Einstein gravity coupled to a massless scalar field in arbitrary spacetime dimensions

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We obtain the exact static spherically symmetric solution of the coupled Einstein–massless-scalar-field equations for arbitrary spacetime dimensions. The general solution involves two parameters: it is asymptotically flat and the scalar field tends to zero at spacelike infinity. It exhibits a naked singularity unless the scalar field vanishes when it naturally reduces to the Schwarzschild solution in D spacetime dimensions.

I. INTRODUCTION

The idea that we might live in a spacetime possessing more than four dimensions is almost as old as general relativity itself. Introduced by Kaluza and elaborated further by Klein, this counterintuitive notion is now the soul of certain modern unifying theories. As a consequence, an understanding of the Einstein gravity in more than four dimensions gains importance to physicists, as the growing volume of recent literature indicates. With this as our motivation, plus the fact that exact solutions of any theory are always desirable and valuable, we here derive the exact static spherically symmetric solution of the coupled Einstein–massless-scalar-field equations valid for arbitrary dimensions $D \geq 4$ of the spacetime manifold.

Exact solutions of the higher-dimensional Einstein gravity have been previously discussed in the literature. Much of the research has been devoted to finding solutions which represent a realistic Kaluza-Klein cosmology, i.e., solutions (preferably singularity-free) where a D-dimensional space is topologically the product of a four-dimensional expanding world with a $(D-4)$-dimensional (preferably compact) internal space. However, other solutions such as D-dimensional black holes have also been found and examined in detail (see Ref. 1). In our work we will examine spherically symmetric static nonvacuum spacetimes. A massless scalar field will be taken as the source of gravity. Although at first sight such a choice of source would appear to be of purely mathematical interest, further thought suggests the opposite. For example, if in ten-dimensional $N=1$ supergravity[2] one neglects the Kalb-Ramond field, then the bosonic sector[3] of the theory involves a massless scalar field alone. Also, the spectrum of superstring theories always involves a massless scalar field—the so-called dilaton field. We should point out that at $D=4$ dimensions the problem of a self-gravitating massless scalar field has drawn the attention of many researchers. In a series of recent papers, Christodoulou[4] employs a time-dependent massless scalar field to attack some of the unresolved issues of general relativity, such as the formation of an event horizon and cosmic censorship. Almost 30 years ago, Bergmann and Leipnik[5] attempted to construct static solutions to the problem but due to an inappropriate choice of coordinates (they used Schwarzschild-type coordinates) their efforts met with limited success. During the same period, exact spherically symmetric solutions by means of suitable generation techniques were obtained by Buchdahl.[6] In fact his powerful technique enable him to construct the most general two-parameter family of solutions of the static field equations (of course our solution, in the special case of $D=4$, reduces to his). At about the same time Yilmaz[7] also discovered a special class of solutions, which however was a particular case of Buchdahl's general solution. Recent work by Wyman[8] appeared to have exhausted all possible classes of solutions to the problem. Further we should note here that very recently Mannheim and Kazanas[9] have suggested an interpretation of the solutions of the Einstein–massless-scalar-field equation as a possible manifestation of the Higgs vacuum.

A clear message emerging from the above-cited investigations is that the relevant field equations appear easier to be handled in isotropic-type coordinates rather than, say, Schwarzschild-type spherical coordinates. In our analysis we shall employ a D-dimensional version of the former set.

II. THE SOLUTION

We seek the general static and spherically symmetric solution of the Einstein equations coupled to a massless scalar field. We shall assume that the scalar field is static and spherically symmetric, i.e., that it shares the same symmetries with the geometry.

The D-dimensional (we always assume that $D \geq 4$) Einstein equations coupled to a massless scalar field are

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=kT_{\mu\nu}\ ,$$
$$T_{\mu\nu}=(\nabla_\mu\phi)(\nabla_\nu\phi)-\tfrac{1}{2}(\nabla^a\phi)(\nabla_a\phi)g_{\mu\nu}\ , \quad (1)$$

while the scalar field obeys the equation

$$\nabla^\mu \nabla_\mu \phi = 0 \; . \tag{2}$$

Equations (1) are equivalent to

$$R_{\mu\nu} = k(\nabla_\mu \phi)(\nabla_\nu \phi) \; . \tag{3}$$

Adopting a set of isotropic coordinates in D spacetime dimensions we can write the line element in the form

$$ds^2 = -e^f (dt)^2 + e^{-h}[(dr)^2 + r^2 \gamma_{ij}(dx^i)(dx^j)] \; , \tag{4}$$

where $f = f(r)$, $h = h(r)$, while γ_{ij} and x^j are the intrinsic metric and coordinates of a $(D-2)$-dimensional sphere. For the above line element the only nonvanishing components of the Ricci tensor are

$$R^t{}_t = -e^h \left[\frac{\ddot f}{2} + \frac{\dot f^2}{4} + \frac{\dot h \dot f}{4} + \frac{D-2}{2r}\left[1 - \frac{r\dot h}{2}\right]\dot f \right] , \tag{5}$$

$$R^r{}_r = -e^h \left[\frac{\ddot f}{2} + \frac{\dot f^2}{4} + \frac{\dot h \dot f}{4} - \frac{D-2}{2r}(r\dot h)^{\cdot} \right] , \tag{6}$$

$$R^i{}_j = \frac{e^h}{r^2}\delta^i{}_j \left[-\tfrac{1}{2} r \dot f (1 - \tfrac{1}{2} r \dot h) + \frac{r}{2}(r\dot h)^{\cdot} + r\dot h \left[1 - \frac{r\dot h}{4}\right](D-3) \right] . \tag{7}$$

The Laplacian associated with the line element (4) and acting on static and spherically symmetric scalar fields reduces to

$$(e^{f/2 + h(3-D)/2} r^{D-2} \phi_{,r})_{,r} = 0 \; , \tag{8}$$

which admits the first integral

$$\dot\phi = \frac{C}{r^{D-2}} \exp\left[\frac{(D-3)h - f}{2}\right] ; \tag{9}$$

the overdot denotes differentiation with respect to r and C in Eq. (9) is an integration constant.

Next we turn to the Einstein equations. From Eq. (3) it follows that the only nonvanishing component of the Ricci tensor should be $R_{rr} = k\dot\phi^2$. From the equations $R^t{}_t = R^i{}_j = 0$ we form the combination

$$R^t{}_t + (D-3)R^i{}_i = 0 \tag{10}$$

(there is no summation over the index i) which using Eqs. (5) and (7) implies the following first-order differential equation for $y = \dot f - (D-3)\dot h$:

$$\dot y + \frac{y^2}{2} + \frac{2D-5}{r} y = 0 \; . \tag{10'}$$

The integration of the above equation is easy and its general solution is given by

$$y = \frac{4(3-D)A}{r(r^{2D-6} + A)} \; , \tag{11}$$

with A an arbitrary constant of integration. It turns out, a further integration of the above value of y implies

$$f - (D-3)h = \ln\left[\frac{r^{2D-6} + A}{\hat B r^{2D-6}}\right]^2 , \tag{12}$$

with $\hat B$ another integration constant, and therefore by combining Eqs. (9) and (12) we get

$$\dot\phi = \hat B C \frac{r^{D-4}}{r^{2D-6} + A} \; . \tag{13}$$

To complete the solution we should determine f and h separately. If we employ the (t,t) component of Eq. (3) we obtain

$$\dot f \left[\frac{\ddot f}{\dot f} + \tfrac{1}{2}\dot f + \tfrac{1}{2}\dot h + \frac{D-2}{r}\left[1 - \frac{r\dot h}{2}\right]\right] = 0 \; . \tag{14}$$

Considering the case $\dot f \neq 0$ and eliminating h and its derivative by virtue of Eqs. (11) and (12) we obtain

$$\dot f = G\hat B \frac{r^{D-4}}{r^{2D-6} + A} \; , \tag{15}$$

where G is another integration constant. If we now substitute our findings, i.e., Eqs. (11), (13), and (15), in the (r,r) component of Eq. (3) we obtain an identity, provided the various integration constants encountered so far satisfy

$$4A(D-2)(D-3) = -\left[kC^2\hat B^2 + \frac{1}{4}\frac{D-2}{D-3}\hat B^2 G^2\right] . \tag{16}$$

This relation indicates that the constant A must be negative, which we shall denote hereafter by

$$A = -r_0^{2D-6} \; .$$

With the above definite choice of the sign of A the remaining relevant integrations needed to complete the solution are rather trivial, yielding the following final form for the massless field and metric, respectively:

$$\phi = \tilde\gamma \ln \frac{r^{D-3} - r_0^{D-3}}{r^{D-3} + r_0^{D-3}} \; , \tag{17}$$

$$e^f = \left[\frac{r^{D-3} - r_0^{D-3}}{r^{D-3} + r_0^{D-3}}\right]^{2\gamma} , \tag{18}$$

$$e^{-h} = \left[1 - \frac{r_0^{2D-6}}{r^{2D-6}}\right]^{2/(D-3)} \left[\frac{r^{D-3} - r_0^{D-3}}{r^{D-3} + r_0^{D-3}}\right]^{-2\gamma/(D-3)} , \tag{19}$$

where

$$\tilde\gamma = \left[\frac{D-2}{(D-3)k}(1-\gamma^2)\right]^{1/2} . \tag{20}$$

In the process we have eliminated the parameters $\hat B$ and G, in favor of a new parameter γ defined by $4(D-3)r_0^{D-3}\gamma = \hat B G$, and we have disregarded all the physically irrelevant integration constants.

When $r \to \infty$, $f \to 0$, $h \to 0$, $\phi \to 0$: Hence the spacetime becomes asymptotically flat and the scalar field vanishes at infinity. In more detail we readily find that

$$e^{f}\simeq 1-4\gamma\left[\frac{r_0}{r}\right]^{D-3},$$

$$e^{-h}\simeq 1+\frac{4\gamma}{D-3}\left[\frac{r_0}{r}\right]^{D-3}, \tag{20'}$$

$$\phi\simeq -2\tilde{\gamma}\left[\frac{r_0}{r}\right]^{D-3}.$$

Thus the higher the dimension, the faster the scalar field vanishes asymptotically.

III. TWO PARTICULAR CASES

Equations (4) and (17)–(20) describe a two-parameter family of solutions of the Einstein–massless-scalar-field equations. The parameter r_0 by construction is required to be positive while, on the other hand, the reality of ϕ demands the following range of γ: $0\le\gamma^2\le 1$. When $\gamma\in[-1,0)$, the solutions, although asymptotically flat, are characterized by "negative" Arnowitt-Deser-Misner (ADM) mass. In particular for $\gamma=-1$, which is a vacuum solution, it describes the D-dimensional Schwarzschild solution with negative mass (i.e., a naked singularity). So we shall demand the following range of γ:

$$0\le\gamma\le 1\ . \tag{21}$$

It is interesting to see the limiting form of the metric and field at the lower and upper bounds of γ. For the latter case, i.e., $\gamma=1$, it is obvious from Eqs. (17)–(20) that $\phi=0$, while the metric reduces to

$$ds^2=-\left[\frac{1-z}{1+z}\right]^2(dt)^2+(1+z)^{4/(D-3)}\times[(dr)^2+r^2\gamma_{ij}(dx^i)(dx^j)]\ , \tag{22}$$

where

$$z=(r_0/r)^{D-3}\ . \tag{23}$$

The solution is recognized as the Schwarzschild solution in D spacetime dimensions,[10] in isotropic coordinates. To verify that we introduce the Schwarzschild coordinate R defined by

$$Z=\frac{z}{(1+z)^2},\quad z=(r_0/r)^{D-3},\quad Z=(r_0/R)^{D-3}\ . \tag{24}$$

The metric (22) becomes

$$ds^2=-\left[1-\frac{4r_0^{D-3}}{R^{D-3}}\right](dt)^2+\left[1-\frac{4r_0^{D-3}}{R^{D-3}}\right]^{-1}(dR)^2+R^2\gamma_{ij}(dx^i)(dx^j)\ , \tag{25}$$

which is the Schwarzschild solution, in Schwarzschild coordinates, in D spacetime dimensions, with mass $M=2r_0^{D-3}$.

For the lower bound of γ, i.e., $\gamma=0$, the obtained solution reads

$$\phi=\left[\frac{D-2}{k(D-3)}\right]^{1/2}\ln\left[\frac{r^{D-3}-r_0^{D-3}}{r^{D-3}+r_0^{D-3}}\right], \tag{26}$$

$$ds^2=-(dt)^2+\left[1-\frac{r_0^{2D-6}}{r^{2D-6}}\right]^{2/(D-3)}(dr^2+r^2\gamma_{ij}dx^idx^j)\ , \tag{27}$$

which is a one-parameter family of solutions, parametrized by r_0. This one-parameter class of solutions could also have been discovered by a direct integration of the field equations, if one had chosen $\dot{f}=\ddot{f}=0$ as the solution of Eq. (14). The solution described by Eqs. (26) and (27) appears rather interesting, and we shall study it in some more detail. It turns out that this solution can be explicitly expressed in Schwarzschild-type coordinates. This can be easily seen if we set

$$R=\frac{1}{r}(r^{2D-6}-r_0^{2D-6})^{1/(D-3)}\Longleftrightarrow r^{D-3}=\tfrac{1}{2}[R^{D-3}+(R^{2D-6}+4r_0^{2D-6})^{1/2}]\ .$$

Using further that

$$\left[1-\left[\frac{r_0}{r}\right]^{2D-6}\right]^{2/(D-3)}=\frac{R^2}{r^2}$$

and

$$dr=rR^{D-4}(dR)(R^{2D-6}+4r_0^{2D-6})^{-1/2} \tag{28}$$

we finally obtain the following one-parameter family of solutions in Schwarzschild coordinates:

$$ds^2=-(dt)^2+\frac{R^{2D-6}(dR)^2}{R^{2D-6}+4r_0^{2D-6}}+R^2\gamma_{ij}(dx^i)(dx^j) \tag{29}$$

and

$$\phi=\left[\frac{D-2}{k(D-3)}\right]^{1/2}\ln\left[\left[1+\frac{4r_0^{2D-6}}{R^{2D-6}}\right]^{1/2}-\frac{2r_0^{D-3}}{R^{D-3}}\right]. \tag{30}$$

The behavior of the above solution for small and large values of the radial coordinate R can be easily exhibited. Calculation, for example, of the scalar curvature of the solution gives

$$\mathcal{R}=\frac{4(D-2)(D-3)r_0^{2(D-3)}}{R^{2(D-2)}}\ . \tag{31}$$

Hence, the spacetime exhibits a naked curvature singularity at $R=0$ (Ref. 11). It is worth pointing out that $R=0$ represents a $(D-2)$-dimensional sphere, as can easily be seen from the definition of R. Except at $R=0$, the solution is everywhere regular and asymptotically Minkowskian. Asymptotically the massless scalar field tends to zero with the behavior

$$\phi \sim \frac{1}{R^{D-3}} . \tag{32}$$

IV. SINGULARITIES IN THE GENERAL CASE

We now return to the two-parameter family of solutions given by Eqs. (4) and (17)–(20). To determine the singularities of this solution we have evaluated the scalar curvature[11] of the metric (4). Working (again) in the isotropic coordinates we have found, after some considerable reductions, that

$$\mathcal{R} = \frac{4(D-2)(D-3) r_0^{2(D-3)} (1-\gamma^2) r^{2(D+2)}}{(r^{D-3}+r_0^{D-3})^{2(D-2+\gamma)/(D-3)} (r^{D-3}-r_0^{D-3})^{2(D-2-\gamma)/(D-3)}} . \tag{33}$$

However as we have mentioned in Sec. III [Eq. (21)], it is required for the parameter γ to take values in the range [0,1]. It is then easy to see that

$$2 < \frac{2(D-2-\gamma)}{D-3} < \frac{2(D-2)}{D-3} . \tag{34}$$

Therefore Eqs. (33) and (34) imply that the $(D-2)$-dimensional surface $r=r_0$ is always a curvature singularity. Since, on the other hand, the time translational Killing field remains timelike for all $r > r_0$, we conclude that the curvature singularity at $r=r_0$ is a naked singularity.

V. CONCLUSIONS

It is well known that in four spacetime dimensions the "chemistry" between gravity and a massless scalar field is problematic: Static, spherically symmetric, asymptotically flat solutions of the coupled Einstein–massless-scalar-field equations either represent a regular black-hole solution with a physically trivial massless exterior field[12] or a nontrivial massless field configuration and a singular pointlike event horizon.[13] To discover this incompatibility one had to fully couple gravity to the massless scalar, in the manner of Einstein. Our investigation has established that the incompatibility between black holes and a massless scalar persists in any $D \geq 4$ spacetime dimensions. The only case when the singularity at $r=r_0$ is absent would correspond to $\gamma=1$. In that case, the scalar field would be absent, while $r=r_0$ would be a $(D-2)$-dimensional regular event horizon, confirming a no-hair theorem for a higher-dimensional Einstein gravity.

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