{8\pi^2}\sum_i\left[\mu_i\bar{R}_i+\frac{\nu_i}{\bar{R}_i}\right]^2\right]^{1/2}+\left[\omega_r+\frac{1}{8\pi^2}\sum_i\left[\mu_i\bar{R}_i-\frac{\nu_i}{\bar{R}_i}\right]^2\right]^{1/2}\right\}^2+\sum_j\rho_j^2\right]^{1/2}, \qquad (44)$$

where $\bar{R}\equiv\alpha'^{-1/2}R$ with the convention $2\pi^2\alpha'=1$. For small $\mu$ this reduces to

$$\beta_0(\mu)\simeq\beta_0+A_{IJ}\mu_I\mu_J\ , \qquad (45)$$

where the $\mu_I$ stands for all the chemical potenials $\mu_i$, $\nu_i$, $\rho_j$ and the coefficients $A_{IJ}$ are determined in terms of the radii from (44).

Using (43) one can perform first the $\beta$ and then the $\mu$ integrations in (40) and (41). The consequence of the $\mu$ dependence to leading order appearing chiefly through $\beta_0(\mu)$ is that $\mu$ goes for a ride in the $\beta$ integrations, which can be performed in the same way as before. The $\mu$ integrations are then evaluated by the saddle-point method. In the large-$E$ limit with finite $q$ and $Q$, the saddle point is at $\mu=0$; therefore, only (45) is needed. For instance, with $C_1\equiv(2\sqrt{\pi})^{d-2D}(\det A)^{-1/2}$, we have[23]

$$\begin{aligned}f(\epsilon,q)&\simeq\frac{C_1CV}{\epsilon^{D+1}}e^{\beta_0\epsilon-(4\epsilon)^{-1}q^TA^{-1}q}\ ,\\ \Omega(E,Q)&\simeq\frac{C_1}{m_0E^D}e^{\lambda_0+\beta_0E-(4E)^{-1}Q^TA^{-1}Q}\ ,\quad d=0\\ &\simeq\frac{C_1CV}{\bar{E}^{(d/2)+1}E^{D-d/2}}e^{\lambda_0V+\beta_0E-(4E)^{-1}Q^TA^{-1}Q}\ ,\quad d\geq 3\ .\end{aligned} \qquad (46)$$

Comparing these with our earlier expressions for $f(\epsilon)$ and $\Omega(E)$, note the presence of additional power suppression factors $\epsilon^{-(D-d/2)}$ and $E^{-(D-d/2)}$, respectively, reflecting the shrinking widths of the $\mu$-saddle point. [The imposition of the conservation of the 16 U(1) charges in the case of the heterotic string will produce an additional factor of $E^{-8}$.] Also note that the $e^{-(4E)^{-1}Q^TA^{-1}Q}$ factor implies that the number of states decreases as $Q$ increases for fixed $E$. This is to be expected because the charges we have considered all cost energy ($E^2 \sim Q^2+\sum_n \alpha_{-n}\alpha_n$), so at nonzero $Q$ less energy (and hence fewer number of states) is available for the oscillators. Will these factors alter our previous conclusions for the inclusive distribution? The answer is no.

It is convenient to introduce a "mixed" description by defining

$$f(\epsilon;\mu)\equiv\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta\epsilon}\tilde{f}(\beta,\mu)$$

and (47)

$$\Omega(E;\mu)\equiv\int_{L-i\infty}^{L+i\infty}\frac{d\beta}{2\pi i}e^{\beta E}Z(\beta,\mu)\ .$$

This corresponds to the "charge-bath" picture mentioned at the beginning of Sec. II, with $\mu$ characterizing the external charge bath. For a neutral charge bath, by setting $\mu=0$, all our previous results then follow.

However, for a strict microcanonical treatment, we can do away with an external charge bath and obtain

$$f(\epsilon,q)=\int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}e^{\mu q}f(\epsilon;\mu)$$

and (48)

$$\Omega(E,Q)=\int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}e^{\mu Q}\Omega(E;\mu)\ .$$

In particular, the distribution $\mathcal{D}(\epsilon;E,Q)$, instead of Eq. (1), can be expressed as

$$\mathcal{D}(\epsilon;E,Q)=\frac{1}{\Omega(E,Q)}\int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}e^{\mu Q}f(\epsilon;\mu)[\Omega(E-\epsilon;\mu)+\delta(E-\epsilon)]\ . \qquad (49)$$

It is then easy to see that, at high-energy densities where $Q/E\to 0$, effects of charge fluctuations about the saddle point $\mu=0$, i.e., those leading to (46), cancel between those from the numerator and the denominator.

More directly, using (46) in (39) gives, e.g., for $d=0$ in the central domain, the distribution function of energy and charge to be

$$\mathcal{D}(\epsilon,q;E,Q)\simeq C_1\frac{1}{\epsilon}\left[\frac{E}{\epsilon(E-\epsilon)}\right]^D\exp\left[-\frac{1}{4(E-\epsilon)}[(\epsilon/E)Q^TA^{-1}Q+(E/\epsilon)q^TA^{-1}q-2Q^TA^{-1}q]\right]\ . \qquad (50)$$

A dramatic simplification occurs upon summing over $q$. We get

$$\mathcal{D}(\epsilon;E,Q)\simeq\frac{1}{\epsilon}\ ,$$

which is the same result as (31). Thus the imposition of conservation laws does not alter the distribution of the total energy among strings in a specified energy range. The same is observed for $d\neq 0$, e.g., for $d\geq 3$, $\mathcal{D}(\epsilon;E,Q)$ is given by (25).

## V. SUMMARY

To summarize we have presented a formalism for determining physical quantities such as the energy spectrum of a free gas. The formalism is valid for the microcanonical description of any noninteracting gas. This was used to determine properties of a string gas at high-energy densities. The nature of the Hagedorn singularity in the free energy as a function of $\beta$ turned out to be crucial in the analysis. It was found that the spectrum depends on the number of noncompact dimensions. For $d=0$ the distribution is flat [Fig. 3(b)], strings of all energies contributing equally to the total energy. For $d\geq 3$ the distribution is peaked at the two ends [Fig. 2(b)] with a single energetic string soaking up most of the energy. These properties are the same when conservation laws are taken into account and are in striking contrast to usual gases of a finite number of species of particles (Fig. 1). For $d=1,2$ energetic strings are exponentially suppressed at finite energy densities under the conditions described.

In this paper we have assumed that all compactified spatial dimensions have a radius of the order $\sqrt{\alpha'}$. As the compact dimensions expand, other singularities (in addition to the Hagedorn singularity $\beta_0$) in the complex $\beta$ plane also become important.[15] As a radius goes to infinity, an infinite number of these singularities accumulate at $\beta_0$ thereby changing the nature of the Hagedorn singularity. This then affects the high-energy behavior of the string gas and it is interesting to compare the energy distribution for the case of compact space with a very large radius with the case in which space is intrinsically noncompact.[18]

At sufficiently low energies only a few modes of the strings are excited and the qualitative behavior of the string gas is like an ordinary gas depicted in Fig. 1. At energies significantly higher than Hagedorn energy densities the behavior is described by Fig. 2 or Fig. 3, which are qualitatively different from Fig. 1. It is clearly of interest to study the Hagedorn region where the description changes. Since asymptotic formulas, e.g., (4), are no longer valid in this region, numerical studies using alternative representations must be adopted. Here again the complex-$\beta$ representation is useful in the context of the free gas. Work along this line is in progress.[24]

Finally, while the free string gas already exhibits novel properties, it is of great interest to study the interacting gas. A number of issues have to be addressed in this context, not least the consequences of gravitational interactions. We hope that the relationship between the thermal partition function and the $S$ matrix of a system discussed in Ref. 25, together with results on string scattering at high energies, (involving extension of Refs. 19 or 20), can be used to provide some insight in the subject of string statistical mechanics.

*Note added.* After the completion of this paper we received Ref. 26 which also independently discusses the use of chemical potentials to impose conservation laws, and recognizes the appearance of additional energy prefactors in $\Omega(E)$ as a consequence. However, its conclusion that a high-energy string gas is always dominated by a single energetic string applies only for $d \geq 3$, as argued in this paper. In the case of $d=0$, for example, strings over almost the entire energy range contribute equally to the total energy of the gas, whether or not momentum and winding number conservation are imposed on the ensemble.

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