Field-theoretic derivation of Wolfenstein's matter-oscillation formula

Palash B. Pal* and T. N. Pham
Centre de Physique Theorique, Ecole Polytechnique, 91128 Palaiseau, France
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We use covariant field-theoretic methods to rederive Wolfenstein's formula for neutrino oscillations within a medium.

More than a decade ago, Wolfenstein[1] pointed out that the patterns of vacuum mixing and oscillation of neutrinos can be significantly modified if the neutrinos pass through a material medium. Recently, Mikheyev and Smirnov[2] have shown that this mechanism might play a crucial role[3] in understanding the solar-neutrino problem. Since then, the subject has received a lot of attention. Various authors[4] have extended the original, two-generation analysis to include the effects of the third generation. The magnitude of the effect in stellar objects other than the Sun has also been discussed.[5]

Despite all the attention received by the topic and despite its potential importance in the field of neutrino physics, there exists no field-theoretical treatment of it in the literature. The original derivation[1] employed a potential picture, which has been used by subsequent authors as well.[3] In this paper we want to rectify this situation by showing how one can use covariant field-theoretical methods to obtain Wolfenstein's formula for neutrino oscillations in matter.

The central argument in Wolfenstein's analysis[1] is that the effective mass of a neutrino changes considerably in a medium owing to interaction with the particles in the medium. The change is different for the electron neutrino ν_e, which has both neutral- and charged-current interactions with the electrons, compared to the ν_μ and the ν_τ, which have only the former kind of interaction. This difference results in changing patterns of neutrino oscillations.

In quantum field theory, the mass of a particle is obtained from the inverse propagator. To keep our discussion clear, we consider Weyl neutrinos such as those in the standard electroweak model. Such neutrinos are massless in the vacuum and hence do not mix with one another. Thus we can discuss the effective mass of the ν_e separately from the other neutrinos. At the tree level, the inverse propagator of such a neutrino is given by

$$iS^{-1}_{0(\nu)}(p)=i\not{p}L \ , \tag{1}$$

where $\not{p}\equiv\gamma_\mu p^\mu$ and $L=\frac{1}{2}(1-\gamma_5)$ is the projection operator for the left-chiral fermions. The presence of this projection operator in the propagator implies that the neutrino can only be left handed, as is appropriate for a Weyl neutrino.

Quantum corrections would add a self-energy Σ to the inverse propagator, so that the full quantum propagator will be

$$iS^{-1}_{(\nu)}(p)=i(\not{p}L-\Sigma) \ . \tag{2}$$

Our goal is to compute this Σ in the presence of a medium and consider its implications for the neutrino masses and mixings.

At the one-loop level, contributions to Σ come, e.g., from the charged-current diagram of Fig. 1 and the neutral-current diagram of Fig. 2. There are other self-energy diagrams, obtained by replacing the gauge boson lines in Figs. 1 and 2 by Higgs-boson lines (physical and unphysical). In the gauge we use, the contributions of the diagrams involving the Higgs bosons are suppressed, compared with those involving the gauge bosons, by powers of m_l/M_W, where l stands generically for charged leptons. The effects of these diagrams are thus smaller and unimportant for our purpose.

These diagrams, of course, are the same as the ones which appear in the vacuum field theory. However, to evaluate them in our case of interest, we should use not the Feynman rules in vacuum, but the rules for field theory in a medium in equilibrium.[5] Better known as *finite-temperature field theory,* the formalism is equally equipped to discuss the effect of a finite density of particles, for example. There are two equivalent formulations of these Feynman rules: viz., in the *imaginary-time* and in the *real-time* formalisms. In this paper we use the latter since it is easier in this formalism to separate the vacuum effects from the effects of the medium.

The Feynman rules for all the vertices in the real-time formalism are identical to the corresponding rules in the vacuum. The propagators, however, are different. For example, the electron propagator is given by[6]

$$iS_e(k)=(\not{k}+m_e)\left[\frac{i}{k^2-m_e^2}-2\pi\delta(k^2-m_e^2)f_F(k\cdot u)\right] , \tag{3}$$

where u^μ is the four-vector denoting the center-of-mass velocity of the medium and f_F denotes the Fermi distribution function

$$f_F(x)=\frac{\theta(x)}{e^{\beta(x-\mu)}+1}+\frac{\theta(-x)}{e^{-\beta(x-\mu)}+1} , \tag{4}$$

where $\beta=1/T$ is the inverse temperature and μ is the chemical potential of the species of particle. The W and Z propagators are also modified in the medium, but let us

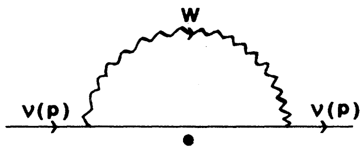

FIG. 1. The contribution to the self-energy of a Weyl neutrino due to its charged-current interactions.

restrict our discussion to the temperature range for which $T \ll M_W$, so that we can neglect the thermal effects in the W and the Z propagators. Then, using the 't Hooft–Feynman gauge to write the W propagator, we can write the amplitude of Fig. 1 as

$$\frac{i}{2}g^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\alpha L i S_e(k) \gamma^\alpha L \frac{1}{(p-k)^2 - M_W^2} , \tag{5}$$

where g is the SU(2) gauge coupling constant. The electron propagator iS_e has been given in Eq. (3). Looking at it, we notice that the first term in the large parentheses gives just the electron propagator in the vacuum. The contribution of that term to the amplitude in (5) will give the wave-function renormalization of the neutrino in the vacuum. In the present context, this is of no importance to us. We want to discuss the effects of the medium, which comes from the second term in the large parentheses of Eq. (3). Using this we can write down the medium contribution to Σ coming from Fig. 1 as

$$-i\Sigma_1 = -ig^2 \int \frac{d^4k}{(2\pi)^3} \frac{\not{k}L}{(p-k)^2 - M_W^2} \times \delta(k^2 - m_e^2) f_F(k \cdot u) . \tag{6}$$

So far, our discussion is completely covariant. Now, in order to perform the integration we go to the rest frame of the medium where $u^\mu = (1, \mathbf{0})$. In this frame we can write

$$\delta(k^2 - m_e^2) = \frac{1}{2\omega_k} [\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)] , \tag{7}$$

where

$$\omega_k \equiv (\mathbf{k}^2 + m_e^2)^{1/2} . \tag{8}$$

To the lowest order in M_W^{-2}, the momentum dependence in the W propagator may be disregarded. The integral of Eq. (6) can then be easily evaluated to obtain

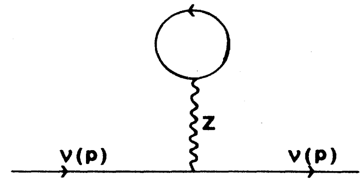

FIG. 2. The contribution to the self-energy of a Weyl neutrino due to its neutral-current interactions.

$$\Sigma_1 = \frac{g^2}{4M_W^2} (n_e - n_{\bar{e}}) \gamma^0 L . \tag{9}$$

To arrive at this form we have used the fact that the number densities of the electrons n_e and of the positrons $n_{\bar{e}}$ are given by

$$n_{\bar{e},e} = 2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\omega_k \pm \mu)} + 1} , \tag{10}$$

where the factor 2 appears because of spin degeneracy. Using the definition of the Fermi constant

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} , \tag{11}$$

we can easily rewrite Eq. (9), this time in a form that is valid in any arbitrary frame:

$$\Sigma_1 = \sqrt{2} G_F (n_e - n_{\bar{e}}) \hat{u} L \equiv A_1 \hat{u} L , \tag{12}$$

for example. Thus, from Eq. (2), we notice that because of charged-current interactions only, the inverse propagator is given by

$$iS_{(\nu)}^{-1}(p) = i(\not{p} - A_1 \not{u}) L , \tag{13}$$

so that

$$iS_{(\nu)}(p) = i \frac{\not{p} - A_1 \not{u}}{(p - A_1 u)^2} L . \tag{14}$$

The physical mass of the neutrino can be obtained by considering the poles of this propagator. For this, we introduce the Lorentz-invariant quantities

$$\mathbb{E} \equiv p \cdot u , \quad \mathbb{P} = \sqrt{\mathbb{E}^2 - p^2} , \tag{15}$$

which can be interpreted as the energy and the magnitude of the three-momentum of the neutrino in the rest frame of the medium. In the vacuum, where $A_1 = 0$, the pole of Eq. (14) is given by

$$p^2 = 0 \quad \text{or} \quad \mathbb{E}^2 - \mathbb{P}^2 = 0 , \tag{16}$$

corresponding to massless neutrinos. In the medium we obtain the poles at

$$p^2 = 2A_1 \mathbb{E} - A_1^2 \quad \text{or} \quad \mathbb{E}^2 - 2A_1 \mathbb{E} + A_1^2 - \mathbb{P}^2 = 0 . \tag{17}$$

The solution to this equation is

$$\mathbb{E} = A_1 \pm \mathbb{P} . \tag{18}$$

To interpret this result, it is best to go to the rest frame of the medium, where $\mathbb{E}$ is the energy of the neutrino and $\mathbb{P}$ is the magnitude of the three-momentum. The positive sign in Eq. (18) corresponds to the particle solution. It shows that, for the same magnitude of the neutrino momentum, the energy of the neutrino is increased by an amount A_1 compared to its value in the vacuum. This is Wolfenstein's result.

The other solution in Eq. (18), corresponding to the negative sign, corresponds to the antiparticle solution. As usual, by changing the sign of the energy, we obtain the energy-momentum relation for the antineutrino as $\mathbb{E} = \mathbb{P} - A_1$, so that the energy of the antineutrino decreases within the medium. This result is also well known in the context of a potential approach to the problem.

So far, we have been discussing the contribution due to Fig. 1 only. A similar analysis can be performed on the diagram in Fig. 2, yielding an extra contribution to the self-energy. It must be noted that now any fermion can circulate in the loop. Of course, the only nontrivial contribution comes from the fermions which constitute part of the background medium. A straightforward calculation shows that the axial-vector part of the fermion coupling to the Z does not contribute to the self-energy of the neutrino. We find that the net effect of adding the diagram in Fig. 2 is to replace the quantity A_1 in Eqs. (12)–(18) by A_1+A_2, where

$$A_2=\sqrt{2}G_F\sum_f (T_3^{(f)}-2Q^{(f)}\sin^2\theta_W)(n_f-n_{\bar f}) , \qquad (19)$$

where $Q^{(f)}$ is the electric charge of the fermion and $T_3^{(f)}$ is the third component of the weak isospin for the left-chiral projection of it. Considering, for example, the solar interior where the temperature is such that only electrons, protons, and neutrons are present and hardly any of their antiparticles, we can reduce Eq. (19) to a simpler form. Since $T_3^{(p)}=\frac{1}{2}=-T_3^{(e)}$ and charge neutrality requires $n_e=n_p$, the contributions of electron and proton cancel in (19), and we are left with only the contribution of the neutron:

$$A_2=-\sqrt{2}G_F\tfrac{1}{2}n_n . \qquad (20)$$

The effective neutrino energy, or equivalently, the effective neutrino mass, is accordingly modified.

There is, of course, one important difference between the neutral-current contribution and the charged-current contribution to the self-energy, as has been noted earlier. If we consider more than one generation of neutrinos propagating through a background which contains electrons but not muons or taons (as is usual for ordinary matter), the charged-current self-energy diagram affects only the electron neutrino, whereas the neutral-current contribution affects all flavors equally. It is well known[1–4] that because of this reason the neutral-current contribution is irrelevant in discussing neutrino mixing. It is relevant only for the discussion of the effective mass of the neutrino.

We have thus derived all the features of Wolfenstein's formula by using covariant field-theoretic methods. It must be emphasized that although our result is the same as the conventional one, some points come out clearly from this derivation. For example, unlike the original derivation, we never assumed that the electrons in the medium are nonrelativistic, so our derivation shows that Eq. (18) is equally applicable at temperatures comparable to, or larger than, the electron mass. Moreover, it also shows how to take a finite positron density into account. Both these considerations might be important for discussing matter oscillations in supernovas, for example.

Note added in proof. Before publication of this paper we came across a paper by D. Notzold and G. Raffelt, Nucl. Phys. **B307**, 924 (1988), where a similar calculation has been performed.

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*On leave from the Department of Physics and Astronomy, University of Massachusetts, Amherst, MA 01003.

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