Initial-condition dependence of inflationary-universe models

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An analytical and numerical study of the effects of initial conditions on the evolution of a classical scalar field is presented. In this model, matter consists of a scalar field and a radiation bath. The initial energy density of the scalar field is taken to be equal to the energy density of one degree of freedom of the radiation bath. Even for this subspace of the space of initial conditions, three very different scenarios are possible. One may get chaotic inflation, dynamical relaxation, or a tension energy-dominated universe which results in no inflation. The phase space for each of these scenarios is determined.

I. INTRODUCTION

Studies of the early Universe have generated a lot of interest among the physics community. There has been steady progress in our understanding of the possible initial conditions which may have existed in the early Universe. Guth's (1981)[1] proposal of an inflationary-universe model (now called the old inflationary universe) made use a homogeneous scalar field ϕ with a nontrivial potential. It was assumed that at high temperature ϕ was trapped in a global minimum of the temperature-dependent effective potential. Gravity was described by a Friedmann-Robertson-Walker (FRW) metric of classical general relativity. As the temperature of the radiation background dropped, the global minimum turned into a metastable local minimum. During this period $V(\phi=0)$ played a role of a cosmological constant. There was a first-order phase transition from the false vacuum to a true vacuum by quantum tunneling. This theory turned out to be inconsistent.[2] Because of the slow tunneling rate, regions of the true vacuum never quite percolated enough. The new inflationary models were proposed to circumvent this problem.[3,4] The crucial difference here is that below a critical temperature T_c the temperature-dependent potential has a local maximum at $\phi=0$ instead of a local minimum. Hence, there is a second-order phase transition which occurs not by bubble nucleation but by spinodal decomposition. The problem of insufficient percolation was avoided.

Further study of new inflationary universe models has taught us that the scalar field, which is responsible for inflation, cannot be in thermal equilibrium with the radiation bath.[5,6] Hence, the use of finite-temperature effective-potential methods is not justified. Since this conclusion is so crucial we will briefly summarize the reasons. By demanding that the energy-density perturbations produced during an inflationary phase be consistent with the observed isotropy of the microwave background, we get a severe upper bound on the value for the self-coupling of the scalar field:[7–10] namely, $\lambda_\phi \leq 10^{-12}$. This in turn implies that the scalar field must be very weakly coupled to every other field because, if there were couplings of the form $L_{\text{int}}=g\phi^2\chi^2$, where ϕ is the scalar field responsible for inflation and χ is some other field, then this interaction would induce one-loop corrections to λ_ϕ of the order $\delta\lambda=O(g^2)$. By demanding that the new value of the effective four-point coupling be bounded by 10^{-12}, we obtain $g \leq 10^{-6}$. With such a small coupling, a typical interaction time between ϕ and χ particles is many orders of magnitude longer than the expansion time of the Universe. Therefore if χ is a field in thermal equilibrium, then there can be no efficient transfer of energy between the ϕ and χ fields. Hence it is inconsistent to assume that the ϕ field was ever in thermal equilibrium in the early Universe.

Since there is no thermal equilibrium, the initial conditions for $\phi(x,t)$ are not unique. In particular, large inhomogeneities should be expected. Different choices of initial conditions will result in a different evolution of the Universe. With this fact established one may analyze the evolution for different categories of initial conditions. At some initial temperature T_i, ρ_ϕ (the energy density of the scalar field) can be either larger or smaller than $(\pi^2/30)T_i^4$. That is, ρ_ϕ is smaller or larger than the energy density of a single spin degree of freedom in thermal equilibrium. Chaotic inflation[6] is a successful inflationary model which falls in the former category and dynamical relaxation[11] is a mechanism for creating inflation in the latter category. In the chaotic inflationary universe model, the scalar field is assumed to start out at a large value of ϕ (and hence with a large potential energy) and homogeneous in space. In dynamical relaxation we start with a plane-wave scalar field configuration which dynamically relaxes to $\phi=0$. If $V(\phi)$ is the standard-double-well potential $\lambda_\phi(\sigma^2-\phi^2)^2$, then one may obtain inflation (these mechanisms will be discussed in more detail in the later sections). Both mechanisms have shortcomings, because in either mechanism, one has to impose *a priori* at some initial temperature T_i certain constraints on ϕ, $\dot{\phi}$, and $\nabla\phi$ to be able to arrive at an inflationary period. An argument against the chaotic inflationary model has been the unnatural largeness of the initial value for the scalar field, and a lack of justification for using a scalar field which is homogeneous on a scale many orders of magnitude larger than the size of the initial Hubble radius. In this paper a systematic study of the evolution of the Universe is carried out for a field satisfying the initial condition $\rho_\phi(t_i) \le (\pi^2/30)T_i^4$ at some initial temperature T_i and initial time t_i. An interesting result is that even within these initial conditions there are some for which the Universe will evolve into an inflationary phase via chaotic inflation.

The organization of this paper is as follows. In Sec. II, the mathematical framework of the theory is briefly reviewed. In Sec. III, we explore sets of possible initial values for ϕ, $\dot\phi$, and $\nabla\phi$, which are consistent with the constraint $\rho_\phi(t_i) \le \rho_{\rm rad}(t_i)$, where t_i is the time corresponding to the beginning of the scenario. It is shown that if one uses the largest amplitude excitation (large ϕ), then energy considerations force upon us to use an extremely homogeneous configuration for the scalar field. In particular, associated with the largest amplitude excitation is a spatial fluctuation wavelength λ which is larger than the size of the initial Hubble radius by a factor of $\lambda_\phi^{-1/4}(\lambda_\phi \le 10^{-12}$ is the dimensionless self-coupling constant for the scalar field). In Sec. IV we calculate the subphase space of the space of the allowed initial conditions which will evolve by dynamical relaxation. Next (Sec. V) we calculated the subspace that will evolve via chaotic inflation and the subspace that will cause the Universe to evolve into a tension energy-dominated universe. Loosely, the phase space which will lead to chaotic inflation is the large amplitude (ϕ) region of the space of the allowed initial conditions. Analytical and numerical results, which include the back-reaction effects, are presented. One analytical prediction is that this transition from the radiation-dominated universe to a chaotic universe will take place within a few expansion times. The prediction is confirmed by numerical simulations. In doing the back-reaction effects (Sec. VI) we have suppressed any spatial gravitational fluctuations by replacing ρ_ϕ by its spatial average $\langle\rho_\phi\rangle_{\rm space}$ in the Einstein equation. In Sec. VII, we take a specific point in phase space which we predict will lead to tension energy domination and discuss its evolution in detail. The main results are as follows: this transition from the radiation energy domination to the tension energy domination again takes place within a few Hubble expansion times, there will be no inflation in this case because the equation of state is $p/\rho = -\frac{1}{3}$ (p is the pressure of the matter field and ρ is the energy density of the matter field). Finally (Sec. VIII) we discuss consequences of mode mixing. We have addressed the question of what happens when we excited one mode (which belonged to the phase space of chaotic inflation) in addition to exciting another mode (which belonged to the phase space for dynamical relaxation). The conclusion is that the Universe will still evolve into a chaotic inflationary phase. There will also be a few initially unexcited modes which will start to grow, but the will not grow enough to interfere with the onset of inflation.

We use units in which $\hbar = c = k_B = 1$. m_P is the Planck mass and G is Newton's constant.

II. PRELIMINARIES

In this paper matter is taken to consist of a scalar field ϕ plus a radiation bath. Gravity is treated classically. Matter influences space-time via the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} , \tag{2.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. It can be decomposed as

$$T_{\mu\nu} = T_{\mu\nu}(\phi) + T_{\mu\nu}(\mathrm{rad}) . \tag{2.2}$$

As usual,

$$T_\mu^\nu(\mathrm{rad}) = \mathrm{diag}(-\rho, +p, +p, +p) \tag{2.3}$$

with equation of state

$$p = \tfrac{1}{3}\rho . \tag{2.4}$$

The energy density $\rho(T)$ is given by

$$\rho = \frac{\pi^2}{30} n(T) T^4 . \tag{2.5}$$

Here $n(T) = n_b(T) + \frac{7}{8} n_f(T)$ and $n_f(T)$ $[n_b(T)]$ is the number of fermionic [bosonic] spin degrees of freedom at temperature T. In the standard model, $n(T)$ can be as large as 100 in the early Universe. In this paper we keep $n(T)$ as a free parameter. The energy-momentum tensor of the scalar field is

$$T_{\mu\nu}(\phi) = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu} L , \tag{2.6}$$

where

$$L = \tfrac{1}{2} g^{\mu\nu} \partial_\mu\phi\partial_\nu\phi - V(\phi) ,$$

in particular,

$$-T_0^0 = \rho(\phi) = \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2} a^{-2} (\nabla\phi)^2 + V(\phi) , \tag{2.7}$$

$$T_x^x = p_x(\phi) = \tfrac{1}{2}\dot\phi^2 - V(\phi) + \tfrac{1}{2} a^{-2} (\partial_x\phi)^2 - \tfrac{1}{2} a^{-2} (\partial_y\phi)^2 - \tfrac{1}{2} a^{-2} (\partial_z\phi)^2 , \tag{2.8}$$

and similarly for the other diagonal components. We assume that the only effect of matter on the metric is to determine the time dependence of the scale factor in the FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] , \tag{2.9}$$

where $k = +1$, 0, -1 for a closed, critical, or open universe, and $d\Omega^2$ is the metric on S^2. In a strict sense this assumption is only true if the energy-momentum tensor is homogeneous, because in that case the off-diagonal elements of T_ν^μ are zero. More will be said about this assumption later.

For a flat FRW metric, Einstein's equation gives

$$H^2(t) = \left[\frac{\dot a}{a} \right]^2 = \frac{8\pi G}{3} [\rho_{\rm rad}(t) + \rho_\phi(t)] . \tag{2.10}$$

Since there is no direct coupling between ϕ and the radiation field, it is correct to assume conservation of entropy:

$$n(T) a(T)^3 T^3 = \mathrm{const} . \tag{2.11}$$

In the early Universe (high temperature) before any particle can decouple, $n(T)$ remains constant. We then get the simple relationship $aT=\text{const}$.

The equation of motion for $\phi(x,t)$ is

$$\ddot{\phi}+3H\dot{\phi}-a^{-2}\nabla^2\phi=-V'(\phi)\ . \tag{2.12}$$

Before we proceed to the main subject of this paper we would like to briefly describe the mechanisms for dynamical relaxation and chaotic inflation. In dynamical relaxation,[11] one starts with an initial plain wave for the scalar field $\phi(x,t_i)$ with a double-well potential, e.g. $V(\phi)=\frac{1}{2}\lambda_\phi(\phi^2-\sigma^2)^2$. Since λ_ϕ is very small, the nonlinear force $V'(\phi)$ is smaller than any other force. Neglecting this force, the solution to the equation of motion is rather simple $\phi(x,\tau)\sim[1/a(t)]f(x,\tau)$. Where τ is conformal time and $f(x,\tau)$ is oscillatory in x and τ. The key property of dynamical relaxation is that a^{-1} erases any inhomogeneity which may have existed at the beginning of the Universe. Therefore as $\phi(x,t)\to 0$ and $V(\phi)\to V(0)$, $V(0)=\frac{1}{2}\lambda_\phi\sigma^4$ plays the role of a cosmological constant when ρ_{rad} falls below $V(0)$.

Dynamical relaxation has been studied numerically in Refs. 12 and 13. Thermal fluctuations have been incorporated[14,15] in the analysis, and so have gravitational perturbations.[16] However, back reaction has only been included in the numerical study of Ref. 17.

In chaotic inflation one does not require a double-well potential. One may use a potential of the form $V(\phi)=\frac{1}{2}m^2\phi^2$ or $V(\phi)=\lambda_\phi\phi^4$. However, one requires a nearly homogeneous initial scalar field configuration. In contrast to dynamical relaxation, $\rho(\phi)\simeq V(\phi)\gg\rho_{\text{rad}}$ from the beginning. For a large initial value of ϕ, it turns out that $\phi(t)$ is a very slowly varying function of time. Therefore effectively the scenario starts in an inflationary phase. Chaotic inflation has been extensively developed by Linde and co-workers.[18–20] In particular, it can be shown that the scenario works even when including small gravitational perturbations.[20]

III. INITIAL CONDITIONS

The dynamics of a system does not determine the initial conditions. Initial conditions are free parameters for the system. In general, for different initial conditions the subsequent evolution of the system will be different. When trying to obtain inflation using a scalar field one is faced with two kinds of questions. What were the forces (potential) acting on the scalar field? And given the dynamics, what must have been the initial condition in order for the scalar field to have produced an inflationary period in the early Universe? There are few guides. The scalar field must interact very weakly with itself and with other fields. And to agree with the present observed Hubble expansion of the Universe, the scalar field must today be localized at a minimum of a potential with a very small cosmological constant. However, if there was inflation, then there was a period when the energy density of the Universe was dominated by an almost constant potential term. Hopefully for some large phase space of initial conditions, it is possible to gracefully evolve into and out of an inflationary period without fine-tuning the parameters of the theory. We proceed with these facts in mind.

In this part of the paper we explore sets of possible initial values for ϕ, $\dot{\phi}$, and $\nabla\phi$ consistent with the restriction

$$\rho_\phi(t_i)\le\frac{\pi^2}{30}T_i^4\ , \tag{3.1}$$

where t_i is an initial time corresponding to T_i. As stated previously, there is no justification for taking $\rho_\phi(t_i)\le(1/n)\rho_{\text{rad}}(t_i)$ since the scalar field is not in a thermal equilibrium. But we proceed because one of the unexpected results is that even within this set of initial conditions there is a region of parameter space for which the universe will evolve into a chaotic inflationary phase.

For the scalar field potential we take a simple symmetry-breaking potential

$$V(\phi)=\tfrac{1}{2}\lambda_\phi(\phi^2-\sigma^2)^2\ . \tag{3.2}$$

We start with a single plane-wave scalar field configuration $\phi(x,t_i)=\phi_0\cos(kx+\alpha)$.

Assuming σ is not too large and imposing $V(\phi)\le(1/n)\rho_{\text{rad}}$ at some initial temperature T_i we get

$$\phi_0\le\lambda_\phi^{-1/4}T_i\ . \tag{3.3}$$

And imposing

$$\tfrac{1}{2}(\nabla\phi)^2\le\frac{1}{n(T_i)}\rho_{\text{rad}} \tag{3.4}$$

we get

$$k\phi_0\le T_i^2\ . \tag{3.5}$$

And similarly from the kinetic term we get

$$\dot{\phi}_0\le T_i^2\ . \tag{3.6}$$

A graphic representation of the allowed values for ϕ_0, and k are given in region D of Fig. 1.

Some comments are in order. Associated with the largest possible amplitude $\phi_{\max}=\lambda_\phi^{-1/4}T_i$, is a wave vector $k=\lambda_\phi^{1/4}T_i$, which tells us something about the order of homogeneity of the scalar field. $\lambda\sim k^{-1}=\lambda_\phi^{-1/4}T_i^{-1}$.

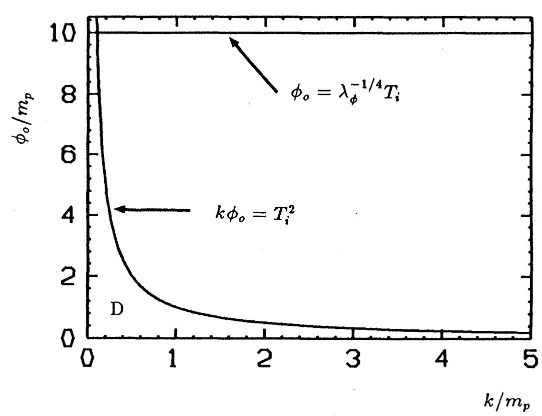

FIG. 1. Sketch of the phase space for $\rho_\phi\le(1/n)\rho_{\text{rad}}$ (region D) for $\dot{\phi}_0\le T_i^2$. The region is bounded by the lines $\phi_0\le\lambda_\phi^{-1/4}T_i$ and $k\phi_0\le T_i^2$ ($n=100$, $T_i=m_P$, $\lambda_\phi=10^{-4}$).

Evaluating this at the Planck temperature ($T_i \sim m_P$) and using $H(T \simeq m_P) = O(1/m_P)$ we arrive at the conclusion that the scalar field is homogeneous on a scale $\lambda_\phi^{-1/4}$ larger than the size of a causal horizon. This is an aesthetically displeasing feature about using such a large initial amplitude of ϕ. Recall that one of the reasons for introducing the inflationary universe models was to try to explain in a causal way existence of structures such as galaxies, clusters, etc. But we have just arrived at the conclusion that when one uses a plane-wave excitation with the largest possible amplitude then one is forced by energy considerations to start with a field configuration which is difficult to reconcile with the usual expectation from causality. As it will be shown later, this initial condition will evolve into chaotic inflation. We will call this initial condition I.

One can arrive at a more general class of initial conditions by imposing that ρ_ϕ and not necessarily every term (i.e., tension, kinetic, and potential energy) be as large as $(1/n)\rho_{\text{rad}}$ at some initial temperature T_i. In particular, the potential term is allowed to be very small but the total scalar field energy density should be $(1/n)\rho_{\text{rad}}$. Then it is no longer necessary to have such a large amplitude ($\phi_0 = \lambda_\phi^{-1/4} T_i$). But whatever amplitude one chooses one should impose a relationship $k\phi_0 \le T_i^2$. A popular choice in the literature is[11–13,17] $\phi_0 = T_i$ and $k = T_i$. Now the scale of homogeneity for the scalar field is about the size of an initial causal horizon:

$$\lambda \sim k^{-1} \sim \frac{1}{T_i} = O(H^{-1}(T_i)) \; . \tag{3.7}$$

It will be shown later that with this initial condition a Universe with a large initial number of particles in thermal equilibrium [i.e., $n(T_i) \gg 1$] will evolve into a tension energy-dominated Universe within a few expansion times. We will call this initial condition II.

A third set of initial conditions which we have studied is $\phi \le 0.1 T_i$, and $k \le 10 T_i$. It will be shown later that for these initial conditions (initial condition III) the field will evolve into an inflationary phase via dynamical relaxation.

IV. DYNAMICAL RELAXATION, INITIAL CONDITION III

It is very instructive to look at the range of parameter space for which one can get dynamical relaxation. As stated earlier, dynamical relaxation is a mechanism for localizing $\phi(x,t)$ close to zero. Such a mechanism is one possible way of generating inflation because $V(\phi=0)$ can act as a cosmological constant when the radiation energy density falls below it. When dynamical relaxation is taking place $\rho_\phi \ll \rho_{\text{rad}}$ until the onset of inflation. Consider a plane-wave initial condition, $\dot\phi(x,t_i) = \dot\phi_0 \cos kx$ and $\phi(x,t_i) = \phi_0 \sin kx$ for some yet unspecified k, ϕ_0, and $\dot\phi_0$. Assuming $V'(\phi)$ is negligible in the equation of motion we get

$$\phi(x,\tau) = \frac{f(x,\tau)}{a(\tau)} \tag{4.1}$$

with $a(t) \sim t^{1/2}$, and

$$\begin{aligned} f(x,\tau) &= \phi_0 \sin kx \cos k(\tau - \tau_i) \\ &\quad + \frac{1}{k}(\dot\phi_0 \cos kx + H_0 \phi_0 \sin kx) \\ &\quad \times \sin k(\tau - \tau_i) \; , \end{aligned} \tag{4.2}$$

where τ is conformal time defined by $dt = a(t) d\tau$ and H_0 is the Hubble parameter at t_i:

$$H_0 = 16.6 \left[\frac{n(T_i)}{100} \right]^{1/2} \frac{T_i}{m_P} T_i \; . \tag{4.3}$$

We have also used the fact that for a flat ($k=0$) radiation-dominated FRW metric, the Ricci scalar vanishes. Therefore to arrive at our analytical solution, it was not necessary to add a conformal coupling of the form $\frac{1}{6} R \phi^2$. Notice that it was crucial to start the scenario with the energy density of the radiation field greater than the energy density of the scalar field by a factor of $n(T_i)$. Because in principle both ρ_{rad} and ρ_ϕ will contribute to the form of $a(t)$. But if $n(T_i) \gg 1$, one can neglect any back reaction of $\phi(x,t)$ on $a(t)$ until the onset of inflation.

Now we would like to go back and be more quantitative about the values $\dot\phi_0$, ϕ_0, and k for which dynamical relaxation occurs. We demand that ρ_ϕ be smaller than $(1/n)\rho_{\text{rad}}(t)$ for several Hubble expansion times. In particular, $V(\phi)$ must be smaller than $(1/n)\rho_{\text{rad}}(t)$. After using the fact that both $\phi(x,t)$ and $T(t)$ vary as $a(t)^{-1}$, this implies that $f(x,\tau) \le \lambda_\phi^{-1/4} T_i$. Demanding that each term in $f(x,\tau)$ obeys this inequality we get

$$\phi_0 \le \lambda_\phi^{-1/4} T_i \; , \tag{4.4}$$

$$\frac{\dot\phi_0}{k} \le \lambda_\phi^{-1/4} T_i \; , \tag{4.5}$$

and

$$\frac{\phi_0}{k} \le \frac{1}{16.6} \left[\frac{100}{n} \right]^{1/2} \lambda_\phi^{-1/4} \frac{m_P}{T_i} \; . \tag{4.6}$$

Similarly, we demand that the tension energy be smaller than $(1/n)\rho_{\text{rad}}(t)$. This leads to the conditions

$$\phi_0 k \le T_i^2 \; , \tag{4.7}$$

$$\dot\phi_0 \le T_i^2 \; , \tag{4.8}$$

and

$$\phi_0 \le \frac{1}{16.6} \left[\frac{100}{n} \right]^{1/2} m_P \; . \tag{4.9}$$

For the kinetic energy, the algebra is slightly more messy. But after using $\dot\phi = -H(t)\phi + a(t)^{-2} f'(\tau)$ and $H(t) = \frac{1}{2} t^{-1}$ (which is true for a radiation dominated FRW Universe), we find no new constraints.

The results are plotted in Fig. 2. We can easily see that only a small fraction of the phase space which satisfied $\rho_\phi(t_i) = (1/n)\rho_{\text{rad}}(t_i)$ obeys all the above conditions to evolve via dynamical relaxation.

Figure 3 shows the time evolution from a numerical analysis for initial conditions which lead to dynamical re-

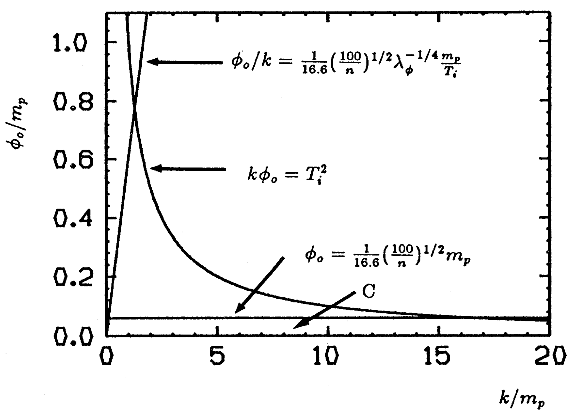

FIG. 2. Sketch of the phase space for dynamical relaxation (region C). The region is bounded by the lines $\phi_0/k \leq (1/16.6)(100/n)^{1/2}\lambda_\phi^{-1/4}m_P/T_i$, $k\phi_0 \leq T_i^2$, and $\phi_0 \leq (1/16.6)(100/n)^{1/2}m_P$ ($n=100$, $T_i=m_P$, $\lambda_\phi=10^{-4}$).

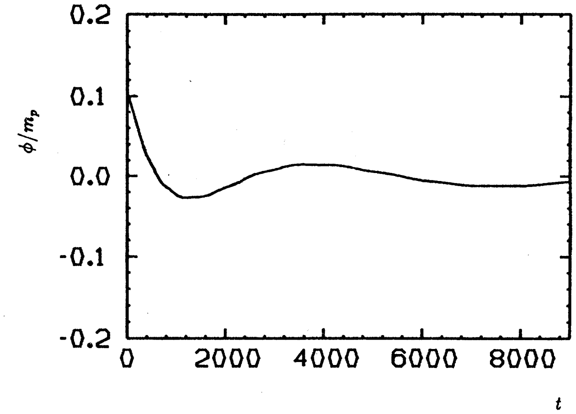

FIG. 4. The value of the scalar field at the point in space corresponding to the maximum of the standing wave as a function of time (two-dimensional slice of Fig. 3).

laxation. It shows $\phi(x,t)$ oscillating with a gradually decreasing amplitude. Figure 4 shows the time evolution of $\phi(x_0)$ where x_0 is the location of the crest of the wave.

The rest of the paper deals with the evolution of ϕ in those parts of phase space for which $\rho_\phi(t_i) \leq (1/n)\rho_{\rm rad}(t_i)$ but which do not obey all the initial condition constraints to evolve by dynamical relaxation.

V. CHAOTIC INFLATION, INITIAL CONDITION I

We now consider in detail consequences of starting with a plane-wave initial condition with $\phi_0=\lambda_\phi^{-1/4}T_i$, $k=\lambda_\phi^{1/4}T_i$, and $\dot\phi_0 \leq T_i^2$. As stated previously, even though the initial condition satisfies $\rho_\phi(t_i)=(1/n)\rho_{\rm rad}(t_i)$, the Universe will quickly evolve into a state with $\rho(\phi)\simeq V(\phi) \gg \rho_{\rm rad}$, which is the initial condition of chaotic inflation. One can learn a lot by just looking at the order of magnitude of various terms in the equation of motion for $\phi(x,t)$:

$$\ddot\phi + 3H\dot\phi - a^{-2}\nabla^2\phi = -V'(\phi) \ . \tag{5.1}$$

The Hubble damping force initially has the magnitude

$$3H_0\dot\phi_0 \sim 50\left[\frac{n}{100}\right]^{1/2}\left[\frac{T_i}{m_P}\right]T_i^3 \tag{5.2}$$

the tension force is

$$\nabla^2\phi_0 \sim k^2\phi_0 \sim \lambda_\phi^{1/4}T_i^3 \tag{5.3}$$

and the nonlinear force has the strength

$$V'(\phi_0)=2\lambda_\phi(\phi_0^2-\sigma^2)\phi_0 \sim 2\lambda_\phi\phi_0^3 \sim 10^{-3}\left[\frac{\lambda_\phi}{10^{-12}}\right]^{1/4}T_i^3 \ . \tag{5.4}$$

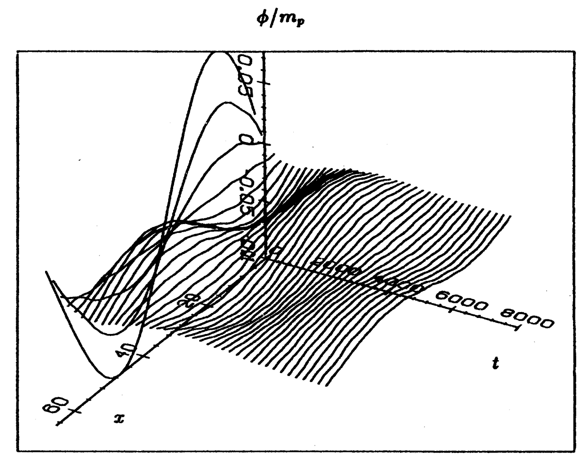

FIG. 3. Three-dimensional plot of the time evolution of $\phi(x)$ for dynamical relaxation. The initial excitation was $\phi(x,t_i)=\phi_0\sin(kx)$ and $\dot\phi(x,t_i)=\dot\phi_0\cos(kx)$. ($\phi_0=T_i/10$, $\dot\phi_0=T_i^2$, $k=10T_i$, $n=100$, $T_i=m_P$, $\lambda_\phi=10^{-12}$, $\sigma=10m_P$, and each time step is $\Delta t=\frac{1}{800}m_P^{-1}$.) At the 800th time step the scale factor is $a=18$, and the Hubble parameter is $H=0.05m_P$.

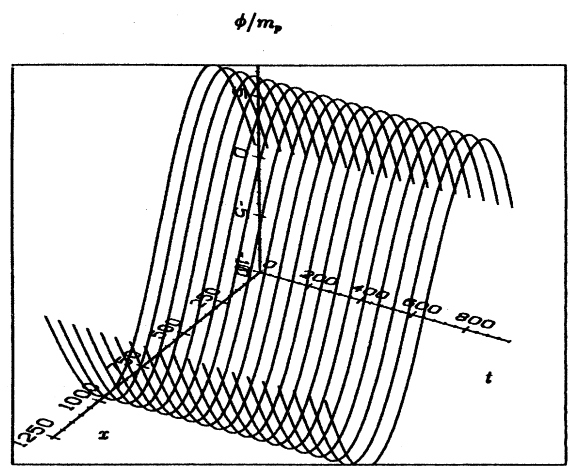

FIG. 5. Three-dimensional plot of the time evolution of $\phi(x)$ for chaotic inflation with back reaction included. The initial excitation was $\phi(x,t_i)=\phi_0\sin(kx)$, and $\dot\phi(x,t_i)=0$. ($\phi_0=\lambda_\phi^{-1/4}T_i$, $k=\lambda_\phi^{1/4}T_i$, $n=100$, $T_i=m_P$, $\lambda_\phi=10^{-4}$, $\sigma=0$, and each time step is $\Delta t=\frac{1}{50}m_P^{-1}$.) $\rho_{\rm rad}=\langle V(\phi)\rangle_{\rm space}$ at approximately the 16th time step. At the 600th time step, the scale factor is $a=3.3\times10^6$, and the Hubble parameter is $H=1.1m_P$.

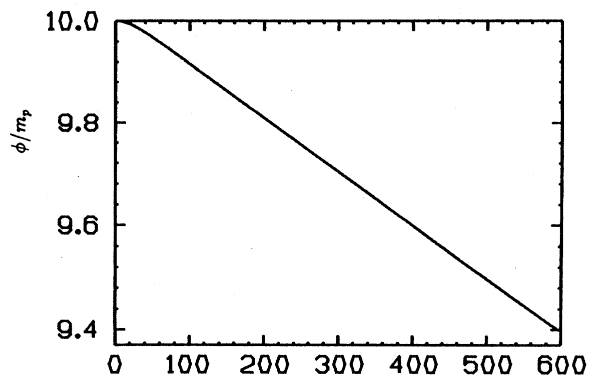

FIG. 6. The value of the scalar field at the point in space corresponding to the maximum of the standing wave as a function of time (two-dimensional slice of Fig. 5).

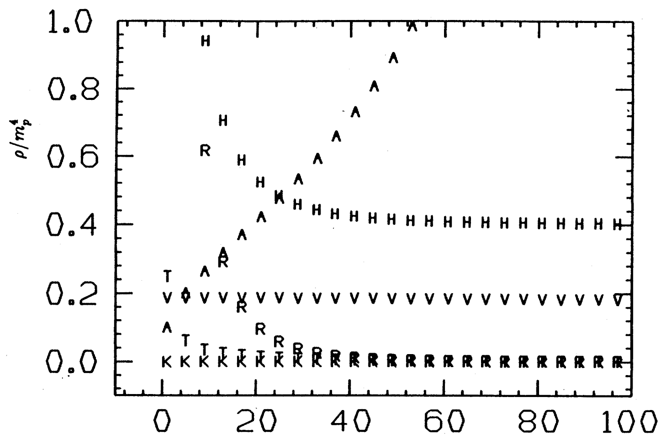

FIG. 7. Graph of the various energy densities and the scale and Hubble parameter as a function of time for chaotic inflation. $\rho_{\rm rad}=R$, $\langle V(\phi)\rangle_{\rm space}=V$, $\langle \text{tension}\rangle_{\rm space}=T$, $\langle\text{kinetic}\rangle_{\rm space}=K$. For reference we have also plotted the scale factor and the Hubble parameter. ($a=\frac{1}{10}$ scale factor, and $H=\frac{1}{3}$ Hubble parameter, with y axes in the unit of m_P.)

Since the nonlinear and tension forces are smaller than the Hubble damping force by a factor of $\lambda_\phi^{1/4}$, one can neglect them in a first approximation. Then the effective equation of motion is

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}\simeq 0 \ . \tag{5.5}$$

And therefore $\dot{\phi}\sim a^{-3}$. Figures 5 and 6 show the numerical solution for $\phi(x,t)$. They show a practically static configuration. This has the immediate consequence that $V_\phi(t)$ cannot stay smaller than $\rho_{\rm rad}(t)$ for a very long time.

If we had started with $\dot{\phi}(t_i)\simeq 0$, then the initial Hubble damping force is smaller than the other two forces which we previously neglected. But our result still holds since starting with a smaller $\dot{\phi}_0$ will not help the system to develop a large enough $\dot{\phi}(t)$ which would be necessary to keep $V_\phi(t)\ll\rho_{\rm rad}(t)$.

Notice that this conclusion was independent of the choice of σ. In fact, as long as σ is chosen so that the magnitude of the nonlinear force does not differ significantly from our previous estimate, then our conclusion will remain unchanged. Since $\lambda_\phi\sim 10^{-12}$, this condition is easily satisfied.

We can obtain a rough estimate for the time t^* when $\rho_{\rm rad}$ becomes comparable to ρ_ϕ. Initially $V_\phi(t_i)\simeq(1/n)\rho_{\rm rad}(t_i)$. At the transition time $V_\phi(t^*)=\rho_{\rm rad}(t^*)$. But since $\dot{\phi}$ goes to zero rapidly, ϕ remains essentially constant. Therefore it is a good approximation to set $V_\phi(t^*)\simeq V_\phi(t_i)$, which implies $a^4(t^*)\simeq n$. For $n=100$, $a(t^*)\simeq 3.2$. Numerically we obtained $a(t^*)\simeq 4$. Using similar approximations one can show that at t^*, the tension energy $T_\phi(t^*)$ is smaller than $\rho_{\rm rad}(t^*)$ and $V_\phi(t^*)$ by a factor of $a(t^*)^2$. And from (5.5) it follows that the kinetic energy, $K_\phi(t^*)$, is smaller than $\rho_{\rm rad}(t^*)$ by a factor of $a(t^*)^6$.

After t^* one can no longer neglect ρ_ϕ in the expression for H. Therefore, back-reaction effects of ρ_ϕ on the evolution of ϕ via H are important. We have numerically computed the evolution of $\phi(x,t)$ until $t>t^*$ including an approximate treatment of the back reaction (see Sec. VI). The results for $H(t)$, $a(t)$, $\rho_{\rm rad}$, $\langle V(\phi)\rangle_{\rm space}$, $\langle K_\phi\rangle_{\rm space}$, and $\langle T_\phi\rangle_{\rm space}$ as function of time are shown in Fig. 7, where "space" means spatial average. We see that $\rho_{\rm rad}$ continues to decrease until it becomes smaller than the tension and kinetic energy. In agreement with the analytical prediction, $V(\phi)$ hardly decreases while $\rho_{\rm rad}$ falls. Notice that for $t>t^*$ [after $\langle V(\phi)\rangle_{\rm space}\simeq\rho_{\rm rad}$], the Hubble parameter $H(t)$ remains approximately constant, which shows that the Universe has entered into an inflationary phase.

After the energy density of the Universe becomes dominated by the potential energy term, the subsequent evolution of ϕ is as determined by the chaotic inflationary scenario. Because the factor a^{-2} in the equation of motion for ϕ [Eq. (5.1)], the tension force quickly becomes negligible for $t>t^*$. For simplicity we shall consider the case of $\sigma=0$. Then using $H=(4\pi G\lambda_\phi/3)^{1/2}\phi^2$ and $V'(\phi)=2\lambda_\phi\phi^3$ the equation of motion becomes

$$\ddot{\phi}+\frac{3\sqrt{4\pi G\lambda_\phi/3}}{m_P}\phi^2\dot{\phi}\simeq -2\lambda_\phi\phi^3 \ . \tag{5.6}$$

An analytical approximation which turns out to be self-consistent is the slow-rolling approximation,[21,22] i.e., neglecting the $\ddot{\phi}$ term. The solution is

$$\phi(x,t)\simeq\phi(x,t_0)\exp\left[-\left[\frac{\lambda_\phi}{3\pi}\right]^{1/2} m_P(t-t_0)\right] \ . \tag{5.7}$$

As a check, it can be shown that the acceleration term is smaller than the other terms (provided that T_i is not too much smaller than m_P):

$$\frac{\ddot{\phi}(t)}{V'_\phi(T)} \simeq \frac{\ddot{\phi}(t)}{3H(t)\dot{\phi}(t)} \simeq \frac{m_P^2}{\phi(t)^2} \simeq 10^{-6}\left[\frac{m_P^2}{T_i^2}\right]\left[\frac{\lambda_\phi}{10^{-12}}\right]^{1/2}\exp\left[2\left[\frac{\lambda_\phi}{3\pi}\right]^{1/2} m_P(t-t_0)\right]. \tag{5.8}$$

Using the Einstein equation for $H(t)$ and (5.7) a straightforward calculation gives

$$a(t)=a(t_0)\exp\left[\frac{\pi}{m_P^2}[\phi(t_0)^2-\phi(t)^2]\right], \tag{5.9}$$

where $t_0 \simeq t^*$ is the time when the Universe starts to be potential domination. For sufficient inflation we need the scale factor to grow by at least e^{60}. A simple calculation gives

$$t_f - t_0 \simeq 30\sqrt{3/\pi}\frac{m_P}{T_i}T_i^{-1}. \tag{5.10}$$

Given this value of $t_f - t_0$ and using (5.7), it is clear that ϕ will be in a slow-rolling phase for the entire duration of inflation. Figures 5–7 are numerical results including the back-reaction effects and without making the slow-rolling approximation.

So far we have discussed only one special point in phase space which evolves via chaotic inflation. A similar analysis shows that for all initial conditions in region A of Fig. 8 (large value of ϕ_0) the time evolution will lead to chaotic inflation. The argument goes as follows. Initially, the Universe is dominated by radiation. Then for general initial conditions, before any back reaction can cause $a(t)$ and $H(t)$ to deviate from that of a radiation era, the solution of the equation of motion is known (4.1)–(4.3) assuming one can neglect the nonlinear force, which can easily be justified. Previously, to get the parameter space for dynamical relaxation, we determined parameters for which the tension, kinetic, and potential energies remained smaller than $(1/n)\rho_{\rm rad}(t)$ until ϕ has relaxed to zero. Here we ask what choices of ϕ_0, k, and $\dot{\phi}_0$, consistent with $\rho_\phi(t_i)=(1/n)\rho_{\rm rad}(t_i)$, will lead to an equation of state dominated by potential energy. A straightforward calculation gives

$$\frac{\phi_0}{k} \geq \frac{1}{5.8}\lambda_\phi^{-1/4}\left[\frac{100}{n}\right]^{1/4}\frac{m_P}{T_i}. \tag{5.11}$$

There is also a condition for which the evolution leads to an equation of state dominated by tension energy. The criterion is

$$\phi_0 \geq \tfrac{1}{2}m_P. \tag{5.12}$$

The subphase space for tension energy domination is given in region B of Fig. 8. We will comment in Sec. VII on what happens if both (5.11) and (5.12) are satisfied.

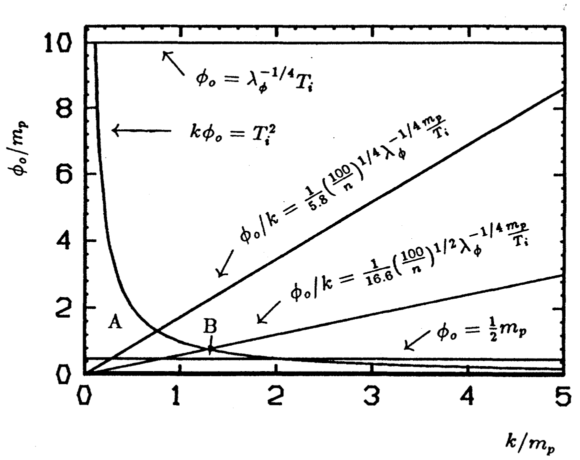

FIG. 8. Sketch of the phase space for chaotic inflation (region A) and for the tension energy-dominated universe (region B). The phase space for chaotic inflation is bounded by the lines $\phi_0 \leq \lambda_\phi^{-1/4}T_i$, $k\phi_0 \leq T_i^2$, and $\phi_0/k \leq (1/5.8)(100/n)^{1/4}\lambda_\phi^{-1/4}m_P/T_i$. The phase space for the tension-dominated Universe is bounded by the lines $k\phi_0 \leq T_i^2$, $\phi_0 \geq \frac{1}{2}m_P$, and $\phi_0/k \leq (1/16.6)(100/n)^{1/2}\lambda_\phi^{-1/4}m_P/T_i$ ($n=100$, $T_i=m_P$, $\lambda_\phi=10^{-4}$).

VI. COMMENTS ON BACK REACTION

As the value of $\rho_{\rm rad}$ falls below ρ_ϕ, it becomes invalid to take $a \sim t^{1/2}$ and $H \sim t^{-1}$. In our numerical simulation we have directly integrated

$$H^2=\frac{8\pi G}{3}(\rho_\phi+\rho_{\rm rad}) \tag{6.1}$$

to obtain $a(t)$. However, this method is only a first step. Since ϕ is inhomogeneous, the metric should also be inhomogeneous. Its scale of inhomogeneity is the same as that of the scalar field. Notice that it is even wrong to write $\ddot{\phi}+3H\dot{\phi}-a^{-2}\nabla^2\phi=-V'(\phi)$, because metric inhomogeneities give rise to extra terms. In order to make (6.1) well defined, we replace ρ_ϕ by its spatial average $\langle\rho_\phi\rangle_{\rm space}$. In this approach, one must be careful when interpreting the results. If the metric enters a de Sitter phase via chaotic inflation, this should be interpreted as the region of space where ϕ is large starting to inflate. This is the standard interpretation of chaotic inflationary models.

VII. TENSION ENERGY DOMINATION, INITIAL CONDITION II

In this part of the paper we discuss in detail consequences of the initial condition $\phi_0=k=T_i$ and $\dot{\phi}_0 \leq T_i^2$ and $n(T_i)=100$ (for $T_i=m_P$). As stated earlier, this initial condition will evolve into a tension energy-dominated Universe. The region of phase space which gives an equation of state dominated by tension energy is given in region B of Fig. 8. In this case, there will be no inflation. Recall that the condition for inflation is $p/\rho < -\frac{1}{3}$ in order for the Hubble radius to contract in comoving coordinates.[21,22] Here one should substitute $p \rightarrow \frac{1}{3}\langle p_x+p_y+p_z\rangle_{\rm space}$ and $\rho \rightarrow \langle\rho\rangle_{\rm space}$. But when the tension energy domination occurs $p \simeq \langle -\frac{1}{6}(\nabla\phi)^2a^{-2}\rangle_{\rm space}$. And $\rho \simeq \langle \frac{1}{2}(\nabla\phi)^2a^{-2}\rangle_{\rm space}$;

therefore, $p/\rho=-\frac{1}{3}$. This implies that the Hubble radius, in comoving coordinates, will not contract but will have a constant size.

Again it is helpful to look into the order of magnitude of the various forces acting on ϕ. The Hubble damping force is

$$3H_0\dot{\phi}_0 \simeq 50\left[\frac{n}{100}\right]^{1/2} T_i^3 , \tag{7.1}$$

the tension force is

$$\nabla^2\phi_0 \simeq T_i^3 , \tag{7.2}$$

and the nonlinear force has magnitude

$$V'(\phi_0) \simeq 10^{-12}\left[\frac{\lambda_\phi}{10^{-12}}\right] T_i \max\{\sigma^2,\phi_0^2\} . \tag{7.3}$$

Since the nonlinear force is so small, we can neglect it, at least initially. And since the scenario starts with $\rho_\phi(t_i)=(1/n)\rho_{\rm rad}(t_i)$, at least initially it is correct to assume that both a and H will evolve as in the radiation-dominated era. Within this approximation one can solve the equation of motion exactly as Eqs. (4.1) and (4.2).

We just restate the solution with appropriate coefficients. For $\phi(x,t_i)=T_i\sin T_i x$, $\dot{\phi}(x,t_i)=T_i^2\cos T_i x$, and $k=T_i$ we get $\phi(x,\tau)=[1/a(\tau)]f(x,\tau)$ where

$$f(x,\tau)=T_i\left[\frac{H_0}{T_i}\sin T_i x \,\sin T_i(\tau-\tau_i) + \sin T_i(x+\tau-\tau_i)\right] . \tag{7.4}$$

Substituting this solution into the expressions for the various energy densities, one can see that the solution will soon break down because the tension energy density becomes comparable to the radiation energy density. Therefore the assumption of no back reaction becomes invalid. This happens because of the large value of

$$\frac{H_0}{T_i}=16.6\left[\frac{n}{100}\right]^{1/2}\frac{T_i}{m_P} . \tag{7.5}$$

To estimate the time t^* when this transition occurs, we take the case $\dot{\phi}_0=0$. The result for $\dot{\phi}_0=T_i^2$ is similar since the large Hubble damping force quickly suppresses $\dot{\phi}$. t^* is given by equating $T_\phi(t)$ with $\rho_{\rm rad}(t)$. Using $T_\phi(t_i)=(1/n)\rho_{\rm rad}(t_i)$ we obtain $a(t^*)\simeq\sqrt{n}$.

After t^* back-reaction effects are again important via H. The numerical analysis shows that $\rho_{\rm rad}(t)$ continues to fall until it becomes smaller than the kinetic energy of the scalar field. Graphically the field configuration appears as a simple strongly damped oscillation (Figs. 9 and 10). Notice that the conclusion is largely independent of the value of σ. Our results will stand unchanged as long as σ is chosen so that our previous conclusion about the smallness of the nonlinear force in comparison to the other forces is not altered. This condition will be satisfied as long as $\sigma \le \lambda_\phi^{-1/2}T_i$. But after ϕ has decreased its amplitude such that $\phi(x,t)\ll\sigma$ and the nonlinear force becomes non-negligible, then different choices for σ will have effects on the evolution of $\phi(x,t)$.

Some comments are in order . From our simplistic derivation of different parameter spaces [Fig. 8, Eqs. (5.11) and (5.12)], it appears as if there is a large region which satisfies both the criteria for potential-energy domination and the criteria for tension energy domination. It turns out that potential-energy domination occurs first. Granted that the potential domination occurs first so $\rho_\phi+\rho_{\rm rad}\simeq V(\phi)$, then because ϕ hardly varies in time, the potential will act as a cosmological constant. Therefore, the scale factor will start to grow exponentially. But since the tension energy is proportional to a^{-2}, it will decrease exponentially and will never be important.

Finally, it should be noted that the region of tension energy domination can be made to disappear for various choices of n, and T_i.

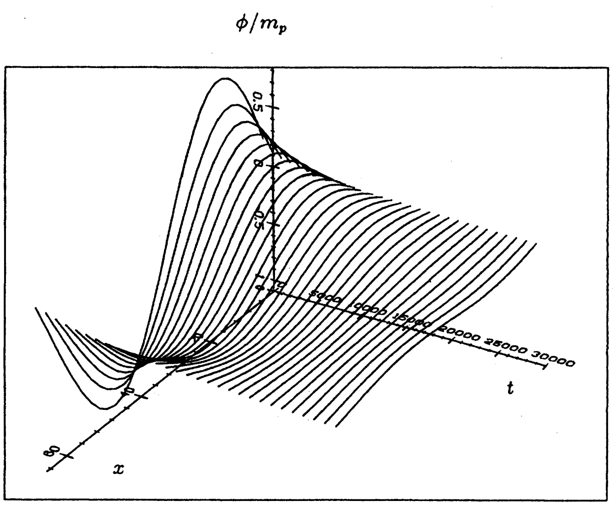

FIG. 9. Three-dimensional plot of the time evolution of $\phi(x)$ for the tension-dominated Universe with back reaction included. The initial excitation was $\phi(x,t_i)=\phi_0\sin(kx)$ and $\dot{\phi}(x,t_i)=0$. ($\phi_0=T_i$, $k=T_i$, $n=100$, $T_i=m_P$, $\lambda_\phi=10^{-12}$, $\sigma=10m_P$, and each time step is $\Delta t=\frac{1}{70}m_P^{-1}$.) The tension energy starts to dominate at approximately the 300th time step. At the 20000th time step, the scale factor is $a=200$, and the Hubble parameter is $H=1.71\times10^{-3}m_P$.

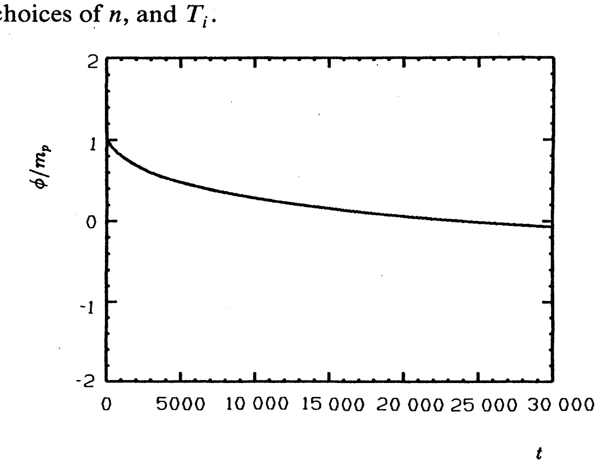

FIG. 10. The value of the scalar field at the point in space corresponding to the maximum of the standing wave as a function of time (two-dimensional slice of Fig. 9).

VIII. MODE MIXING

So far the analysis has been limited to initial single-mode excitations. We would now like to discuss consequences of exciting more than one mode. We are interested in answers to specific questions. How do initially unexcited modes become excited? How do excited modes interfere with each other? If we excite one small k (belonging to the phase space for chaotic inflation) and one or more of large k (belonging to the phase space for dynamical relaxation) will the collective excitations still cause the Universe to evolve into a chaotic inflationary phase?

To our knowledge, no analytical analysis for multimode excitations has been given in the literature. We decompose ϕ (a real scalar field) into its Fourier modes. For numerical purposes we put our Universe in a box. Therefore,

$$\phi(x,t)=\sum_k \phi_k(t)e^{ik\cdot x}\ , \tag{8.1}$$

where k can take on the values $k_xL=2\pi n_x$, $n_x=0,1,2,\ldots$, and L is the size of the box. Because ϕ is real $\phi_k^*=\phi_{-k}$. Using

$$\frac{1}{V}\int e^{ik\cdot x}e^{-ik'\cdot x}d^3x=\delta_{k,k'}\ , \tag{8.2}$$

we Fourier transform the equation of motion (2.12) and obtain

$$\ddot{\phi}_k+3H\dot{\phi}_k+a^{-2}k^2\phi_k =2\lambda_\phi\left[\sigma^2\phi_k-\sum_{k_1,k_2}\phi_{k_1}\phi_{k_2}\phi_{k-k_1-k_2}\right]. \tag{8.3}$$

Let us consider the case where we have excited two modes. In particular $k_1=\lambda_\phi^{1/4}T_i$, $\phi_{k_1}=\lambda_\phi^{-1/4}T_i$ (belonging to the phase space for chaotic inflation) and $k_2=10T_i$, $\phi_{k_2}=0.1T_i$ (belonging to the phase space for dynamical relaxation).

One important fact one should keep in mind is that since ϕ is real, for every ϕ_{k_i} excitation there is also an excitation $\phi_{-k_i}=\phi_{k_i}^*$. The equation of motion for the k_1 mode is

$$\ddot{\phi}_{k_1}+3H\dot{\phi}_{k_1}+a^{-2}k_1^2\phi_{k_1} =2\lambda_\phi[\sigma^2\phi_{k_1}-(|\phi_{k_1}|^2+|\phi_{k_2}|^2)\phi_{k_1}+\text{smaller terms}]\ . \tag{8.4}$$

The largest term in the expression for the nonlinear force is smaller than the tension force by $\lambda_\phi^{1/4}$. But we know from Sec. V that with these numbers ϕ_{k_1} will not change very much in the course of time. Therefore, we conclude that k_1 mode will again cause the potential energy to dominate and give rise to chaotic inflation.

The equation of motion for the k_2 mode is

$$\ddot{\phi}_{k_2}+3H\dot{\phi}_{k_2}+a^{-2}k_2^2\phi_{k_2} =2\lambda_\phi[\sigma^2\phi_{k_2}-(|\phi_{k_2}|^2+|\phi_{k_1}|^2)\phi_{k_2}+\text{smaller terms}]\ . \tag{8.5}$$

It is clear that even the largest term in the nonlinear force is smaller than the tension force by $\lambda_\phi^{1/2}$. Therefore k_2 will evolve as a free mode. But the evolution of ϕ_{k_2} will be somewhat different from the dynamical relaxation case. The reason being that here the Hubble parameter H will soon be determined by ϕ_{k_1} in the expression for $V(\phi)$ in ρ_ϕ, whereas in a dynamical relaxation case, $H(t)\sim t^{-1}$ until the onset of inflation.

Were there no nonlinear effects, then initially unexcited modes would remain unexcited. Modes which will to first order become excited by nonlinear effects are $\pm 3\mathbf{k}_1, \pm 3\mathbf{k}_2$, $\pm(2\mathbf{k}_1\pm\mathbf{k}_2)$, $\pm(2\mathbf{k}_2\pm\mathbf{k}_1)$. Of these initially unexcited modes which we anticipate to get excited, the nonlinear force for $\pm 3\mathbf{k}_1$ mode is the largest. Therefore, we expect the growth for ϕ_{3k_1} to be the largest. The reason is clear. The largest nonlinear force term which can appear is $\lambda_\phi\phi_{k_1}^3$ with $\phi_{k_1}=\lambda_\phi^{1/4}T_i$. The equation of motion for ϕ_{3k_1} is

$$\ddot{\phi}_{3k_1}+3H\dot{\phi}_{3k_1}+a^{-2}3k_1^2\phi_{3k_1} =2\lambda_\phi[\sigma^2\phi_{3k_1}-(|\phi_{k_2}|^2+|\phi_{k_1}|^2)\phi_{3k_1}-{\phi_{k_1}}^3] +\text{smaller terms}\ . \tag{8.6}$$

Since initially ϕ_{3k_1} and $\dot{\phi}_{3k_1}$ are zero, in a first approximation we can neglect all the terms containing them. We now use a crude upper estimate:

$$\langle V(\phi)\rangle_{k_i}\le V(\phi)_{\max}\simeq 2\lambda_\phi^{1/4}T_i^3\ , \tag{8.7}$$

where $\langle\ \rangle_k$ means a Fourier amplitude of the kth mode. Take the case where $\mathbf{k}_i=3\mathbf{k}_1$. Then the equation of motion becomes

$$\ddot{\phi}_{3k_1}(x,t)\le 2\lambda_\phi|\phi_{k_1}(x,t)|^3\le 2\lambda_\phi^{1/4}T_i^3\ . \tag{8.8}$$

The solution with initial conditions $\phi_{3k_1}(t_i)=\dot{\phi}_{3k_1}(t_i)=0$ is

$$\phi_{3k}(t)\le\lambda_\phi^{1/4}T_i^3(t-t_i)^2\ . \tag{8.9}$$

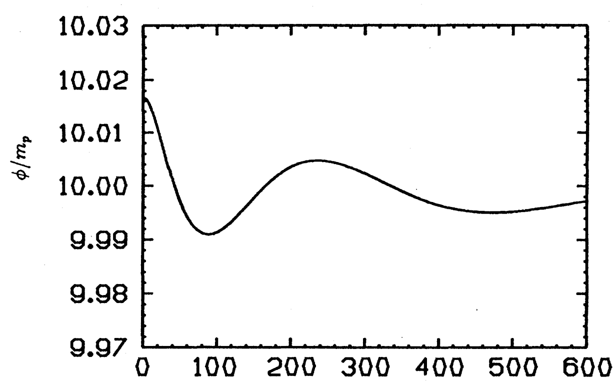

FIG. 11. The value of the scalar field, at the point in space corresponding to the maximum of the standing wave, as a function of time for two mode excitations [$k_1=\lambda_\phi^{1/4}T_i$, $k_2=50T_i$, $\phi(x,t_i)=\phi_{k_1}\sin(k_1x)+\phi_{k_2}\sin(k_2x)$, $\phi_{k_1}=\lambda_\phi^{-1/4}T_i$, $\phi_{k_2}=\frac{1}{50}T_i$, $n=100$, $T_i=m_P$, $\lambda_\phi=10^{-4}$, $\sigma=10m_P$, and each time step is $\Delta t=\frac{1}{800}m_P^{-1}$].

To see how large this amplitude will be by the end of inflation t_f, use (5.10) and obtain

$$\phi_{3k_1}(t_f) \le C \times \frac{m_P}{T_i} m_P \left[\frac{\lambda_\phi}{10^{-12}} \right]^{1/4} . \tag{8.10}$$

where C is a constant of the order 1. This is very small compared to

$$\phi_{k_1}(t_f) \simeq \lambda_\phi^{-1/4} T_i \exp\left[\frac{-30}{\pi} \left[\frac{m_P}{T_i} \right]^2 \lambda_\phi^{1/2} \right] .$$

Therefore, we can conclude that the ϕ_{3k_1} mode will not grow enough to interfere with initially excited evolution of the k_1 mode.

For the other modes, the growth will be less because the growth producing nonlinear forces are smaller for them. The tension force for a large-k mode like $3\mathbf{k}_2$ will not be any problem because it will quickly be suppressed by the large a^{-2} factor. Figure 11 shows the time evolution from a numerical analysis for two-mode excitation. In agreement with the analytical prediction the two modes essentially evolve independently; hence, their collective evolution appears as a simple superposition.

IX. CONCLUSION

In this paper an analytical numerical study of the evolution of a singlet scalar field has been given. All the initial conditions studied satisfy $\rho_\phi(t_i) = (1/n)\rho_{\mathrm{rad}}(t_i)$. It is shown that even within this regime there is a region of phase space for which the Universe will enter into chaotic inflation. The results of this paper further quantifies the phenomenological feasibility of chaotic inflationary models.

ACKNOWLEDGMENTS

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