New variables for gravity: Inclusion of matter

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The Lagrangian and Hamiltonian formulations of general relativity in terms of soldering forms and self-dual connections are extended to include matter sources and the cosmological constant. For matter sources we consider minimally coupled Klein-Gordon fields, complex- and Grassmann-valued Dirac fields, and Yang-Mills fields. Somewhat surprisingly, in spite of the derivative coupling in the spin-half fields, the use of only the self-dual part of the connection as a basic variable does not lead to spurious equations or inconsistencies. Furthermore, as in the source-free case considered earlier, all equations of the theory are polynomial in terms of these variables. Therefore, the framework has several potential applications especially to the nonperturbative canonical quantization program.

I. INTRODUCTION

In this paper we continue the development of a nonperturbative approach to canonical gravity. (For a review, see Ref. 1.) The first step in this program was a reformulation of general relativity.[2,3] The key idea there was to perform a canonical transformation on the gravitational phase space to pass on to new variables in terms of which the basic equations of the theory simplify considerably. To see how this arises it is convenient to shift the perspective slightly and begin with complex general relativity. That is, let us first consider complex four-metrics on a real four-manifold $\mathcal{M}$, which has the topology of $\Sigma\times\mathbb{R}$, for some three-manifold Σ. Then, the new canonically conjugate pair $(\tilde{\sigma}^{a}{}_{A}{}^{B}, A_{aA}{}^{B})$ consists of a soldering form $\tilde{\sigma}^{a}{}_{A}{}^{B}$ (of density weight 1) for SU(2) spinors on Σ, and a connection one-form $A_{aA}{}^{B}$, with values in the (complexified) Lie algebra of SU(2). The soldering form is in essence the square root of the three-metric while $A_{aA}{}^{B}$ represents, in any solution to field equations, a potential for the self-dual part of the Weyl curvature. It turns out that the constraint functionals as well as the Hamiltonian of the theory are *polynomial* in these variables. To obtain real—Euclidean or Lorentzian—relativity one has to restrict oneself to the appropriate "real section" of the complex phase space. This is accomplished by imposing suitable reality conditions. These conditions are again polynomial: they just require that the (density-weighted) three-metric $-\mathrm{tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b}$ constructed from $\tilde{\sigma}^{a}{}_{A}{}^{B}$ and its Poisson brackets with the constraint functionals be real.[4] In the canonical quantization program, one can first ignore the reality conditions, solve the quantum constraints, and *then* incorporate these conditions as Hermitian-adjointness relations on the appropriate operators on the Hilbert space of physical states. Considerable progress has been made in solving the quantum constraints exactly (both in quantum cosmology[5] and full quantum gravity[6]) because of simplicity of their expressions.

The general framework involving new canonical variables was first obtained using Hamiltonian methods indicated above. However, a manifestly covariant Lagrangian formulation was soon given independently by Samuel[7] and by Jacobson and Smolin.[8] (See also the Appendix of Ref. 9.) This formulation is better suited for inclusion of matter terms. For, in the Lagrangian formulation, one has only to choose the basic dynamical variables and their couplings. Given the source-free Lagrangian formulation and general facts about gravitational coupling, this is a relatively easy choice to make. In the Hamiltonian formulation, on the other hand, one has to guess the appropriate canonical transformation and this becomes rather complicated once Dirac fields are brought in. Therefore, we shall use the work of Samuel, Jacobson, and Smolin as our point of departure. The purpose of this paper is to extend their framework by including matter sources and to analyze the resulting algebra of constraints and Hamiltonians. We shall find that the key features of the source-free framework are preserved in the extension. More precisely, we shall see that the constraint functionals, the Hamiltonians, and the reality conditions continue to retain their polynomial form in the basic canonical variables for matter and gravity and that the close relation with Yang-Mills theory also continues to hold. Consequently, one can continue to borrow techniques from Yang-Mills theory and QCD to analyze physical predictions of general relativity and quantum gravity.[9,10]

We begin, in Sec. II, by introducing the basic fields and the Lagrangian on a four-manifold $\mathcal{M}$, topologically $\Sigma\times\mathbb{R}$. As in the source-free case,[7–9] we use a first-order formalism and express the gravitational part of the action in terms of the four-dimensional SL(2,$\mathbb{C}$) soldering form, $\sigma^{a}{}_{A}{}^{A'}$, the unprimed spin (or "internal") connection, ${}^4A_{aA}{}^{B}$, and a possible cosmological constant Λ. As in Ref. 8 we restrict the soldering form to be anti-Hermitian from the beginning so that *the space-time metric will always be real* [with signature $(-+++)$]. The matter fields consist of minimally coupled, massive Klein-Gordon field ϕ, massive Dirac field $(\xi^{A},\eta_{A'})$, and Yang-Mills connection one-form ${}^4\mathbf{A}_{a}$ (with an arbitrary internal gauge group). It is easy to extend the framework to include suitable interaction terms between Klein-Gordon, Yang-Mills, and Dirac fields and to allow several distinct fields of any given spin. In particular, one could replace the real Klein Gordon field by a Higgs multiplet and/or let the Dirac field have an additional, Yang-Mills internal index. As in the source-free case, our total action is *complex*. One would *a priori* expect that this would lead to *twice* as many equations as one wants. In the source-free case, the extra equation turns out to be simply the Bianchi identity. We shall show that the situation remains unaltered despite the fact that the Dirac field now couples to the *complex-valued* self-dual connection ${}^4A_{aA}{}^B$. Thus, the main result of Sec. II is that, even though our action depends only on the self-dual part of the spin connection, its variation does not lead to spurious equations of motion.

In Sec. III we carry out the variation explicitly and write down the resulting equations of motion in a space-time language. We find that while the Klein-Gordon and Yang-Mills equations are the expected ones, the Dirac equation contains a cubic term in the spin-$\frac{1}{2}$ fields. This is not surprising: since we are using a first-order formalism, one would suspect that the outcome would resemble the Einstein-Cartan theory rather than the minimally coupled Einstein-Dirac. However, it is not *a priori* clear that our theory is equivalent to the Einstein-Cartan theory since, unlike in the usual Palatini formulation, our Lagrangian depends only on the self-dual part of the spin connection. Given any solution to the field equations, however, we can extend the connection to primed spinors *and* vectors by requiring that it be real and annihilate the space-time metric. Then, the connection develops the standard torsion terms of the Einstein-Cartan theory. That is, as far as solutions to the classical equations of motion are concerned, our Lagrangian is equivalent to the standard first-order Einstein-Cartan Lagrangian. Now, it is well known[11] that the Einstein-Dirac theory can be recovered in a first-order framework by adding a quartic term to the Dirac Lagrangian. We show that this is also the case for our self-dual Lagrangian by exhibiting the required quartic term.

One often uses Grassmann-valued fermion fields in path integrals. Therefore, it is desirable to let our Dirac fields be either complex or Grassmann valued in the discussion of the action. This is straightforward to achieve: the discussion of Secs. II and III is completely insensitive to the choice. In the Grassmann case, one just has to be careful with orderings and use consistently the ordering of spin-$\frac{1}{2}$ fields specified in the paper.

In Sec. IV we pass to a Hamiltonian formulation through a Legendre transform. All the canonically conjugate variables are now fields on the three-manifold Σ. These are subject to four types of constraints: *the gravitational Gauss law, the Yang-Mills Gauss law, the vector and the scalar constraints.* We exhibit these constraints in terms of the canonically conjugate variables. They are all polynomial. For the convenience of the reader we list them in the form of a table.

In Sec. V we examine the Poisson-brackets algebra generated by these constraints. The gravitational Gauss-law constraint generates internal rotations on (unprimed) spinor indices and provides a representation of the Lie algebra of local SU(2) transformations. Similarly, the Yang-Mills Gauss-law constraint provides a representation of the Lie algebra of local gauge transformations. A linear combination (with "q-number coefficients") of the vector constraint and the gravitational Gauss law generates diffeomorphisms on the three-manifold Σ. This combination will be referred to as the *diffeomorphism constraint.* The Poisson-brackets algebra of diffeomorphism constraints provides us with a representation of the Lie algebra of the diffeomorphism group of Σ. Thus, the Gauss law and diffeomorphism (or vector) constraints generate a closed Poisson-brackets algebra which mirrors the "kinematic symmetries" associated with (suitable bundles constructed on) the three-manifold Σ. The geometrical interpretation of these constraints also trivializes the task of computing their Poisson brackets with scalar constraints. Thus, the only nontrivial computation left is that of the Poisson brackets between scalar constraints themselves. As expected, the bracket is a vector constraint whose coefficient is a "q number." Thus, although the constraints are of first class, their algebra is open in the sense of Becchi, Rouet, Stora, and Tyutin (BRST).

Section VI discusses Hamiltonians. As is well known, if Σ is compact, the Hamiltonian consists only of a linear combination of constraints, basically because we do not have a background space-time geometry. A more interesting situation occurs in the asymptotically flat case. (For details, see, e.g., Ref. 1.) Therefore, in this section we restrict ourselves to a noncompact three-manifold Σ and canonically conjugate fields thereon satisfying suitable boundary conditions. In this case, constraints only generate those canonical transformations which are asymptotically identity. In particular, they do *not* generate space-time translations. Thus, the Hamiltonians are distinct from constraints. To obtain Hamiltonians one must add suitable surface terms to constraints functionals. As one might expect, these are precisely the surface integrals that define the Arnowitt-Deser-Misner energy-momentum. We conclude this section by discussing the reality conditions that must be imposed on the basic canonical variables in order to ensure that we are dealing with *real* general relativity. These arise as direct consequences of equations of motion and definitions of the canonically conjugate variables. *They are also polynomial in the basic variables.*[4]

Our conventions are as follows. Throughout we use Penrose's abstract index notation.[12] The signature of the space-time metric is $(-+++)$. Usually, the stem letters of various fields and/or their index structure are used to distinguish between space-time, "four-dimensional" fields on M, and spatial "three-dimensional" fields on Σ. Thus, while g_{ab} and $\sigma^a{}_A{}^{A'}$ are the space-time metric and the corresponding SL(2,$\mathbb{C}$) soldering form on M, q_{ab}, and $\sigma^a{}_A{}^B$ are the spatial metric and the SU(2) soldering form on Σ. When there is a possibility of an ambiguity, the "four-dimensional" fields carry an explicit prefix 4. Thus, ${}^4A_{aA}{}^B$ is the space-time connection while $A_{aA}{}^B$ is its pullback to Σ. The torsion-free derivative operator compatible with $\sigma^a{}_A{}^{A'}$ is denoted by ∇, while the one compatible with $\sigma^a{}_A{}^B$ is denoted by D. The derivative operator on unprimed spinors, defined by the gravitational self-dual connection ${}^4A_{aA}{}^B$ is denoted by ${}^4\mathcal{D}$ while that defined by the Yang-Mills connection ${}^4\mathbf{A}$ is denoted by ${}^4\mathbf{D}$. The relation between the "four-dimensional" SL(2,$\mathbb{C}$) spinors and the "three-dimensional" SU(2) spinors is as follows. Recall first that, given a Hermitian metric, $G_{AA'}(=\bar{G}_{A'A})$, the primed SL(2,$\mathbb{C}$) spinors define a Hermitian conjugation operation on the unprimed ones via $(\xi^\dagger)_A := G_A{}^{A'}\bar{\xi}_{A'}$. Given a spacelike submanifold Σ with a unit normal field $n^{AA'} := \sigma_a^{AA'} n^a$, we will choose the required Hermitian metric to be $G^{AA'} = i\sqrt{2}\, n^{AA'}$. Consequently, the "dagger" operation will satisfy: $(\xi^\dagger)^A \xi_A \geq 0$; and $[(\xi^\dagger)^\dagger]_A = -\xi_A$. Finally, while in the main body of the paper we have set $G=c=1$, at the end of Sec. III, we briefly indicate how to restore factors of G. In the present framework they appear in rather unexpected terms so that care is needed in taking the weak-field or strong-coupling limits.[13]

After this work was completed we learned that some of the results presented in this paper had been obtained independently by other authors. Jacobson[14] has discussed the coupling to Dirac fields briefly, while S. Koshti (private communication) has analyzed some features of the coupling to Klein-Gordon fields.

II. SELF-DUAL ACTION AND REALITY OF EQUATIONS

Fix a four-manifold $\mathcal{M}$, with topology $\Sigma \times \mathbb{R}$ for some three-manifold Σ. The total Lagrangian density $\mathcal{L}_T$ on $\mathcal{M}$ will be a sum of several pieces, each of which is a scalar density of weight 1:

$$\mathcal{L}_T = \mathcal{L}_E + \mathcal{L}_C + \mathcal{L}_{\mathrm{KG}} + \mathcal{L}_D + \mathcal{L}_{\mathrm{YM}} . \tag{2.1}$$

Here $\mathcal{L}_E$ is the gravitational part of the Lagrangian density, $\mathcal{L}_C$ is the cosmological constant term, and $\mathcal{L}_{\mathrm{KG}}$, $\mathcal{L}_D$, and $\mathcal{L}_{\mathrm{YM}}$ are the matter Lagrangian densities for the spin-0 Klein-Gordon field, the spin-$\frac{1}{2}$ Dirac field, and the spin-1 Yang-Mills field, respectively. As in the source-free case,[7–9] we use a first-order framework for the gravitational part. Thus, $\mathcal{L}_E \equiv \mathcal{L}_E(\sigma^a{}_A{}^{A'}, {}^4A_{aA}{}^B)$ is a functional of an anti-Hermitian soldering form $\sigma^a{}_A{}^{A'}$ and a connection ${}^4A_{aA}{}^B$ which acts only on the unprimed spinor (or, rather, *internal*) indices. (Note that, being anti-Hermitian, $\sigma^a{}_{AA'}$ satisfies $\bar{\sigma}^a{}_{AA'} = -\sigma^a{}_{A'A}$.) It is given by

$$\mathcal{L}_E(\sigma^a{}_A{}^{A'}, {}^4F_{abA}{}^B) = ({}^4\sigma)\sigma^a{}_A{}^{A'}\sigma^b{}_{BA'}{}^4F_{ab}{}^{AB} , \tag{2.2a}$$

where $({}^4\sigma)$ is the determinant of the inverse soldering form and ${}^4F_{abA}{}^B$ is the curvature tensor of ${}^4A_{aA}{}^B$. The soldering form $\sigma^a{}_A{}^{A'}$ defines a real four-metric g^{ab} of signature $(-+++)$ via $g^{ab} = \sigma^a{}_{AA'}\sigma^{bAA'}$ and a (unique) torsion-free derivative operator ∇ which acts on unprimed and primed spinors as well as tensor indices. The connection ${}^4\mathcal{D}$ defined by ${}^4A_{aA}{}^B$ via ${}^4\mathcal{D}_a\lambda_A := \partial_a\lambda_A + {}^4A_{aA}{}^B\lambda_B$ bears no relation to ∇ at this stage. Indeed, ${}^4\mathcal{D}$ can only act on unprimed spinors. It is a somewhat remarkable fact that none of the essential equations of the theory require the availability of a specific extension of ${}^4\mathcal{D}$ to tensors or primed spinors. Thus, although for computational convenience we may occasionally extend ${}^4\mathcal{D}$ in an appropriate way, it should be borne in mind that the framework does *not* require the knowledge of *any* derivative operator on tensors or primed spinors. The cosmological constant Λ contributes via

$$\mathcal{L}_C(\sigma^a{}_A{}^{A'}) := \Lambda({}^4\sigma) . \tag{2.2b}$$

The Klein-Gordon and Yang-Mills Lagrangian densities are the familiar ones:

$$\mathcal{L}_{\mathrm{KG}}(\sigma^a{}_A{}^{A'}, \phi) := 4\pi({}^4\sigma)(g^{ab}\partial_a\phi\partial_b\phi + \mu^2\phi^2) , \tag{2.2c}$$

$$\mathcal{L}_{\mathrm{YM}}(\sigma^a{}_A{}^{A'}, {}^4\mathbf{A}_a) := \tfrac{1}{2}({}^4\sigma)g^{ac}g^{bd}\mathrm{tr}\,{}^4\mathbf{F}_{ab}{}^4\mathbf{F}_{cd} , \tag{2.2d}$$

where tr denotes the trace over the (suppressed) Yang-Mills internal indices. Finally, we shall take the Dirac Lagrangian density to be

$$\mathcal{L}_D(\sigma^a{}_A{}^{A'}, {}^4A_{aA}{}^B, \xi^A, \bar{\xi}^{A'}, \bar{\eta}^A, \eta^{A'}) := \sqrt{2}({}^4\sigma)\left[\sigma^a{}_{AA'}[\xi^{A'}{}^4\mathcal{D}_a\xi^A - ({}^4\mathcal{D}_a\bar{\eta}^A)\eta^{A'}] + \frac{im}{\sqrt{2}}(\bar{\eta}_A\xi^A - \bar{\xi}^{A'}\eta_{A'})\right] , \tag{2.2e}$$

where the spin-$\frac{1}{2}$ fields ξ^A and $\bar{\eta}^A$, can be either complex or Grassmann valued. Note that $\mathcal{L}_D$ contains derivatives only of unprimed spinors and that these fields are minimally coupled to gravity. This coupling is the same as in Yang-Mills theory except that the "Higgs scalars" are now spin-$\frac{1}{2}$ fields of one chirality; the Yang-Mills internal index is replaced by the unprimed spinor index. Note also that while all matter fields couple to $\sigma^a{}_A{}^{A'}$, only the Dirac field couples to ${}^4A_{aA}{}^B$. Because of this, much of the conceptual and computational nontriviality of our analysis lies only in the gravitational and Dirac parts, $\mathcal{L}_E$ and $\mathcal{L}_D$, of the total Lagrangian density $\mathcal{L}_T$.

Note first that while $\mathcal{L}_C$, $\mathcal{L}_{\mathrm{KG}}$, and $\mathcal{L}_{\mathrm{YM}}$ are manifestly real, $\mathcal{L}_E + \mathcal{L}_D$ is not, owing to the absence of the primed spin connection ${}^4\bar{A}_{aA'}{}^{B'}$, the complex conjugate of ${}^4A_{aA}{}^B$. This gives rise to the possibility that equations of motion may not be *real;* i.e., the Lagrangian density considered here may not lead to the familiar equations for the gravitational and Dirac fields. Indeed, *a priori*, it is quite possible that the variation of $\mathcal{L}_E + \mathcal{L}_D$ may give rise to spurious equations which may even lead to inconsistencies. Therefore, before proceeding with a detailed analysis of the theory, let us make a detour to ensure that the complex nature of $\mathcal{L}_E + \mathcal{L}_D$ does not give rise to such unforeseen problems.

To analyze this issue it is particularly convenient to use a general fact about variational principles. Let $S(x^i, y^\alpha)$ be an action depending on two types of dynamical variables, x^i and y^α. Solutions to dynamical equations are extrema of S with respect to both x^i and y^α. Let us suppose that the equations $\partial S/\partial y^\alpha = 0$ admit *unique* solutions $y_0^\alpha(x)$ for each choice of x^i so that the surface in the x-y space representing the solutions to this equation can be coordinatized by x^i. Then the pullback, $\underline{S}(x) := S(x^i, y_0^\alpha(x))$, of the action to the solution set has the property that its extrema are precisely the extrema of the full action $S(x^i,y^\alpha)$. In our case, the role of y^α will be played by the connections ${}^4A_{aA}{}^B$ and the role of x^i by the remaining variables, $\sigma^a{}_A{}^{A'}$, ξ^A, $\bar{\xi}^{A'}$, $\bar{\eta}^A$, $\eta^{A'}$, ϕ, and ${}^4\mathbf{A}_a$, that occur in $\mathcal{L}_T$. We shall first show that the extrema of $\mathcal{L}_T$ with respect to variations in ${}^4A_{aA}{}^B$, i.e., solutions to the equations of motion of ${}^4A_{aA}{}^B$, exist and are unique for each choice of the remaining variables. Therefore, equations of motion for $\sigma^a{}_A{}^{A'}$ and for the matter fields can be obtained by extremizing the pullback $\underline{S}_T(\sigma^a{}_{AA'},\xi^A,\ldots)$ of the full action to the space of solutions ${}^4A_{aA}{}^B$. We shall show that this pullback is real, ensuring that equations of motion for $\sigma^a{}_A{}^{A'}$, ϕ, and ${}^4\mathbf{A}_a$ are real, and those for ξ_A and $\bar{\eta}_A$ are the complex conjugates of the equations for $\bar{\xi}^{A'}$ and $\eta^{A'}$.

Let ∇ denote the unique torsion-free connection compatible with $\sigma^a{}_A{}^{A'}$. Then $({}^4\mathcal{D}_a-\nabla_a)\lambda_A={}^4C_{aA}{}^B\lambda_B$ for some ${}^4C_{aA}{}^B$. Hence, varying the total action

$$S_T:=\int d^4x\,\mathcal{L}_T(\sigma,{}^4A,\phi,{}^4\mathbf{A},\xi,\bar{\xi},\bar{\eta},\eta) \tag{2.3}$$

with respect to ${}^4A_{aA}{}^B$ keeping all other fields (including $\sigma^a{}_A{}^{A'}$) fixed is equivalent to varying S_T with respect to ${}^4C_{aA}{}^B$. The corresponding equation of motion for ${}^4C_{aA}{}^B$ (or ${}^4A_{aA}{}^B$), $\delta S_T/\delta^4C_{aA}{}^B=0$, yields

$$2\sigma^{[m}{}_M{}^{A'}\sigma^{a]}{}_{A'A}{}^4C_{aN}{}^A+2\sigma^{[m}{}_N{}^{A'}\sigma^{a]}{}_{A'A}{}^4C_{aM}{}^A =-\frac{i}{2}(\sigma^m{}_{MA'}k_N{}^{A'}+\sigma^m{}_{NA'}k_M{}^{A'}) \tag{2.4a}$$

with

$$k^{AA'}:=-i\sqrt{2}(\bar{\xi}^{A'}\xi^A-\bar{\eta}^A\eta^{A'})\ . \tag{2.4b}$$

This algebraic equation has the unique solution

$${}^4C_a{}^{AB}=\frac{i}{4}k^{(A}{}_{D'}\sigma_a{}^{B)D'}\ . \tag{2.5}$$

Thus, given any choice of field configurations $\sigma^a{}_A{}^{A'}$, ξ^A, and $\bar{\eta}^{A'}$, the connection ${}^4\mathcal{D}$ (or ${}^4A_{aA}{}^B$) is uniquely determined. Therefore, to obtain equations of motion for the remaining fields, we need only consider the reduced action $\underline{S}_T$, obtained by substituting the solution (2.5) for ${}^4A_{aA}{}^B$ in S_T. Simplifications occur only in the gravitational and the Dirac parts of S_T. The curvature tensor ${}^4F_{abA}{}^B$ in $\mathcal{L}_E$ can now be expressed in terms of the curvature tensor of $\sigma^a{}_A{}^{A'}$ and the field $k^{AA'}$ defined above, while the connection ${}^4\mathcal{D}$ in $\mathcal{L}_D$ can be replaced by the torsion-free connection ∇ compatible with $\sigma^a{}_A{}^{A'}$ and $k^{AA'}$. The reduced gravitational action is given by

$$\underline{\mathcal{L}}_E=({}^4\sigma)\left[-\tfrac{1}{2}{}^4R-\frac{3i}{4}\nabla_ak^a-\tfrac{3}{16}k^ak_a\right]\ , \tag{2.6a}$$

where ${}^4R$ is the scalar curvature of the four-metric $g^{ab}=\sigma^a{}_{AA'}\sigma^{bAA'}$ and $k^a:=\sigma^a{}_{AA'}k^{AA'}$, and the reduced Dirac action, by

$$\underline{\mathcal{L}}_D=\frac{1}{\sqrt{2}}({}^4\sigma)\sigma^a{}_{AA'}[\bar{\xi}^{A'}\nabla_a\xi^A-(\nabla_a\bar{\xi}^{A'})\xi^A+\bar{\eta}^A\nabla_a\eta^{A'}-(\nabla_a\bar{\eta}^A)\eta^{A'}]+({}^4\sigma)im(\bar{\eta}_A\xi^A-\bar{\xi}^{A'}\eta_{A'})+\tfrac{3}{8}({}^4\sigma)k^ak_a +\frac{i}{2}({}^4\sigma)\nabla_ak^a\ . \tag{2.6b}$$

Note that $\underline{\mathcal{L}}_E$ now depends not only on the gravitational variable $\sigma^a{}_A{}^{A'}$ but also on spin-$\frac{1}{2}$ fields because we have solved for ${}^4A_{aA}{}^B$ in terms of the soldering form and Dirac fields. Since $\mathcal{L}_C$, $\mathcal{L}_{\mathrm{KG}}$, and $\mathcal{L}_{\mathrm{YM}}$ do not involve ${}^4A_{aA}{}^B$, the reduction has no effect on these terms. Combining all terms, we now have

$$\underline{S}_T=\int d^4x(\underline{\mathcal{L}}_E+\underline{\mathcal{L}}_D+\mathcal{L}_C+\mathcal{L}_{\mathrm{KG}}+\mathcal{L}_{\mathrm{YM}})\ . \tag{2.6c}$$

Since the vector field k^a is real by its definition, each term in the integrand is manifestly real, except for a total divergence $-(i/4)({}^4\sigma)\nabla_ak^a$. Since total divergences do not affect equations of motion, we conclude that equations of motion arising from (2.6c)—and, hence, from (2.3)—for g_{ab}, ϕ, and ${}^4\mathbf{A}_a$ are all real and those for the spin-$\frac{1}{2}$ fields ξ^A and $\bar{\eta}^A$ are complex conjugates of the equations for $\bar{\xi}^{A'}$ and $\eta^{A'}$. Thus, even though S_T is complex, it does *not* give rise to any spurious equations of motion.

III. EULER-LAGRANGE EQUATIONS

Let us now vary the total action S_T to obtain the Euler-Lagrange equations of motion for the soldering form and matter fields.

The variation with respect to the scalar field ϕ and the Yang-Mills potential $\mathbf{A}_a$ is straightforward and yields the expected equations for these fields:

$$g^{ab}\nabla_a\nabla_b\phi-\mu^2\phi=0 \tag{3.1}$$

and

$${}^4\mathbf{D}_a{}^4\mathbf{F}^{ab}\equiv\nabla_a{}^4\mathbf{F}^{ab}+[{}^4\mathbf{A}_a,{}^4\mathbf{F}^{ab}]=0\ , \tag{3.2}$$

where, as before, ∇ is the unique torsion-free derivative operator compatible with the metric g^{ab}. Variations with respect to the Dirac fields yield

$$\begin{aligned}
&\tilde{\sigma}^a{}_{AA'}{}^4\mathcal{D}_a\xi^A-\frac{im}{\sqrt{2}}({}^4\sigma)\eta_{A'}=0\ ,\\
&{}^4\mathcal{D}_a(\tilde{\sigma}^a{}_{AA'}\eta^{A'})-\frac{im}{\sqrt{2}}({}^4\sigma)\xi_A=0\ ,\\
&{}^4\mathcal{D}_a(\tilde{\sigma}^a{}_{AA'}\bar{\xi}^{A'})-\frac{im}{\sqrt{2}}({}^4\sigma)\bar{\eta}_A=0\ ,\\
&\tilde{\sigma}^a{}_{AA'}{}^4\mathcal{D}_a\bar{\eta}^A-\frac{im}{\sqrt{2}}({}^4\sigma)\bar{\xi}_{A'}=0\ .
\end{aligned} \tag{3.3}$$

To compare these equations with the standard ones, let us express ${}^4\mathcal{D}$ in terms of ∇ and the spinor fields using the equation of ${}^4\mathcal{D}$ obtained in the previous section. We then obtain

$$\sigma^a{}_{AA'}\left[\nabla_a - \frac{3i}{8}k_a\right]\xi^A = \frac{im}{\sqrt{2}}\eta_{A'}\ ,$$
$$\sigma^a{}_{AA'}\left[\nabla_a + \frac{3i}{8}k_a\right]\eta^{A'} = \frac{im}{\sqrt{2}}\xi_A\ , \tag{3.4}$$

and the complex conjugate equations for $\bar{\xi}^{A'}$ and $\bar{\eta}^{A}$, where, as before, $k_a = -i\sqrt{2}\,\sigma_a^{AA'}(\bar{\xi}_{A'}\xi_A - \bar{\eta}_A\eta_{A'})$. These are *not* the standard minimally coupled spin-$\frac{1}{2}$ equations owing to the presence of terms involving k_a. Since these terms are cubic in spin-$\frac{1}{2}$ fields, (3.4) are not even linear in Dirac fields. We will return to this point at the end of this section.

To obtain the gravitational field equations, we have to vary S_T with respect to $\sigma^a{}_{AA'}$. Since each piece in S_T depends on $\sigma^a{}_{AA'}$, let us carry out the variation term by term. We have

$$\frac{\delta S_E}{\delta\sigma^a{}_{AA'}} = -({}^4\sigma)(2\sigma^b{}_B{}^{A'}\,{}^4F_{ab}{}^{AB} + \sigma^d{}_D{}^{D'}\sigma^b{}_{BD'}\,{}^4F_{db}{}^{DB}\sigma_a{}^{AA'})\ , \tag{3.5a}$$

$$\frac{\delta S_C}{\delta\sigma^a{}_{AA'}} = -\Lambda({}^4\sigma)\sigma_a{}^{AA'}\ , \tag{3.5b}$$

$$\frac{\delta S_{\rm KG}}{\delta\sigma^a{}_{AA'}} = 8\pi({}^4\sigma)[\sigma^{bAA'}\partial_a\phi\partial_b\phi - \tfrac{1}{2}\sigma_a{}^{AA'}(g^{bd}\partial_b\phi\partial_d\phi + \mu^2\phi^2)]\ , \tag{3.5c}$$

$$\frac{\delta S_{\rm YM}}{\delta\sigma^a{}_{AA'}} = 2({}^4\sigma)\mathrm{tr}(\sigma^{bAA'}g^{cd}\mathrm{tr}\,{}^4\mathbf{F}_{ac}\,{}^4\mathbf{F}_{bd} - \tfrac{1}{4}\sigma_a{}^{AA'}g^{cd}g^{mn}\mathrm{tr}\,{}^4\mathbf{F}_{cm}\,{}^4\mathbf{F}_{dn})\ , \tag{3.5d}$$

$$\frac{\delta S_D}{\delta\sigma^a{}_{AA'}} = \sqrt{2}({}^4\sigma)\left[\bar{\xi}^{A'}\,{}^4\mathcal{D}_a\xi^A - ({}^4\mathcal{D}_a\bar{\eta}^A)\eta^{A'} - \frac{i}{2\sqrt{2}}\sigma_a{}^{AA'}\nabla_b k^b\right]. \tag{3.5e}$$

[Here, in the last equation, we have used the equation of motion (3.4) for spin-$\frac{1}{2}$ fields.] Thus, the Euler-Lagrange equations for $\sigma^a{}_A{}^{A'}$ say that the sum of the right-hand sides of these equations should vanish. The numerical factors in front of the matter terms have been chosen to agree with the conventions for stress energy in the literature.[15] Set

$$H_{ab} = -({}^4\sigma)^{-1}\sigma_{bAA'}\frac{\delta S_E}{\delta\sigma^a{}_{AA'}} \tag{3.6a}$$

and

$$E_{ab}(\text{matter}) = [8\pi({}^4\sigma)]^{-1}\sigma_{bAA'}\frac{\delta S_{\text{matter}}}{\delta\sigma^a{}_{AA'}}\ . \tag{3.6b}$$

Note that $E_{ab}(\mathrm{KG})$ and $E_{ab}(\mathrm{YM})$ are the standard stress-energy tensors of the Klein-Gordon and Yang-Mills fields. The field equation for $\sigma^a{}_A{}^{A'}$ now becomes

$$H_{ab} + \Lambda g_{ab} = 8\pi E_{ab}(\text{matter})\ . \tag{3.7}$$

Let us now focus on the gravity-spin-$\frac{1}{2}$ part, $S_E + S_D$, of the total action S_T. We can simplify (3.7) further by substituting for ${}^4\mathcal{D}$ and ${}^4F_{abA}{}^B$ in terms of $\sigma^a{}_A{}^{A'}$ and Dirac fields. We have

$$H_{ab} = G_{ab} - \tfrac{1}{16}g_{ab}(k^ck_c + 8i\nabla_c k^c) + \frac{i}{2}\nabla_a k_b + \tfrac{1}{4}\epsilon_{ab}{}^{cd}\nabla_c k_d - \tfrac{1}{8}k_a k_b \tag{3.8a}$$

and

$$8\pi E_{ab}(D) = \sqrt{2}\,\sigma_{bAA'}[\bar{\xi}^{A'}\nabla_a\xi^A - (\nabla_a\bar{\eta}^A)\eta^{A'}] + \tfrac{1}{8}g_{ab}(k^ck_c - 4i\nabla_c k^c) - \tfrac{1}{8}k_a k_b\ , \tag{3.8b}$$

where G_{ab} is the Einstein tensor of the metric g_{ab}. Hence, if we had only the Dirac field as the matter source, the field equation for g^{ab} would have been

$$G_{ab} = \frac{1}{\sqrt{2}}\sigma_{bAA'}[\bar{\xi}^{A'}\nabla_a\xi^A - (\nabla_a\bar{\xi}^{A'})\xi^A + \bar{\eta}^A\nabla_a\eta^{A'} - (\nabla_a\bar{\eta}^A)\eta^{A'}] + \tfrac{3}{16}g_{ab}k^ck_c - \tfrac{1}{4}\epsilon_{ab}{}^{cd}\nabla_c k_d\ , \tag{3.9}$$

where we have expressed the term $(i/2)\nabla_a k_b$ in terms of Dirac fields. We see that, as expected from our discussion in the previous section, the right-hand side is manifestly real. Unfortunately, it is not manifestly symmetric. However, since the equations of motion obtained from S_T are the same as those obtained from the reduced action $\underline{S}_T$ of (2.6), modulo equations of motion for matter fields, (3.9) is just the result of setting the variation of $\underline{S}_T$ with respect to g^{ab} equal to zero. Thus, equations of motion for matter ensure that the right-hand side of (3.9) is in fact symmetric in indices a and b.

Since the equations of motion (3.4) for spin-$\frac{1}{2}$ fields contain a cubic term, and since the effective stress-energy term for spin-$\frac{1}{2}$ fields [i.e., the right-hand side of (3.9)] contains a quartic term in these fields, the equations of motion obtained above are *not* the standard Einstein-Dirac equations. This is not surprising since we are using a first-order formalism and since the usual Palatini first-order formalism leads to Einstein-Cartan theory rather than the standard Einstein-Dirac theory. In the Palatini framework one can just add a quartic term to the action to recover the Einstein-Dirac equations. What is the situation in the present case? Let us return to the reduced action $\underline{S}_T$ of (2.6). $\underline{S}_T$ contains precisely one term, $+\frac{3}{16}k^ak_a$, involving Dirac fields, which fails to be quadratic in these fields. Therefore, one would expect that the removal of this term from S_T would lead us to the Einstein-Dirac theory. This expectation is correct. Set

$$S_T' = S_T - \tfrac{3}{16}\int d^4x\,({}^4\sigma)k^ak_a\ . \tag{3.10}$$

Since the added term does not involve ${}^4A_{aA}{}^B$, the solution (2.5) for the equation of motion of ${}^4A_{aA}{}^B$ remains unaffected. Therefore, $\underline{S}'_T$ is obtained from $\underline{S}_T$ simply by removing quartic terms, $k^a k_a$, from $\underline{S}_T$. The new reduced action $\underline{S}'_T$ is precisely the usual Einstein-Dirac action. The new equations of motion for spin-$\frac{1}{2}$ fields are simply

$$\sigma^a{}_{AA'}\nabla_a\xi^A=\frac{im}{\sqrt{2}}\eta_{A'} \text{ and } \sigma^a{}_{AA'}\nabla_a\eta^{A'}=\frac{im}{\sqrt{2}}\xi_A \qquad (3.4')$$

and their complex conjugates, while the new equations for g^{ab} are

$$G_{ab}+\Lambda g_{ab}=\frac{1}{\sqrt{2}}\sigma_{bAA'}[\bar{\xi}^{A'}\nabla_a\xi^A-(\nabla_a\bar{\xi}^{A'})\xi^A+\bar{\eta}^A\nabla_a\eta^{A'}-(\nabla_a\bar{\eta}^A)\eta^{A'}]-\tfrac{1}{4}\epsilon_{ab}{}^{cd}\nabla_c k_d+8\pi E_{ab}(\mathrm{KG})+8\pi E_{ab}(\mathrm{YM}) , \qquad (3.9')$$

where, as before, $E_{ab}(\mathrm{KG})$ and $E_{ab}(\mathrm{YM})$ are the standard[15] stress-energy tensors for the Klein-Gordon and Yang-Mills fields. Thus, if we had used S'_T as our total action, we would have obtained the standard equations of motion for all fields, including, in particular, spin-$\frac{1}{2}$ fields. Note however, that it is only the action S_T that admits a direct extension to supergravity.[16]

A number of remarks are in order.

(i) As we have seen, the action S_E+S_D leads to the Einstein-Cartan theory. Why then did we not encounter torsion terms in the derivative operator? It is because we have formulated the theory using *only* the self-dual connection ${}^4\mathcal{D}$ which does not act on tensors. Nonetheless, our description is complete and there is no compelling reason to extend its action to tensors or unprimed spinors. However, let us suppose that we do extend the action by requiring that it be real and that it annihilate the space-time metric g_{ab}. Then, the resulting connection does admit torsion. The torsion tensor is, however, governed entirely by the spinor fields: it is given by $T_{ab}{}^c=\frac{1}{2}\epsilon_{mab}{}^c k^m$, where ϵ_{abcd} is the unique alternating tensor determined by the metric g_{ab}, and k^a (as before) is the vector field determined by the soldering form and the spinor fields.

(ii) Since it it is often convenient to let the spinor fields be Grassmann valued while performing path integrations, we have tailored our discussion of action to allow for this possibility. Thus, all our results up to this point hold irrespective of whether spinor fields take their values in complexes or Grassmannians. In the latter case, however, one has to follow consistently the ordering we have chosen for Dirac fields.

(iii) For simplicity we have set $G=1$ in this paper. However, since factors of G are reshuffled in the passage to new variables, let us discuss briefly how G would have entered various expressions had it not been set equal to one. In this framework G plays the role of the Yang-Mills coupling constant. (However, since it is dimensionful, dimensions of ${}^4A_{aA}{}^B$ and ${}^4F_{abA}{}^B$ in the gravitational case are different from those in the Yang-Mills theory.) Thus, ${}^4\mathcal{D}$ has the expression ${}^4\mathcal{D}_a\xi_A=\partial_a\xi_A+G\,{}^4A_{aA}{}^B\xi_B$ and ${}^4F_{abA}{}^B$ is given by ${}^4F_{abA}{}^B=2\partial_{[a}{}^4A_{b]A}{}^B+G[{}^4A_a,{}^4A_b]_A{}^B$. As a result, ${}^4F_{abA}{}^B$ has the dimensions of Lagrangian density. Since $\sigma^a{}_A{}^{A'}$ is dimensionless, the expression of the gravitational action does not have a multiplicative factor of G. Since matter fields have conventional dimensions there is no factor of G whatsoever in the matter action, except through ${}^4\mathcal{D}$ (which appears only in the Dirac Lagrangian). Thus, the only term in the action which has an explicit G dependence is the cosmological term, which has to be multiplied by an overall factor of $1/G$. In the absence of a cosmological-constant term the only G dependence in the entire action comes through the expressions of ${}^4F_{abA}{}^B$ (in S_E) and ${}^4\mathcal{D}$ (in S_D) in terms of ${}^4A_{aA}{}^B$.

(iv) Why is there no surface term in the expression of the gravitational action S_E? It is because we are using a first-order formalism. Thus, in the expression of S_E, we have treated $\sigma^a{}_A{}^{A'}$ and ${}^4A_{aA}{}^B$ as independent variables. In particular, while deriving the equation of motion for ${}^4A_{aA}{}^B$ we keep ${}^4A_{aA}{}^B$ fixed on the boundary (and $\sigma^a{}_A{}^{A'}$ fixed throughout the volume). Therefore, there is no need to add a surface term to obtain the correct equations of motion from variations. In the second-order formalism a surface term is essential: since the soldering form $\sigma^a{}_A{}^{A'}$ is the only dynamical variable in this case, one is allowed to keep only $\sigma^a{}_A{}^{A'}$—rather than both $\sigma^a{}_A{}^{A'}$ and its derivatives—fixed on the boundary while performing the variations. How can we then reconcile the fact that, on solving for ${}^4A_{aA}{}^B$, we obtained a reduced second-order action $\underline{S}_T$ without any surface term? The answer is that the reduction procedure comes with a prescription on how to do variations. For paths which lie entirely in the solution set of $\delta S_T/\delta^4A_a=0$, fixing $\sigma^a{}_A{}^{A'}$ on the boundary automatically fixes certain derivatives of $\sigma^a{}_A{}^{A'}$ on the boundary as well, so that a surface term is not needed in the action for the variation to give the correct equation of motion.

(v) We shall see in the next three sections that, in the Hamiltonian description, the reality conditions, the constraints, and the Hamiltonians are all polynomial in the basic canonical variables. Our Lagrangian, on the other hand, depends nonpolynomially on $\sigma^a{}_A{}^{A'}$ through the determinant (${}^4\sigma$) of the inverse of $\sigma^a{}_A{}^{A'}$. Samuel has pointed out that in the source-free case, this situation can be remedied by using ${}^4\sigma_{aA}{}^{A'}$ as the basic variable in place of $\sigma^a{}_A{}^{A'}$. (See, e.g., pp. 88 and 89 of Ref. 1.) If we include matter fields, on the other hand, this strategy does not work since the matter Lagrangian explicitly contains the contravariant space-time metric. Is there an alternative strategy?

IV. 3+1 DECOMPOSITION

In this section we introduce a foliation in the space-time manifold $\mathcal{M}$ and carry out a $3+1$ decomposition of the action to pass on to the Hamiltonian framework.

Let us introduce on $\mathcal{M}$ a smooth function t whose gradient is nowhere vanishing and whose level surfaces Σ_t are each diffeomorphic to Σ. Let t^a be a smooth vector field on $\mathcal{M}$ with an affine parameter t, i.e., which satisfies

$t^a\nabla_a t=1$. Given a soldering form $\sigma^a{}_{AA'}$ on $\mathcal{M}$ with respect to which each level surface Σ_t is spacelike, denote by n^a the future-directed, unit, timelike vector field everywhere orthogonal to Σ_t. Denote the induced, positive-definite metric on Σ_t by q_{ab} $(=g_{ab}+n_a n_b)$, and obtain the lapse and shift fields N and N^a by projecting t^a into and orthogonal to Σ_t; $t^a=Nn^a+N^a$. Using $i\sqrt{2}\,n^a\sigma_{aAA'}(:=i\sqrt{2}\,n_{AA'}:=G_{AA'})$ as the Hermitian metric for SL(2,$\mathbb{C}$) spinors, we can identify unprimed SL(2,$\mathbb{C}$) spinors on $\mathcal{M}$ with SU(2) spinors on Σ_t and, at the same time, introduce a dagger operation on these spinors: we set $G_A{}^{A'}\bar{\xi}_{A'}=(\xi^\dagger)_A$, the Hermitian conjugate of ξ_A. Using this identification we can now introduce a soldering form $\sigma^a{}_A{}^B$ on SU(2) spinors:

$$\sigma^a{}_{AB}:=-i\sqrt{2}\,\sigma^a{}_{(A}{}^{A'}n_{B)A'}\ . \tag{4.1a}$$

This form is automatically Hermitian, $(\sigma^a{}_{AB})^\dagger=\sigma^a{}_{AB}$, and trace-free, $\sigma^a{}_A{}^A=0$. It is also nondegenerate; it defines an isomorphism between the space of second-rank, trace-free Hermitian spinors ξ_{AB} at any point of Σ_t and the tangent space to Σ_t at that point. Finally, it serves as the square root of the three-metric q_{ab} on Σ_t: $q^{ab}=-\mathrm{tr}\sigma^a\sigma^b$. We can "invert" (4.1a) to express $\sigma^a{}_A{}^{A'}$ in terms of $\sigma^a{}_A{}^B$ and $n^{AA'}$:

$$\sigma^a{}_{AA'}=-i\sqrt{2}\,\sigma^a{}_{AB}n^B{}_{A'}-n^a n_{AA'}\ . \tag{4.1b}$$

Finally, we note a useful identity

$$n_{AA'}n_B{}^{A'}=-\tfrac{1}{2}\epsilon_{AB} \tag{4.2}$$

which, in particular, implies that $\sigma^a{}_{AB}$ defined by (4.1a) is already projected into the three-surfaces Σ_t.

We can now obtain a 3+1 decomposition of the action S_T. Let us begin with the gravitational part S_E. Using (4.1b) in the gravitational Lagrangian density $\mathcal{L}_E$ and substituting $N^{-1}(t^a-N^a)$ for n^a we get

$$\begin{aligned}({}^4\sigma)\sigma^a{}_A{}^{A'}\sigma^b{}_{BA'}{}^4F_{ab}{}^{AB}&=({}^4\sigma)\mathrm{tr}(-i\sqrt{2}\,n^a\sigma^b\,{}^4F_{ab}-\sigma^a\sigma^b\,{}^4F_{ab})\\&=({}^4\sigma)\mathrm{tr}\{-i\sqrt{2}\,N^{-1}\sigma^b[\mathcal{L}_t{}^4A_b-{}^4\mathcal{D}_b({}^4A\cdot t)]-\sigma^a\sigma^b\,{}^4F_{ab}+i\sqrt{2}\,N^{-1}N^a\sigma^b\,{}^4F_{ab}\}\ ,\end{aligned} \tag{4.3a}$$

where $({}^4A\cdot t)$ denotes $t^a\,{}^4A_{aA}{}^B$; $\mathcal{L}_t{}^4A_b$ is the Lie derivative of ${}^4A_{aA}{}^B$ where internal indices are treated as if they were scalars; $\mathcal{D}_a:=q_a{}^b\,{}^4\mathcal{D}_b$ is the pullback of ${}^4\mathcal{D}$ to the three-surface; and where we have used the identity $t^a\,{}^4F_{ab}=\mathcal{L}_t{}^4A_b-{}^4\mathcal{D}_b({}^4A\cdot t)$. To further simplify this expression we define

$$\tilde{\sigma}^a{}_A{}^B:=(\sigma)\sigma^a{}_A{}^B\ \text{ and }\ \underset{\sim}{N}:=(\sigma)^{-1}N\ , \tag{4.4}$$

where (σ) denotes the inverse of $\det(\sigma^a{}_A{}^B)$. Then, in terms of $g\equiv\det(g_{ab})$ and $q\equiv\det(q_{ab})$ we have $({}^4\sigma)=\sqrt{-g}=N\sqrt{q}=N(\sigma)$. (Note that, by convention, a tilde over a tensor denotes a density of weight one and a tilde below a tensor denotes a density of weight minus one.) Using these fields we obtain

$$\begin{aligned}({}^4\sigma)\sigma^a{}_A{}^{A'}\sigma^b{}_{BA'}{}^4F_{ab}{}^{AB}&\\=\mathrm{tr}[-i\sqrt{2}\,\tilde{\sigma}^b\mathcal{L}_t{}^4A_b&+i\sqrt{2}\,\tilde{\sigma}^b\mathcal{D}_b({}^4A\cdot t)\\&+i\sqrt{2}\,N^a\tilde{\sigma}^b\,{}^4F_{ab}-\underset{\sim}{N}\tilde{\sigma}^a\tilde{\sigma}^b\,{}^4F_{ab}]\ .\end{aligned} \tag{4.3b}$$

Finally, we introduce pullbacks, $A_{aA}{}^B$ and $F_{abA}{}^B$, to Σ_t of ${}^4A_{aA}{}^B$ and ${}^4F_{abA}{}^B$ respectively; $A_{aA}{}^B:=q_a{}^{b\,4}A_{bA}{}^B$ and $F_{abA}{}^B:=q_a{}^c q_b{}^d\,{}^4F_{cdA}{}^B$. Then, (4.3) can be expressed in terms only of "three-dimensional" fields:

$$S_E=\int dt\int_{\Sigma_t}d^3x\ \mathrm{tr}[-i\sqrt{2}\,\tilde{\sigma}^b\mathcal{L}_tA_b+i\sqrt{2}\,\tilde{\sigma}^b\mathcal{D}_b({}^4A\cdot t)+i\sqrt{2}\,N^a\tilde{\sigma}^bF_{ab}-\underset{\sim}{N}\tilde{\sigma}^a\tilde{\sigma}^bF_{ab}]\ , \tag{4.5}$$

where we have used the identity $\mathcal{L}_tq_a{}^b=0$ to write $\mathrm{tr}(\tilde{\sigma}^b\mathcal{L}_t{}^4A_b)$ as $\mathrm{tr}(\tilde{\sigma}^b\mathcal{L}_tA_b)$ in the first term. This is the expression we were seeking. It is clear from its form that if we use $A_{aA}{}^B$ as the "configuration variable," apart from a numerical factor, $\tilde{\sigma}^a{}_A{}^B$ is the canonically conjugate momentum. Note also that time derivatives of ${}^4A\cdot t$, N^a, and $\underset{\sim}{N}$ do not appear in the action. We shall see that this continues to be the case even when the matter contributions to the action are included. Therefore, they will play the role of Lagrange multipliers in the theory. Their variations will lead to the constraint equations. From (4.5) one can also read off the gravitational contributions to the constraint equations. We can repeat the above procedure for matter fields. Let us begin with the Dirac action S_D. As in the gravitational case we have to replace spinor fields ($\bar{\xi}^{A'}$ and $\eta^{A'}$) with primed indices by the corresponding Hermitian conjugate fields:

$$(\xi^\dagger)^A=-\bar{\xi}^{A'}G^A{}_{A'}\ \text{ and }\ (\bar{\eta}^\dagger)^A=-\eta^{A'}G^A{}_{A'}\ . \tag{4.6}$$

Using (4.1b) and (4.6) and substituting $N^{-1}(t^a-N^a)$ for n^a we obtain

$$\begin{aligned}({}^4\sigma)\sigma^a{}_{AA'}\bar{\xi}^{A'}\,{}^4\mathcal{D}_a\xi^A&\\=-\underset{\sim}{N}(\sigma)\tilde{\sigma}^a{}_A{}^B(\xi^\dagger)_B\mathcal{D}_a\xi^A&-\frac{i}{\sqrt{2}}(\sigma)(\xi^\dagger)_A\mathcal{L}_t\xi^A\\&+\frac{i}{\sqrt{2}}(\sigma)({}^4A\cdot t)_B{}^A(\xi^\dagger)_A\xi^B\\&+\frac{i}{\sqrt{2}}(\sigma)N^a(\xi^\dagger)_A\mathcal{D}_a\xi^A\ ,\end{aligned} \tag{4.7}$$

where, as before, the Lie derivatives treat internal indices as scalars and $\mathcal{D}_a:=q_a{}^b\,{}^4\mathcal{D}_b$. Transcribing this result for $({}^4\sigma)\sigma^a{}_{AA'}\eta^{A'}\,{}^4\mathcal{D}_a\bar{\eta}^A$ and using the identity $\bar{\xi}^{A'}\eta_{A'}=(\xi^\dagger)^A(\bar{\eta}^\dagger)_A$ we obtain

$$S_D=\int dt\int_{\Sigma_t} d^3x\{-\sqrt{2}\,\underset{\sim}{N}(\sigma)\tilde{\sigma}^{a}{}_{A}{}^{B}[(\xi^\dagger)_B\mathcal{D}_a\xi^A-(\bar{\eta}^\dagger)_B\mathcal{D}_a\bar{\eta}^A]-i(\sigma)[(\xi^\dagger)_A\mathcal{L}_t\xi^A-(\bar{\eta}^\dagger)_A\mathcal{L}_t\bar{\eta}^A]$$
$$+i(\sigma)({}^4A\cdot t)_B{}^A[(\xi^\dagger)_A\xi^B-(\bar{\eta}^\dagger)_A\bar{\eta}^B]+i(\sigma)N^a[(\xi^\dagger)_A\mathcal{D}_a\xi^A-(\bar{\eta}^\dagger)_A\mathcal{D}_a\bar{\eta}^A]$$
$$+\underset{\sim}{N}(\sigma)^2 im[\xi^A\bar{\eta}_A-(\xi^\dagger)^A(\bar{\eta}^\dagger)_A]\}\ . \tag{4.8}$$

This is the desired form of the action. Again, apart from numerical factors, we can pick out $[\xi^A,(\sigma)(\xi_A)^\dagger]$ and $[\bar{\eta}^A,(\sigma)(\bar{\eta}_A)^\dagger]$ as the canonically conjugate pairs and read off the contribution of Dirac fields to the constraint equations.

The treatment of the contributions due to the cosmological constant, the Klein-Gordon and the Yang-Mills fields is straightforward. The final $3+1$ forms of these terms are

$$S_C=\int dt\int_{\Sigma_t} d^3x\ \underset{\sim}{N}(\sigma)^2\Lambda\ , \tag{4.9}$$

$$S_{\rm KG}=\int dt\int_{\Sigma_t} d^3x\,(4\pi)[-\underset{\sim}{N}\,{\rm tr}(\tilde{\sigma}^a\tilde{\sigma}^b)\partial_a\phi\partial_b\phi-\underset{\sim}{N}^{-1}(\mathcal{L}_t\phi-N^a\partial_a\phi)^2+\underset{\sim}{N}(\sigma)^2m^2\phi^2]\ , \tag{4.10}$$

$$S_{\rm YM}=\int dt\int_{\Sigma_t} d^3x\left[\underset{\sim}{N}^{-1}(\sigma)^{-2}{\rm tr}(\tilde{\sigma}^a\tilde{\sigma}^b){\rm tr}\left[\mathcal{L}_t\,\mathbf{A}_a-\mathbf{D}_a({}^4\mathbf{A}\cdot t)-\frac{N^m}{2}\mathbf{B}_{ma}\right]\left[\mathcal{L}_t\,\mathbf{A}_b-\mathbf{D}_b({}^4\mathbf{A}\cdot t)-\frac{N^n}{2}\mathbf{B}_{nb}\right]\right.$$
$$\left.+\tfrac{1}{8}\underset{\sim}{N}(\sigma)^{-2}{\rm tr}(\tilde{\sigma}^a\tilde{\sigma}^c){\rm tr}(\tilde{\sigma}^b\tilde{\sigma}^d){\rm tr}\mathbf{B}_{ab}\mathbf{B}_{cd}\right]\ , \tag{4.11}$$

where $\mathbf{A}_a:=q_a{}^{b\,4}\mathbf{A}_b$, $\mathbf{D}_a:=q_a{}^{b\,4}\mathbf{D}_b$, and $\mathbf{B}_{ab}:=2q_a{}^cq_b{}^{d\,4}\mathbf{F}_{cd}$. ($\mathbf{B}_{ab}$ is the dual of the magnetic field of $\mathbf{A}_a$.) We can now collect the phase-space variables. They are all represented by fields defined intrinsically on the spacelike three-manifolds Σ_t. We choose our "configuration variables" to be $A_{aA}{}^B$, ϕ, ξ^A, $\bar{\eta}^A$, and $\mathbf{A}_a$. Then, from the expression of the Lagrangian, it follows that their canonically conjugate momenta are given, respectively, by $-i\sqrt{2}\tilde{\sigma}^a{}_A{}^B$, $\tilde{\pi}:=-(8\pi)\underset{\sim}{N}^{-1}(\mathcal{L}_t\phi-\mathcal{L}_N\phi)$, $\tilde{\pi}_A:=-i(\sigma)(\xi^\dagger)_A$, $\tilde{\omega}_A:=i(\sigma)(\bar{\eta}^\dagger)_A$, and

$$\tilde{\mathbf{E}}^a:=-2\underset{\sim}{N}^{-1}(\sigma)^{-2}({\rm tr}\tilde{\sigma}^a\tilde{\sigma}^b)$$
$$\times(\mathcal{L}_t\,\mathbf{A}_b-\mathbf{D}_b({}^4\mathbf{A}\cdot t)-\tfrac{1}{2}N^m\mathbf{B}_{mb})\ .$$

(If Σ is noncompact, these canonically conjugate fields are subject to certain asymptotic fall-off conditions which we will specify in the next section.) Note that the Lagrangian also contains other variables: $\underset{\sim}{N}$, N^a, ${}^4A\cdot t$, and ${}^4\mathbf{A}\cdot t$. However, since the time derivatives of these variables never occur, they play the role of Lagrange multipliers. They do not obey any dynamical equations of motion and we can fix their values by a gauge choice. We now carry out variation of the action with respect to these variables. The variation with respect to $\underset{\sim}{N}$ yields

$$\frac{\delta S_E}{\delta \underset{\sim}{N}}=-{\rm tr}(\tilde{\sigma}^a\tilde{\sigma}^bF_{ab}),\quad \frac{\delta S_C}{\delta\underset{\sim}{N}}=(\sigma)^2\Lambda\ ,$$

$$\frac{\delta S_{\rm KG}}{\delta\underset{\sim}{N}}=-4\pi\,{\rm tr}(\tilde{\sigma}^a\tilde{\sigma}^b)\partial_a\phi\partial_b\phi+\frac{1}{16\pi}\tilde{\pi}^2+4\pi(\sigma)^2\mu^2\phi^2\ ,$$

$$\frac{\delta S_D}{\delta\underset{\sim}{N}}=-i\sqrt{2}\,\tilde{\sigma}^a{}_A{}^B(\tilde{\pi}_B\mathcal{D}_a\xi^A+\tilde{\omega}_B\mathcal{D}_a\bar{\eta}^A)$$
$$+(\sigma)^2im\,\xi^A\bar{\eta}_A-im\,\tilde{\pi}^A\tilde{\omega}_A\ ,$$

$$\frac{\delta S_{\rm YM}}{\delta\underset{\sim}{N}}=\tfrac{1}{8}(\sigma)^{-2}{\rm tr}(\tilde{\sigma}^a\tilde{\sigma}^c){\rm tr}(\tilde{\sigma}^b\tilde{\sigma}^d){\rm tr}(\mathbf{E}_{ab}\mathbf{E}_{cd}+\mathbf{B}_{ab}\mathbf{B}_{cd})\ ,$$

where $\mathbf{E}_{ab}$ is the dual of the Yang-Mills electric field. The variation with respect to N^a yields

$$\frac{\delta S_E}{\delta N^a}=i\sqrt{2}\,{\rm tr}(\tilde{\sigma}^bF_{ab}),\quad\frac{\delta S_C}{\delta N^a}=0\ ,$$

$$\frac{\delta S_{\rm KG}}{\delta N^a}=-\tilde{\pi}\partial_a\phi\ ,$$

$$\frac{\delta S_D}{\delta N^a}=-(\tilde{\pi}_A\mathcal{D}_a\xi^A+\tilde{\omega}_A\mathcal{D}_a\bar{\eta}^A)\ ,$$

$$\frac{\delta S_{\rm YM}}{\delta N^a}=-\tfrac{1}{2}\underset{\sim}{\eta}_{abc}{\rm tr}(\tilde{\mathbf{E}}^b\tilde{\mathbf{B}}^c)\ \ [=-{\rm tr}(\tilde{\mathbf{E}}^b\mathbf{F}_{ab})]\ ,$$

where $\underset{\sim}{\eta}_{abc}$ is the (c-number) Levi-Civita form-density. Next, we carry out the variation with respect to ${}^4A\cdot t$. The only nonzero terms are

$$\frac{\delta S_E}{\delta({}^4A\cdot t)_A{}^B}=-i\sqrt{2}\,\mathcal{D}_b\tilde{\sigma}^b{}_B{}^A\ ,$$

$$\frac{\delta S_D}{\delta({}^4A\cdot t)_A{}^B}=-\tfrac{1}{2}(\tilde{\pi}_B\xi^A+\tilde{\pi}^A\xi_B+\tilde{\omega}_B\bar{\eta}^A+\tilde{\omega}^A\bar{\eta}_B)\ .$$

Finally, the only nonvanishing variation with respect to ${}^4\mathbf{A}\cdot t$ comes from the Yang-Mills action:

$$\frac{\delta S_{\rm YM}}{\delta({}^4\mathbf{A}\cdot t)}=\mathbf{D}_b\tilde{\mathbf{E}}^b\ .$$

Summing each of the above variations and setting the results equal to zero, we obtain four constraint equations:

$$\tilde{\tilde{C}}(\tilde{\sigma},A;\tilde{\pi},\tilde{\omega},\xi,\bar{\eta};\tilde{E},\mathbf{A}):=\frac{\delta S_T}{\delta\underset{\sim}{N}}=0\ , \tag{4.12a}$$

$$\tilde{C}_a(\tilde{\sigma},A;\tilde{\pi},\tilde{\omega},\xi,\bar{\eta};\tilde{E},\mathbf{A}):=\frac{\delta S_T}{\delta N^a}=0\ , \tag{4.12b}$$

$$\tilde{C}_A{}^B(\tilde{\sigma},A;\tilde{\pi},\tilde{\omega},\xi,\bar{\eta};\tilde{E},\mathbf{A}):=\frac{\delta S_T}{\delta({}^4A\cdot t)_B{}^A}=0\ , \tag{4.12c}$$

$$\tilde{\mathbf{C}}(\tilde{\sigma},A;\tilde{\pi},\tilde{\omega},\xi,\bar{\eta};\tilde{E},\mathbf{A}):=\frac{\delta S_T}{\delta({}^4\mathbf{A}\cdot t)}=0\ , \tag{4.12d}$$

TABLE I. Gravitational and matter contributions to constraints.

Constraint	Lagrange multiplier	Gravity	Dirac	Klein-Gordon	Yang-Mills	C.C.
$\tilde{C}_{AB}$	$t\cdot{}^4A^{AB}$	$i\sqrt{2}\mathcal{D}_b\tilde{\sigma}^b{}_{AB}$	$\tilde{\pi}_{(A}\xi_{B)}+\tilde{\omega}_{(A}\bar{\eta}_{B)}$	0	0	0
$\tilde{C}_a$	N^a	$i\sqrt{2}\,\mathrm{tr}(\tilde{\sigma}^bF_{ab})$	$-(\tilde{\pi}_A\mathcal{D}_a\xi^A+\tilde{\omega}_A\mathcal{D}_a\bar{\eta}^A)$	$-\tilde{\pi}\partial_a\phi$	$-\frac{1}{2}\underset{\sim}{\eta}_{abc}\mathrm{tr}(\tilde{\mathbf{E}}^b\tilde{\mathbf{B}}^c)$	0
$\tilde{\tilde{C}}$	$\underset{\sim}{N}$	$-\mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^bF_{ab})$	$-i\sqrt{2}\,\tilde{\sigma}^a{}_A{}^B(\tilde{\pi}_B\mathcal{D}_a\xi^A+\tilde{\omega}_B\mathcal{D}_a\bar{\eta}^A)$ $+im(\sigma^2\xi^A\bar{\eta}_A-\tilde{\pi}^A\tilde{\omega}_A)$	$-4\pi\,\mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b)\partial_a\phi\partial_b\phi$ $+\frac{(\tilde{\pi})^2}{16\pi}+4\pi\sigma^2\mu^2\phi^2$	$\frac{1}{8(\sigma^2)}\mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^c)\mathrm{tr}(\tilde{\sigma}^b\tilde{\sigma}^d)$ $\times\mathrm{tr}(\mathbf{E}_{ab}\mathbf{E}_{cd}+\mathbf{B}_{ab}\mathbf{B}_{cd})$	$\sigma^2\Lambda$
$\tilde{\mathbf{C}}$	$t\cdot{}^4\mathbf{A}$	0	0	0	$\mathbf{D}_m\tilde{\mathbf{E}}^m$	0

$\tilde{\tilde{C}}=0$ is called the *scalar* constraint, $\tilde{C}_a=0$ is called the *vector* constraint, and $\tilde{C}_A{}^B=0$ ($\tilde{\mathbf{C}}=0$) is called the Einstein (Yang-Mills) *Gauss-law* constraint. For the convenience of the reader we have collected the contributions to these constraints from the various parts of the action in Table I. The expressions of the Gauss-law and vector constraints are manifestly polynomial in the basic canonical variables. In the expression of the scalar constraint, on the other hand, the Yang-Mills contribution, as written, fails to be polynomial due to the appearance of the multiplicative factor $(\sigma)^{-2}$. Fortunately, however, since

$$(\sigma)^2=q=-\frac{1}{3\sqrt{2}}\underset{\sim}{\eta}_{abc}\mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b\tilde{\sigma}^c) \tag{4.13}$$

is polynomial in $\tilde{\sigma}^a$, we can just multiply the scalar constraint equation by $(\sigma)^2$, thereby restoring the polynomial character of *all* constraint equations. [Note, incidently, that, had there been a relative multiplicative factor of (σ) or $(\sigma)^{-1}$ between matter terms, this procedure would have failed.] Thus, the presence of Yang-Mills fields leads us to scalar constraint with density weight *four.* Therefore, in the presence of Yang-Mills fields, a minor modification occurs in the discussion of dynamics: the lapse function must now be a density of weight minus three. In the absence of Yang-Mills fields we can proceed in the same manner as in the source-free case and continue to use a lapse with density weight minus one.

V. CONSTRAINT ALGEBRA

In this section we shall discuss the Poisson-brackets algebra generated by constraints. To compute these brackets in the case when Σ is noncompact, one needs to keep track of the precise boundary conditions satisfied by the fields since several integration by parts are involved. (See, e.g., Chap. II.2 in Ref. 1 for a general discussion of this issue.) Therefore, let us begin by specifying these conditions. In the case when Σ is compact, one can just ignore these conditions and the subsequent discussion of surface integrals.

For simplicity, let us suppose that Σ has only one asymptotic region, i.e., that the complement of a compact subset of Σ is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$. Let $\tilde{e}^a$ be a flat soldering form in this complement. Then, for reasons explained in Refs. 1 and 3, the gravitational variables $\tilde{\sigma}^a{}_A{}^B$ and $A_{aA}{}^B$ will be required to satisfy

$$\tilde{\sigma}^a=\left[1+\frac{M(\theta,\phi)}{r}\right]^2\tilde{e}^a+O(1/r^2)$$

and

$$\tilde{e}^aA_a=O(1/r^3)\,,$$

$$A_a+\tfrac{1}{3}(\mathrm{tr}A_me^m)e_a=O(1/r^2)\,,$$

where r is the radial coordinate defined by the flat metric $\mathring{q}_{ab}(=-\mathrm{tr}e_ae_b)$ and where the fall-off refers to the Cartesian components of fields in the chart defined by this flat metric. For matter fields we will simply require that all fields and their canonically conjugate momenta should fall off as $1/r^2$. This will in particular ensure that the stress energy will fall off as $(1/r^4)$. These are not the weakest conditions necessary for our framework. However, these are the simplest to work with. (We can, in particular, weaken the conditions on the Yang-Mills potential to allow a nonzero magnetic charge by a more subtle choice of conditions.) Note that these boundary conditions are adapted to asymptotically Minkowskian space-times. That is, we are considering the sector of the theory for which Minkowski space can be thought of as the "classical vacuum." *Thus, from now on, we will set the cosmological constant to zero.* In the spatially noncompact situation now under consideration, we could also have used conditions which refer to asymptotically anti–de Sitter space-times.[17] We expect that the main results of this and the following section will go through in that case as well.

The phase-space Γ will now consist of fields satisfying these boundary conditions (in addition to the obvious algebraic conditions on their indices). The symplectic structure on Γ is specified by the following fundamental (nonvanishing) Poisson brackets:

$$\begin{aligned}
&\{\tilde{\sigma}^a{}_{AB}(x),A_b{}^{CD}(y)\}=-\frac{i}{\sqrt{2}}\delta(x-y)\delta_b{}^a\delta_{(A}{}^C\delta_{B)}{}^D\,,\\
&\{\tilde{\pi}_A(x),\xi^B(y)\}=\delta(x-y)\delta_A{}^B\,,\\
&\{\tilde{\omega}_A(x),\bar{\eta}^B(y)\}=\delta(x-y)\delta_A{}^B\,,\\
&\{\tilde{\pi}(x),\phi(y)\}=\delta(x-y)\,,\\
&\{\tilde{\mathbf{E}}^a{}_{\mathbf{AB}}(x),\mathbf{A}_b{}^{\mathbf{CD}}(y)\}=\delta(x-y)\delta_b{}^a\delta_{\mathbf{AB}}{}^{\mathbf{CD}}\,,
\end{aligned}\tag{5.1}$$

where boldface uppercase latin indices (**A**,**B**, etc.) are the internal indices of Yang-Mills fields.

Next, we define constraint functionals by smearing the left-hand sides of constraint equations (4.12) with suitable fields. Let us set

$$C_{\underset{\sim}{N}}=i\sqrt{2}\int_{\Sigma}d^3x\,\underset{\sim}{N}\tilde{\tilde{C}}\ , \quad (5.2a)$$

$$\mathbb{C}_{\vec{N}}=-\int_{\Sigma}d^3x\,N^a\tilde{C}_a\ , \quad (5.2b)$$

$$C_{N,\mathbf{N}}=\int_{\Sigma}d^3x\,\mathrm{tr}(N\tilde{C}+\mathbf{N}\tilde{\mathbf{C}})\ , \quad (5.2c)$$

where $\underset{\sim}{N}$, $\vec{N}\equiv N^a$, $N_A{}^B$, and $\mathbf{N}_{\mathbf{A}}{}^{\mathbf{B}}$ are the smearing fields. Recall that, to qualify as generators of canonical transformations, functions on phase space must be differentiable. It turns out that, to ensure that the constraint functionals have this property, one must impose boundary conditions on the smearing fields as well. (See, e.g., Refs. 1 and 3.) It turns out that the appropriate condition is that they must go to zero at infinity [as $(1/r)$].

We now wish to find the canonical transformations that these constraint functionals generate. To do this we use the fundamental Poisson-brackets relations (5.1) and the properties

$$\{A,B\}=-\{B,A\}\ , \quad (5.3a)$$

$$\{A,B+\lambda C\}=\{A,B\}+\lambda\{A,C\}\ , \quad (5.3b)$$

$$\{A,BC\}=B\{A,C\}+\{A,B\}C\ , \quad (5.3c)$$

obeyed by all Poisson brackets.

Let us begin with the Gauss-law constraints. For conciseness we will treat the gravitational and the Yang-Mills Gauss laws simultaneously. The gravitational part can be recovered by setting $\mathbf{N}$ equal to zero in (5.2c) and the Yang-Mills part by setting $N=0$. An integration by parts of (5.2c) yields

$$\begin{aligned}C_{N,\mathbf{N}}=\int_{\Sigma_t}d^3x(&i\sqrt{2}\,\mathrm{tr}\{(\partial_bN+[A_b,N])\tilde{\sigma}^b\}\\&-N_A{}^B(\tilde{\pi}_B\xi^A+\tilde{\omega}_b\bar{\eta}^A)\\&-\mathrm{tr}\{(\partial_b\mathbf{N}+[\mathbf{A}_b,\mathbf{N}])\tilde{\mathbf{E}}^b\})\ . \quad (5.4)\end{aligned}$$

Therefore, (5.1) and (5.3) immediately imply

$$\begin{aligned}&\{C_{N,\mathbf{N}},\tilde{\sigma}^a{}_A{}^B\}=[N,\tilde{\sigma}^a]_A{}^B,\quad \{C_{N,\mathbf{N}},A_{aA}{}^B\}=-\mathcal{D}_aN_A{}^B\ ,\\&\{C_{N,\mathbf{N}},\tilde{\pi}_A\}=N_A{}^B\tilde{\pi}_B,\quad \{C_{N,\mathbf{N}},\tilde{\omega}_A\}=N_A{}^B\tilde{\omega}_B\ ,\\&\{C_{N,\mathbf{N}},\xi^A\}=-\xi^BN_B{}^A,\quad \{C_{N,\bar{N}}\bar{\eta}^A\}=-\bar{\eta}^BN_B{}^A\ , \quad (5.5)\\&\{C_{N,\mathbf{N}},\phi\}=0,\quad \{C_{N,\mathbf{N}},\tilde{\pi}\}=0\ ,\\&\{C_{N,\mathbf{N}},\tilde{\mathbf{E}}^a{}_{\mathbf{A}}{}^{\mathbf{B}}\}=[\mathbf{N},\tilde{\mathbf{E}}^a]_{\mathbf{A}}{}^{\mathbf{B}},\quad \{C_{N,\mathbf{N}},\mathbf{A}_{a\mathbf{A}}{}^{\mathbf{B}}\}=-\mathbf{D}_a\mathbf{N}_{\mathbf{A}}{}^{\mathbf{B}}\ .\end{aligned}$$

Thus, the infinitesimal canonical transformations generated by $C_{N,\mathbf{N}}$ are precisely the infinitesimal rotations on SU(2) spinor indices by $N_A{}^B$ and infinitesimal rotations of Yang-Mills indices by $\mathbf{N}_{\mathbf{A}}{}^{\mathbf{B}}$. This geometric interpretation gives us immediately all the Poisson brackets between the Gauss-law and other constraints. In particular, we have

$$\{C_{N,\mathbf{N}},C_{M,\mathbf{M}}\}=-C_{[N,M],[\mathbf{N},\mathbf{M}]}\ \text{ and }\ \{C_{N,\mathbf{N}},C_{\underset{\sim}{M}}\}=0\ . \quad (5.6)$$

(The brackets between the vector and the Gauss-law constraints is also zero.) It turns out[18] that our vector constraint itself does not have direct geometrical interpretation. A combination of this constraint with the Gauss law does. Let us therefore define a new constraint functional $C_{\vec{N}}$ as

$$C_{\vec{N}}:=\mathbb{C}_{\vec{N}}-\int_{\Sigma}d^3x\,\mathrm{tr}[N^a(A_a\tilde{C}+\mathbf{A}_a\tilde{\mathbf{C}})] \quad (5.2d)$$

(i.e., we have effectively set $N_A{}^B=N^aA_{aA}{}^B$, $\mathbf{N}_{\mathbf{A}}{}^{\mathbf{B}}=N^a\mathbf{A}_{a\mathbf{A}}{}^{\mathbf{B}}$). To compute the infinitesimal canonical transformations generated by this new constraint, it is convenient to rewrite it. We have

$$\begin{aligned}C_{\vec{N}}=\int_{\Sigma}d^3x[&-i\sqrt{2}\,\mathrm{tr}(N^a\tilde{\sigma}^bF_{ab})+N^a(\tilde{\pi}_A\partial_a\xi^A+\tilde{\omega}_A\partial_a\bar{\eta}^A)+N^a\tilde{\pi}\partial_a\phi+\mathrm{tr}(N^a\tilde{\mathbf{E}}^b\mathbf{F}_{ab})+i\sqrt{2}\,\mathrm{tr}(N^aA_a\mathcal{D}_b\tilde{\sigma}^b)\\&-\mathrm{tr}(N^A\mathbf{A}_a\mathbf{D}_b\tilde{\mathbf{E}}^b)]\\=\int_{\Sigma}d^3x[&-i\sqrt{2}\,\mathrm{tr}(\tilde{\sigma}^b\mathcal{L}_{\vec{N}}A_b)+(\tilde{\pi}_A\mathcal{L}_{\vec{N}}\xi^A+\tilde{\omega}_A\mathcal{L}_{\vec{N}}\bar{\eta}^A)+\tilde{\pi}\mathcal{L}_{\vec{N}}\phi+\mathrm{tr}(\tilde{\mathbf{E}}^b\mathcal{L}_{\vec{N}}\mathbf{A}_b)]\ , \quad (5.7)\end{aligned}$$

where we used the result that

$$\begin{aligned}\mathrm{tr}(N^a\tilde{\mathbf{E}}^b\mathbf{F}_{ab}-N^a\mathbf{A}_a\mathbf{D}_b\tilde{\mathbf{E}}^b)&=\mathrm{tr}(N^a\tilde{\mathbf{E}}^b(2\partial_{[a}\mathbf{A}_{b]}+[\mathbf{A}_a,\mathbf{A}_b])-\mathbf{D}_b(N^a\mathbf{A}_a\tilde{\mathbf{E}}^b)+\mathbf{D}_b(N^a\mathbf{A}_a)\tilde{\mathbf{E}}^b)\\&=\mathrm{tr}(N^a\tilde{\mathbf{E}}^b\partial_a\mathbf{A}_b+\tilde{\mathbf{E}}^b\mathbf{A}_a\partial_bN^a-\partial_b(N^a\mathbf{A}_a\tilde{\mathbf{E}}^b))\\&=\mathrm{tr}(\tilde{\mathbf{E}}^b\mathcal{L}_{\vec{N}}\mathbf{A}_b)-\mathrm{tr}(\partial_b(N^a\mathbf{A}_a\tilde{\mathbf{E}}^b)) \quad (5.8)\end{aligned}$$

and also the fact that surface terms which arise in the simplification vanish because of our boundary conditions. It follows immediately from (5.7) that canonical transformations generated by the new constraint correspond precisely to the diffeomorphisms generated by the smearing field N^a on the three-manifold Σ (with spinor and Yang-Mills indices treated as scalars). We have

$$\{C_{\vec{N}},\tilde{\sigma}^a{}_A{}^B\}=\mathcal{L}_{\vec{N}}\tilde{\sigma}^a{}_A{}^B,\quad \{C_{\vec{N}},A_{aA}{}^B\}=\mathcal{L}_{\vec{N}}A_{aA}{}^B,\ \text{ etc. }, \quad (5.9)$$

for all the dynamical variables. Therefore, the new constraint (5.2d) will be referred to as the *diffeomorphism* constraint. Now, we can once again use the geometrical interpretation of the canonical transformation to deduce the Poisson brackets between (5.2d) and other constraints. We have

$$\{C_{\vec{N}},C_{M,\mathbf{M}}\}=-C_{\mathcal{L}_{\vec{N}}M,\mathcal{L}_{\vec{N}}\mathbf{M}}\ , \quad (5.6a)$$

$$\{C_{\vec{N}},C_{\vec{M}}\}=-C_{[\vec{N},\vec{M}]}\ \text{ where }\ [\vec{N},\vec{M}]^a\equiv\mathcal{L}_{\vec{N}}M^a\ , \quad (5.6b)$$

$$\{C_{\vec{N}}, C_{\underset{\sim}{M}}\} = -C_{\mathcal{L}_{\vec{N}}\underset{\sim}{M}} \ . \tag{5.6c}$$

Thus, by a judicious choice of constraints, one can compute all but one of the possible Poisson brackets between constraint functionals. The brackets yet to be evaluated is that between two scalar constraints. This computation is somewhat long. However, it is considerably simpler than the corresponding computation in terms of the more traditionally used gravitational canonical variables since the functionals themselves are polynomial in the new variables. The final result is the expected one: the Poisson brackets is a vector constraint. A straightforward evaluation of the brackets using only (5.3) yields

$$\{C_{\underset{\sim}{N}}, C_{\underset{\sim}{M}}\} = \int_\Sigma d^3x (\underset{\sim}{N}\partial_a \underset{\sim}{M} - \underset{\sim}{M}\partial_a \underset{\sim}{N})\mathrm{tr}(\tilde{\sigma}^{\,b}\tilde{\sigma}^{\,c}\tilde{\sigma}^{\,a} - \tilde{\sigma}^{\,a}\tilde{\sigma}^{\,b}\tilde{\sigma}^{\,c})F_{bc} \ .$$

To show that the right-hand side is a linear combination of constraints, we add and subtract $(\underset{\sim}{N}\partial_a \underset{\sim}{M} - \underset{\sim}{M}\partial_a \underset{\sim}{N})\mathrm{tr}\tilde{\sigma}^{\,b}\tilde{\sigma}^{\,a}\tilde{\sigma}^{\,c}F_{bc}$ and use the fact that since $\tilde{\sigma}^{\,a}$ are trace-free in the internal indices, they satisfy $\tilde{\sigma}^{\,(a}\tilde{\sigma}^{\,b)} = \frac{1}{2}\mathrm{tr}\tilde{\sigma}^{\,a}\tilde{\sigma}^{\,b}$. We then have

$$\{C_{\underset{\sim}{N}}, C_{\underset{\sim}{M}}\} = \mathbb{C}_{\vec{K}} \quad (= C_{\vec{K}} + C_{K\cdot A, K\cdot \mathbf{A}}) \ , \tag{5.6d}$$

where

$$K^a := -2(\underset{\sim}{N}\partial_b \underset{\sim}{M} - \underset{\sim}{M}\partial_b \underset{\sim}{N})\mathrm{tr}(\tilde{\sigma}^{\,a}\tilde{\sigma}^{\,b}) \ .$$

Note that in this calculation we did *not* have to use nondegeneracy of $\tilde{\sigma}^{\,a}$. In fact, none of our evaluations of Poisson brackets are altered by the possible degeneracy. This is an important point to which we shall return in Sec. VII.

The smearing field K^a in (5.6) depends on the dynamical variables $\tilde{\sigma}^{\,a}{}_A{}^B$. Thus, although the constraints are of *first class* in the Dirac-Bergmann terminology, they do not generate a proper Lie group. In particular, as has been emphasized in the literature, e.g., by Bergmann and Komar,[19] the algebra of constraints is *not* isomorphic to the Lie algebra of the obvious "gauge group" of tetrad gravity (i.e., the semidirect product of the group of "internal" tetrad rotations with the four-dimensional diffeomorphism group of $\Sigma\times\mathbb{R}$). In the BRST terminology, the algebra is *open*. Note, however, that the structure functions—and, hence, also the BRST charge[18]—are polynomial in the canonical variables. Finally, although the constraint functionals now contain contributions from matter fields, the structure of the constraint algebra is identical to that in the source-free case.[1,3]. As has been emphasized (especially by Hojman, Teitelboim, and Kuchař)[20] this comes about because the algebra has its roots in geometrodynamics.

VI. HAMILTONIANS AND REALITY CONDITIONS

We are now ready to discuss dynamics. The analysis here is quite similar to that carried out previously in the source-free case.[1,3] New issues arise only in the discussion of reality conditions, where it is the presence of spinor fields that adds new twists. Therefore, for brevity we shall focus on the Einstein-Dirac system and only comment at the end on the effect of other matter sources.

As mentioned in Sec. V, constraint functions $C_{\underset{\sim}{N}}$ and $C_{\vec{N}}$ are differentiable on the phase space only when the lapse and shift fields, $\underset{\sim}{N}$ and N^a tend to zero at infinity. The lapse-shift pairs generating space-time translations, on the other hand, tend to nonzero (constant) values at infinity. Therefore, constraints do *not* generate canonical transformations corresponding to space-time translations; they are not the Hamiltonians of the theory. However, the geometrical interpretation of the canonical transformations generated by the constraints suggests that the Hamiltonians should be closely related to the constraints. This is indeed the case. More precisely, we have the following. Let $\underset{\sim}{T}$ be a smooth scalar density of weight -1 which equals $(\det\mathring{q}_{ab})^{-1/2}$ outside some compact subset of Σ and let T^a be a vector field which is a translational Killing field of $\mathring{q}_{ab}$ outside come compact set. The "time evolution" defined by the geometric lapse function $\underset{\sim}{T}\cdot(\det q)^{1/2}$ and the diffeomorphism generated by T^a on Σ provide us with one-parameter families of mappings of the phase space onto itself. One can verify that these mappings preserve the symplectic structure. One can therefore compute the corresponding generating functionals. These are obtained by adding suitable boundary terms to the constraint functions. We have

$$H_{\underset{\sim}{T}}(A,\tilde{\sigma},\xi,\tilde{\pi},\tilde{\eta},\tilde{\omega}) = \lim_{S\to\Sigma}\left[-\int_S d^3x\, \underset{\sim}{T}\tilde{\tilde{C}} - 2\oint_{\partial S} dS_a \underset{\sim}{T}\tilde{\sigma}^{\,[a}\tilde{\sigma}^{\,b]}A_b\right] \tag{6.1}$$

and

$$H_{\vec{T}}(A,\tilde{\sigma},\xi,\tilde{\pi},\tilde{\eta},\tilde{\omega}) = \lim_{S\to\Sigma}\left[\int_S d^3x\, T^a\tilde{C}_a - 2\sqrt{2}\,i\oint_{\partial S} dS_a T^{[a}\tilde{\sigma}^{\,b]}A_b\right] , \tag{6.2}$$

where the integral is first evaluated on a finite portion S of Σ and the limit of the result is then taken as S expands out to fill all of Σ. (To see how this subtlety arises, see, e.g., pp. 50–52 of Ref. 1.) These are the Hamiltonians generating asymptotic translations, i.e., dynamics.

Note that, on the constraint surface, the numerical values of the Hamiltonians are given just by the surface integrals. Since these integrals explicitly involve $A_{aA}{}^B$, from one's experience in Yang-Mills theory, one might conclude that they are not gauge invariant. This is, however, *not* the case because our gravitational boundary conditions are different from those normally used in the Yang-Mills theory: since the connection $A_{aA}{}^B$ is now assumed to fall off as $1/r^2$, the gauge invariance is in fact assured. Furthermore, because of this fall-off, we can replace the $\tilde{\sigma}^{\,a}{}_A{}^B$ in the surface integrals by its asymptotic value $\tilde{\mathring{\sigma}}^{\,a}{}_A{}^B$. A simple calculation then shows that, when expressed in terms of the three-metric and the extrinsic curvature of the three-manifold Σ, these integrals reduce to the familiar Arnowitt-Deser-Misner (ADM) expressions.[3] Thus, in particular, while the surface integral in (6.1) is a *holomorphic* function of the connection $A_{aA}{}^B$, and therefore complex valued at a generic point of the phase space, its restriction to the constraint surface is in fact real (and, with our conventions, negative). Note also that, as is usual in theories without background structures, the energy and momentum integrals do not have an explicit dependence on matter fields. Matter fields make their presence felt in the expressions of the constraints and thus contribute indirectly to the asymptotic values of the gravitational field on which the energy and momentum integrals depend directly.

Let us examine in detail the infinitesimal canonical transformations generated by these Hamiltonians. As expected, those generated by $H_{\vec{T}}$ are the Lie derivatives of the dynamical variables along the asymptotic translations T^a:

$$\{H_{\vec{T}},\tilde{\sigma}^{a}\}=\mathcal{L}_{T}\tilde{\sigma}^{a} \text{ and } \{H_{\vec{T}},A_a\}=\mathcal{L}_{\vec{T}}A_a \ . \tag{6.3}$$

$H_{\underset{\sim}{T}}$ on the other hand generates evolution equations: denoting $\{H_{\underset{\sim}{T}},f\}$ by $\dot{f}$, we have

$$\begin{aligned}
&\dot{\tilde{\sigma}}^{m}{}_{MN}=\sqrt{2}\,i\mathcal{D}_a(\underset{\sim}{T}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]})_{MN}-\underset{\sim}{T}\tilde{\sigma}^{m}{}_{(M}{}^{A}\xi_{N)}\tilde{\pi}_A-\underset{\sim}{T}\tilde{\sigma}^{m}{}_{(M}{}^{A}\bar{\eta}_{N)}\tilde{\omega}_A\ ,\\
&\dot{A}_m{}^{MN}=\frac{i}{\sqrt{2}}\underset{\sim}{T}[\tilde{\sigma}^{b},F_{mb}]^{MN}-\underset{\sim}{T}(\tilde{\pi}^{(M}\mathcal{D}_m\xi^{N)}+\tilde{\omega}^{(M}\mathcal{D}_m\bar{\eta}^{N)})-\tfrac{1}{2}m\underset{\sim}{T}(\xi^{A}\bar{\eta}_A)(\underset{\sim}{\eta}_{mbc}\tilde{\sigma}^{b}\tilde{\sigma}^{c})^{MN}\ ,\\
&\dot{\xi}^{M}=\sqrt{2}\,i\underset{\sim}{T}\tilde{\sigma}^{a}{}_{A}{}^{M}\mathcal{D}_a\xi^{A}+im\underset{\sim}{T}\tilde{\omega}^{M},\quad \dot{\tilde{\pi}}_M=\sqrt{2}\,i\mathcal{D}_a(\underset{\sim}{T}\tilde{\sigma}^{a}{}_{M}{}^{B}\tilde{\pi}_B)+im\sigma^2\underset{\sim}{T}\bar{n}_M\ ,
\end{aligned}\tag{6.4}$$

where, as before, $\underset{\sim}{\eta}_{abc}$ is the metric independent Levi-Civita tensor density of weight minus one. We will need these equations in the analysis of the reality conditions that follows.

Let us begin with reality conditions in the source-free case. In the previous work on the subject it has often been stated that these conditions are nonpolynomial in the basic variables. We shall first show that, while there was no computational error, this conclusion is nevertheless incorrect: *the conditions are in fact polynomial.* Recall, first that these conditions arise because we now want to restrict ourselves to the real section of the complex phase space. Let us return, for a moment, to the geometrodynamical variables, the three-metric q_{ab} and the extrinsic curvature K_{ab}. One can, if one so desires, begin with complex-valued fields (q_{ab},K_{ab}), set up the constraint and evolution equations and, at the end, take the real section of the complex phase space by imposing the reality condition: $q_{ab}=q^*_{ab}$ and $K_{ab}=K^*_{ab}$ (or, equivalently, $\dot{q}_{ab}=\dot{q}^*_{ab}$). These conditions are automatically preserved under time evolution. We want to impose a similar restriction on the phase space spanned by our pairs $(A_{aA}{}^{B},\tilde{\sigma}^{a}{}_{A}{}^{B})$. Hermiticity of $\tilde{\sigma}^{a}{}_{A}{}^{B}$ guarantees the reality of the (densitized) three-metric $\tilde{\tilde{q}}^{ab}:=-\mathrm{tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b}$. The additional condition we need is that its time derivative $(\tilde{\tilde{q}}^{ab})^{\cdot}$, also be real. Using (6.4) we have

$$(\tilde{\tilde{q}}^{ab})^{\cdot}=-\sqrt{2}\,i\underset{\sim}{T}(\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{a]})\tilde{\sigma}^{b}+\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{b]})\sigma^{a}]\ .$$

Since $\underset{\sim}{T}$ appears as an overall multiplicative factor in this equation and since the Poisson brackets of H_{T} with $\tilde{\tilde{q}}^{ab}$ gives just the Lie derivative of $\tilde{\tilde{q}}^{ab}$, the reality of $\tilde{\tilde{q}}^{ab}$ and of its time derivative under *arbitrary* real lapse and shift evolutions is ensured by requiring

$$\begin{aligned}
&(\mathrm{tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b})^*=[\mathrm{tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b})\ ,\\
&[\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{a]})\tilde{\sigma}^{b}+\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{b]})\tilde{\sigma}^{a}]^*\\
&\qquad=-[\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{a]})\tilde{\sigma}^{b}+\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{b]})\tilde{\sigma}^{a}]\ .
\end{aligned}\tag{6.5}$$

This is the required reality condition in the source-free case. It is clearly polynomial in the basic canonical variables. Indeed, $\tilde{\tilde{q}}^{ab}$ is just quadratic, and, its time derivative, being its Poisson brackets with the Hamiltonian which is at worst quartic, is also at worst quartic in $\tilde{\sigma}^{a}{}_{A}{}^{B}$ and $A_{aA}{}^{B}$. Again, if the reality condition is imposed initially on a pair $(A_{aA}{}^{B},\tilde{\sigma}^{a}{}_{A}{}^{B})$ satisfying constraints, under the Hamiltonian flow, it is preserved in time. Why then was it previously stated that these conditions are nonpolynomial? It is because, in the earlier work, one insisted on expressing them in terms of the geometrodynamical variables, the metric and the extrinsic curvatures, and the nonpolynomial dependence entered in the transition from the new to the old variables. More precisely, the situation is the following. Let $\tilde{\sigma}^{a}{}_{A}{}^{B}$ be nondegenerate and let D denote the unique torsion-free derivative operator which annihilates $\tilde{\sigma}^{a}{}_{A}{}^{B}$. The difference between $\mathcal{D}$ and D is captured in a field $\Pi_{aA}{}^{B}$: $(\mathcal{D}_a-D_a)\lambda_A=(i/\sqrt{2})\Pi_{aA}{}^{B}\lambda_B$. Then, using the evolution equation of $\tilde{\tilde{q}}^{ab}$, it is easy to check that $\dot{q}_{ab}=-2\underset{\sim}{T}\,\mathrm{tr}\Pi_{(a}\tilde{\sigma}_{b)}$. (Thus, $K_{ab}=-\mathrm{tr}\Pi_{(a}\sigma_{b)}$.) Hence, the second of the reality conditions may be replaced by Hermiticity of $\Pi_{aA}{}^{B}$, i.e., of $i(A_{aA}{}^{B}-\Gamma_{aA}{}^{B})$, where $\Gamma_{aA}{}^{B}$ is the spin connection of D. However, since $\Gamma_{aA}{}^{B}$ is nonpolynomial in $\tilde{\sigma}^{a}{}_{A}{}^{B}$, this last condition is also nonpolynomial. Put differently, the extrinsic curvature K_{ab} is a nonpolynomial function of the basic phase-space variables so that the condition that K_{ab} be real is also nonpolynomial. However, $(\tilde{\tilde{q}}^{ab})^{\cdot}=(\det q)(K^{ab}-Kq^{ab})$ *is* polynomial and so is the condition that it be real. Since there is no reason to revert to the geometrodynamical variables in the formulation of the reality conditions, we shall just use (6.5) as the reality conditions in the source-free case.

Let us now return to the Einstein-Dirac system. The reality conditions on $\tilde{\sigma}^{a}{}_{A}{}^{B}$ and $A_{aA}{}^{B}$ are again that $\tilde{\tilde{q}}^{ab}$ and its time derivatives be real. However, since the time evolution equation of $\tilde{\sigma}^{a}{}_{A}{}^{B}$ now involves the spinor fields as well, the explicit form of the condition is modified to

$$(\mathrm{tr}\tilde{\sigma}^a\tilde{\sigma}^b)^* = (\mathrm{tr}\tilde{\sigma}^a\tilde{\sigma}^b)\ ,$$

$$\left[\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{a]})\tilde{\sigma}^b + \mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{b]})\tilde{\sigma}^a - \frac{i}{\sqrt{2}}(\xi^A\tilde{\pi}_A + \tilde{\eta}^A\tilde{\omega}_A)\tilde{\tilde{q}}^{ab}\right]^* = -\left[\mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{a]})\tilde{\sigma}^b + \mathrm{tr}\mathcal{D}_m(\tilde{\sigma}^{[m}\tilde{\sigma}^{b]})\tilde{\sigma}^a - \frac{i}{\sqrt{2}}(\xi^A\tilde{\pi}_A + \tilde{\eta}^A\tilde{\omega}_A)\tilde{\tilde{q}}^{ab}\right]\ . \tag{6.6a}$$

There are, however, additional reality conditions involving spinor fields which arise simply from the definitions of the spinor field momenta, $\tilde{\pi}_A = -i\sigma\xi^\dagger_A$ and $\tilde{\omega}_A = i\sigma\bar{\eta}^\dagger_A$. Note, however, that these spinors reality conditions need to be imposed even in the traditional frameworks though they are often not stated explicitly. Without these conditions the observable currents that arise as quadratic combinations of spinor fields will *not* be real in the phase-space description and the evolution of the initial data will *not* lead to real space-time geometry. Thus, the idea of first introducing a complex phase space and then looking at a real section thereof is rather general; it is not peculiar to the use of new variables. Furthermore, in the specific case of spinor fields now under consideration, the conditions are precisely the ones that arise in the traditional framework. In terms of observable "currents," these conditions are

$$\begin{aligned}
(\tilde{\sigma}^a{}_{AB}\xi^A\tilde{\pi}^B)^* &= (\tilde{\sigma}^a{}_{AB}\xi^A\tilde{\pi}^B)\ ,\\
(\tilde{\sigma}^a{}_{AB}\bar{\eta}^A\tilde{\omega}^B)^* &= (\tilde{\sigma}^a{}_{AB}\bar{\eta}^A\tilde{\omega}^B)\ ,\\
(\tilde{\sigma}^a{}_{AB}\tilde{\pi}^A\tilde{\pi}^B)^* &= (\sigma)^2(\tilde{\sigma}^a{}_{AB}\xi^A\xi^B)\ ,\\
(\tilde{\sigma}^a{}_{AB}\tilde{\omega}^A\tilde{\omega}^B)^* &= (\sigma)^2(\tilde{\sigma}^a{}_{AB}\bar{\eta}^A\bar{\eta}^B)\ .
\end{aligned} \tag{6.6b}$$

The reality conditions for the Einstein-Dirac system are thus given by the two sets of Eqs. (6.6a) and (6.6b).

These conditions have been formulated in terms of physical, tensorial quantities, while the phase space is defined in terms of spinorial fields. Therefore, to single out the real section of the phase space explicitly, we have to eliminate some ambiguities by fixing conventions. This can be achieved as follows. First, we consider the complex phase space spanned by the fields $(A_{aA}{}^B, \tilde{\sigma}^a{}_A{}^B, \xi^A, \tilde{\pi}_A, \bar{\eta}^A, \tilde{\omega}_A)$ with the Poisson-brackets relations given by (5.1). Then, on objects with internal indices, we introduce a "dagger operation" satisfying relations (i) $(a\alpha_A + b\beta_A)^\dagger = a^*\alpha^\dagger_A + b^*\beta^\dagger_A$; (ii) $(\alpha^\dagger_A)^\dagger = -\alpha_A$, (iii) $(\alpha^A)^\dagger\alpha_A \geq 0$; (iv) $(\epsilon_{AB})^\dagger = \epsilon_{AB}$; and, (v) $(\alpha_A\beta_B)^\dagger = \alpha^\dagger_A\beta^\dagger_B$, for all fields α_A and β_A and complex functions a and b. The field $\alpha^\dagger_A$ will be called the adjoint of α_A. (There is considerable freedom in the initial choice of this operation. But once chosen, it is to be kept fixed.) Now, let us consider the reality condition on $\tilde{\sigma}^a$. Let $\tilde{\sigma}^a_0$ be such that $\tilde{\tilde{q}}^{ab}_0 := -\mathrm{tr}\tilde{\sigma}^a_0\sigma^b_0$ is real, and nonnegative. Then, we can find a unique equivalence class of $\tilde{\sigma}^a$, each element of which is Hermitian and satisfies $-\mathrm{tr}\tilde{\sigma}^a\tilde{\sigma}^b = \tilde{\tilde{q}}^{ab}_0$, where two $\tilde{\sigma}^a$ are considered as equivalent if they are related by an SU(2) group element. [Note that the initial $\tilde{\sigma}^a_0$ may not be in this equivalence class; there may be a GL(2,$\mathbb{C}$) transformation relating $\tilde{\sigma}^a_0$ and $\tilde{\sigma}^a$.] Each element of the equivalence class can lie on the real section of the phase space. Next, consider the spinor fields ξ^A and $\tilde{\pi}_A$. [The treatment of the pair $(\bar{\eta}^A, \tilde{\omega}_A)$ is completely analogous.] Let us suppose that this pair satisfies the reality condition (6.6b) on currents with respect to any one of the Hermitian $\tilde{\sigma}^a{}_A{}^B$ obtained above. Then, it is easy to show that we have $\tilde{\pi}_A = \pm i\sigma\xi^\dagger_A$. Of these two disjoint branches, we pick one by requiring that $i\tilde{\pi}^A\xi_A \geq 0$. We now have the appropriate conditions for $\tilde{\sigma}^a{}_A{}^B$ and the Dirac fields to lie on the real section of the phase space. It only remains to single out the compatible $A_{aA}{}^B$. For this, we use the second of Eq. (6.6a). Thus, a point $(\tilde{\sigma}^a{}_A{}^B, A_{aA}{}^B, \xi^A, \tilde{\pi}_A, \bar{\eta}^A, \tilde{\omega}_A)$ will lie on the real section if $\tilde{\sigma}^a{}_A{}^B$ and the Dirac fields satisfy the conditions given above *and* if $A_{aA}{}^B$ is such that the second of Eq. (6.6a) ensuring the reality of $(\tilde{\tilde{q}}^{ab})^\cdot$ is satisfied. If a point of the phase space lies on the real section as well as the constraint hypersurface, its Hamiltonian evolution yields a real, Lorentzian space-time with Dirac spin-$\frac{1}{2}$ sources.

Finally, let us consider the effect of inclusion of (real) Klein-Gordon and Yang-Mills sources. These fields contribute to the expressions of the constraint functions and hence also to the volume term in the Hamiltonian. Moreover, as remarked in Sec. IV, the presence of Yang-Mills sources forces us to use for lapse fields densities of weight minus three. But these changes are minor and have no bearing on any of the conceptual issues. In particular, the inclusion of these bosonic sources has no effect on the reality conditions on $\tilde{\sigma}^a{}_A{}^B$ and $A_{aA}{}^B$.

Let us conclude this section with a few remarks.

(i) As in the source-free case, while the Hamiltonian (6.1) preserves the reality of tensorial quantities such as $\tilde{\tilde{q}}^{ab}$ and the Dirac currents, it does not preserve Hermiticity relations between the spinorial quantities such as $\tilde{\sigma}^a{}_A{}^B$ or the Dirac fields themselves. The spinorial fields are in general "gauge rotated" by SL(2,$\mathbb{C}$) transformations. However, again as in the source-free case, this can be easily corrected by adding to this Hamiltonian a suitable multiple of the Gauss-law constraint. The Hermiticity preserving Hamiltonian is given by

$$H'_{\underset{\sim}{T}} = H_{\underset{\sim}{T}} + \lim_{S\to\Sigma}\left[\frac{i}{\sqrt{2}}\int_S (D_a\underset{\sim}{T})\tilde{\sigma}^a{}_{AB}\tilde{C}^{AB}\right]\ , \tag{6.1'}$$

where $H_{\underset{\sim}{T}}$ is the Hamiltonian defined in (6.1) and $\tilde{C}_A{}^B$ is the Gauss constraint [of Eq. (4.12c)].

(ii) Let us see how the reality conditions given above arise from the space-time geometry of solutions to the field equations. Let us suppose that we are given an anti-Hermitian soldering form $\sigma^a{}_A{}^{A'}$, a connection one-form

${}^4A_{aA}{}^B$, and spin-$\frac{1}{2}$ fields $(\xi^A,\eta^{A'})$, satisfying the field equations of Sec. III on a real four-manifold $\Sigma\times\mathbb{R}$. Denote by ∇ the torsion-free derivative operator compatible with $\sigma^a{}_A{}^{A'}$; by ${}^4\mathcal{D}$ the derivative operator defined by ${}^4A_{aA}{}^B$; and by D the torsion-free derivative operator on a three-dimensional submanifold Σ of M compatible with $\sigma^a{}_A{}^B$. Then, by equations of motion (2.4) and (2.5) we know that $({}^4\mathcal{D}_a-\nabla_a)\alpha_A={}^4C_{aA}{}^B\alpha_B$, and, by the definition of extrinsic curvature K_{ab}, it follows that $(\underset{\leftarrow}{\nabla}_a-D_a)\alpha_A=(i/\sqrt{2})K_{aA}{}^B\alpha_B$ where $K_{aA}{}^B=K_{ab}\sigma^b{}_A{}^B$. Therefore, using the fact that the pullback of ${}^4\mathcal{D}$ to Σ is $\mathcal{D}$, we have $(\mathcal{D}_a-D_a)\alpha_A\equiv(A_{aA}{}^B-\Gamma_{aA}{}^B)\alpha_B=[C_{aA}{}^B+(i/\sqrt{2})K_{aA}{}^B]\alpha_B$, where, as before, $\Gamma_{aA}{}^B$ is the spin connection one-form defined by D, and $C_{aA}{}^B=q_a{}^{b4}C_{bA}{}^B$. If we now express $C_{aA}{}^B$ in terms of $\tilde{\sigma}^a{}_A{}^B$, the Dirac fields, and their momenta on Σ, we obtain an expression for the extrinsic curvature K_{ab} in terms of the phase-space variables:

$$\frac{i}{\sqrt{2}}K_a{}^{AB}=A_a{}^{AB}-\Gamma_a{}^{AB}+\frac{i}{2\sqrt{2}(\sigma)^2}(\tilde{\pi}^M\xi^{(A}+\tilde{\omega}^M\bar{\eta}^{(A})\tilde{\sigma}_a{}^{B)}{}_M\ .$$

When $\tilde{\sigma}^a{}_B{}^B$ and the Dirac fields satisfy the reality conditions [i.e., first equation in (6.6a) and Eq. (6.6b)], the reality condition on $A_{aA}{}^B$ [i.e., the second equation in (6.6a)] is precisely the requirement that K_{ab} be real.

(iii) Let us return to the difference in the Einstein-Cartan and the Einstein-Dirac systems discussed in Sec. III. How does this difference manifest itself in the Hamiltonian description? Since the two Lagrangians differ from each other by a term which is quartic in Dirac fields, the two Hamiltonians also differ by the same term. However, since the term does not contain any derivative couplings, the relations between Hermitian adjoints of spinor fields and their momenta as well as the equation of motion for $\tilde{\sigma}^a{}_A{}^B$ remain unchanged. Therefore, the reality conditions are also unaffected. (One can also arrive at this conclusion from the space-time viewpoint mentioned above. Since the equation of motion of $A_{aA}{}^B$ and the definitions of momenta conjugate to the Dirac fields are the same in the two theories, the expression of the extrinsic curvature in terms of the phase-space variables remains unaffected by the quartic term. Hence, the condition that the extrinsic curvature be real is also unaffected.) Thus, the only difference in the two theories is that their scalar constraints and hence also the volume terms in their Hamiltonians differ by a quartic combination of Dirac fields.[21]

VII. DISCUSSION

A primary motivation behind the "new variables" framework as a whole comes from the possibility that the microphysics of the gravitational interaction may be simpler to formulate in terms of the "Yang-Mills type of variables" than the standard geometrodynamical ones. Thus, for example, in the source-free case, using these new variables Rovelli and Smolin were able to obtain[6] a large class of physical states, i.e., solutions to all quantum constraints. By contrast, as far as the full theory is concerned, not a single physical state is known in the more thoroughly investigated metric representation of quantum geometrodynamics. There are two reasons underlying this success of the new framework. First, a significant technical simplification occurs because constraints are polynomial in terms of $A_{aA}{}^B$ and $\tilde{\sigma}^a{}_A{}^B$. The second and perhaps more important reason is that the use of new variables opens up fresh directions in the canonical quantum gravity program. In particular, the "Yang-Mills formulation" of general relativity enables one to introduce two new representations of quantum states—the connection representation in which states are functionals of the connection one-form $A_{aA}{}^B$ and the loop representation in which they arise as functions on the loop space of the three-manifold Σ—which in turn facilitates the problem of solving the quantum constraints. The viewpoint underlying the present program is that this shift of emphasis from geometrodynamics to gauge theory has a deep significance. Observables associated with, e.g., the parallel transport of "spinors" around closed loops are to be regarded as the fundamental quantities in the Planck regime while the space-time geometry is to be regarded as a secondary concept which comes on its own only in the semiclassical and classical approximations. This viewpoint is supported by the recent results in $2+1$ gravity.[22,23]

An important check on the viability of these ideas is whether or not the attractive features of the framework survive the introduction of matter, for it is often the case that elegant features of source-free gravity are destroyed when sources are brought in. Let us therefore examine this question in some detail in the light of results obtained in this paper. In the spirit of the program, it is natural to regard $A_{aA}{}^B$ as the configuration variable and $\tilde{\sigma}^a{}_A{}^B$, its conjugate momentum, as being analogous to the Yang-Mills electric fields. Therefore, in quantum theory we wish to represent $\tilde{\sigma}^a{}_A{}^B$ by an operator of the type $\delta/\delta A_{aA}{}^B$. But such a representation is permissible only if the classical momentum variable takes values in a vector space, i.e., in the present case, only if $\tilde{\sigma}^a{}_A{}^B$ is allowed to become degenerate. Therefore, to import gauge theory ideas into canonical quantum gravity, we must make sure that the classical Hamiltonian theory itself is meaningful if $\tilde{\sigma}^a{}_A{}^B$ is degenerate. Now, in Sec. II, we began by assuming that $\sigma^a{}_A{}^{A'}$, and hence the four-metric g_{ab}, are nondegenerate. The nondegeneracy was essential to perform the Legendre transform and to pass to the Hamiltonian description. However, in the final Hamiltonian formulation, which is complete in itself, the requirement can be dropped entirely. The symplectic structure, the constraint and evolution equations, the Hamiltonian, and the reality conditions, all continue to be meaningful even when we allow $\tilde{\sigma}^a{}_A{}^B$ to become degenerate. A key question is whether the proof that the constraints are first class depends on nondegeneracy,[24] for even if the constraints themselves are polynomial in the basic canonical pair, the structure functions in the Poisson algebra may involve the inverse of $\tilde{\sigma}^a{}_A{}^B$. Fortunately, this does *not* happen. Thus, the key features which enabled one to introduce the connection and the loop representations in the source-free case are robust and survive the coupling to matter sources. Furthermore, since every equation in the final canonical description is polynomial in the basic variables, one hopes that the Jacobson-Rovelli-Smolin solutions to quantum constraints can be extended to include sources. Recently, Rovelli has extended the loop representation to pure Yang-Mills theory (private communication). Therefore, there is now an attractive possibility on the horizon that the use of loop representation may enable one to construct, in a unified way, the nonperturbative quantum theory of all four basic interactions.

The above discussion brings out the fact that the Hamiltonian framework presented in Secs. V and VI is in fact a slight generalization of Einstein's theory with matter sources, reducing to it in the case when $\tilde{\sigma}^{a}{}_{A}{}^{B}$ is nondegenerate. The new equations do admit a wider class of solutions in which $\tilde{\sigma}^{a}{}_{A}{}^{B}$ is degenerate. In particular, there is no *a priori* reason why $\tilde{\sigma}^{a}{}_{A}{}^{B}$ could not vanish identically even though $A_{aA}{}^{B}$ and its curvature do not vanish. These would be solutions in which there is curvature but no metric. While such solutions have no obvious significance in the classical theory, at least when $\tilde{\sigma}^{a}{}_{A}{}^{B}$ vanishes on a set of nonzero measure, they may be of considerable interest in quantum theory. Indeed, in $2+1$ gravity, the most natural description of the quantum vacuum involves precisely the configuration in which $\tilde{\sigma}^{a}{}_{A}{}^{B}$ vanishes identically.[22]

We will conclude by pointing out a curious feature of the Einstein Yang-Mills system in the present framework. Now, the Einstein fields are represented by pairs $(A_a, \tilde{\sigma}^a)$ and the Yang-Mills, by analogous pairs $(\mathbf{A}_a, \tilde{\mathbf{E}}^a)$. Both are subject to Gauss constraints: $\mathcal{D}_a \tilde{\sigma}^a = 0$ and $\mathbf{D}_a \tilde{\mathbf{E}}^a = 0$. The symmetry extends also to the vector constraint which has the form $\mathrm{tr}\tilde{\sigma}^a F_{ab} = (\mathrm{const}) \times \mathrm{tr}\tilde{\mathbf{E}}^a \mathbf{F}_{ab}$. It is tempting to conjecture that this symmetry may be a reflection of a new type of possible unification. Perhaps there is a way to modify the present framework to larger internal symmetry groups so that a part of the new A_a captures the Einstein self-dual connection and the remainder, the Yang-Mills connection one-form, such that the source-free Einstein constraints on the new A_a and its conjugate momentum are equivalent to the Einstein Yang-Mills constraints of Sec. IV. This would be complementary to the Kaluza-Klein type of unification. Now, the space-time dimension would continue to be $3+1$ but the internal-symmetry group would be enlarged to encompass both the Einstein and the Yang-Mills fields and it is the gravitational field that would emerge as a part of an enlarged Yang-Mills type connection rather than the Yang-Mills field emerging as a part of a metric in a higher-dimensional space-time.

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(iii), the answer is in the negative at least in the present framework.

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[24]That this is not just an abstract possibility is illustrated by the following example. In the source-free case, assuming that $\tilde{\sigma}^{a}{}_{A}{}^{B}$ is nondegenerate, the vector and the scalar constraint can be combined to a single expression: $\tilde{\sigma}^{a}\tilde{\sigma}^{b}F_{ab}=0$. The trace of this equation gives the scalar constraint and the trace-free part, the vector constraint. This form of the constraints is again polynomial and has a further attractive feature that, as in $2+1$ gravity, the only free indices are the internal ones. Although the resulting algebra is again of first class, as it must be, the structure functions involve the inverses of $\tilde{\sigma}^{a}{}_{A}{}^{B}$.