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# Pion transitions and models of chiral symmetry

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We describe a set of pion-decay and scattering amplitudes which are described by only two low-energy parameters in the effective chiral Lagrangian of QCD. After a phenomenological analysis of the data, we demonstrate how the effective-Lagrangian framework correlates the many predictions of these reactions which have been made in the literature using a variety of models with chiral symmetry. A comparison with the data then also determines which model represents QCD. Not surprisingly, the winner is a form of vector dominance.

## I. INTRODUCTION

One of the common ways to probe quantum chromodynamics (QCD) is by means of very-high-energy scattering. In this case, because of asymptotic freedom the coupling constant $\alpha_s$ is small enough that a perturbative treatment of hard scattering is possible. There remains, however, dependence upon noncalculable soft physics such as structure functions, intrinsic $p_T$ distributions, and fragmentation models, which must be determined phenomenologically. At high energies then, QCD leads to relations between scattering processes, parametrized by $\alpha_s$ and empirically determined structure functions.[1]

There exists an analogous program to test QCD at very low energies. Here one cannot use perturbation theory in $\alpha_s$, but rather one exploits the symmetries and anomalies of the theory. The symmetries, especially chiral symmetry, predict the forms of possible reactions at low energy. There remains, however, dependence on noncalculable soft physics, such as the pion-decay constant $F_\pi$ and coefficients in an effective chiral Lagrangian, which must be determined experimentally. At low energies then, QCD leads to relations between scattering and decay processes, parametrized by $F_\pi$ and empirically determined low-energy constants.[2]

In this paper we explore a self-contained set of low-energy reactions:

$$\gamma^* \to \pi^+\pi^-, \quad \pi^+\gamma \to \gamma\pi^+ ,$$
$$\pi \to e\nu\gamma, \quad \pi \to e\nu e^+e^-, \quad \pi^0 \to \gamma\gamma .$$

At one level, our motivation is to see how well this low-energy program, chiral perturbation theory in QCD, is working. At a deeper level, however, we are interested in the structure of the chiral Lagrangian that parametrizes low-energy QCD. There exist in the literature many differing predictions for these reactions within a variety of theories, all chirally symmetric. We show how these simply represent different assumptions for the physics which determines the chiral Lagrangian. By comparison with experiment, we demonstrate that nature selects one version of the physics as being superior.

In the next section then, we present a brief overview of the chiral perturbative techniques of Gasser and Leutwyler, while in Sec. III we confront specific general predictions of chiral symmetry with experimental results. In Sec. IV we examine theoretical prediction for chiral expansion parameters within various chiral models and demonstrate how previous (model-dependent) predictions can be understood. Finally, in Sec. V our findings are summarized.

## II. FORMALISM

In the limit that the $u$-, $d$-, $s$-quark masses vanish, QCD is known to possess an exact global $SU(3)_L \times SU(3)_R$ chiral symmetry:[3]

$$q_L \to \exp\left[i\sum_{j=1}^{8} \lambda_j\alpha_j\right] q_L \equiv Lq_L ,$$
$$q_R \to \exp\left[i\sum_{j=1}^{8} \lambda_j\beta_j\right] q_R \equiv Rq_R . \tag{1}$$

Here $q$ refers to the three-component column vector

$$q = \begin{bmatrix} u \\ d \\ s \end{bmatrix} \tag{2}$$

and $\lambda_i$ are the Gell-Mann matrices. Chiral $SU(3)_L \times SU(3)_R$ invariance is dynamically broken to $SU(3)_V$ and Goldstone's theorem requires the existence of eight Goldstone bosons which are identified with $\pi$, $K$, and $\eta$ (Ref. 4). The classical axial $U(1)_A$ transformation

$$q \to e^{i\theta\gamma_5} q \tag{3}$$

is, however, not a symmetry, and leads to the well-known QCD anomaly.[5] The inclusion of quark mass introduces a small explicit breaking of the chiral symmetry, which can be accounted for via a perturbative expansion in the energy.[6] In this paper all of the processes that we study only involve chiral SU(2). Nevertheless, in order to make contact with other work in the field, we shall use the language of chiral SU(3). This involves no loss of accuracy or generality, as the effects of $K$ or $\eta$ can be absorbed in the low-energy constants and do not modify predictions of chiral SU(2) (Ref. 7).

Since QCD possesses this approximate chiral symmetry, the symmetry must be manifested somehow in the interactions of the (pseudo-)Goldstone bosons, and this point has been carefully exploited in recent studies.[2] The strictures that arise from chiral invariance are most succinctly described in terms of the nonlinear order parameter

$$U=\exp\left[\frac{i}{F_\pi}\sum_{j=1}^{8}\lambda_j\phi_j\right], \tag{4}$$

where $\phi_j$ are the pseudoscalar fields and $F_\pi=94$ MeV is the pion-decay constant. Under $SU(3)_L\otimes SU(3)_R$ the matrix $U$ is defined to transform as

$$U\to LUR^\dagger . \tag{5}$$

Including the effects of quark mass, the simplest Lagrangian consistent with chiral, Lorentz, and U(1) gauge invariance is then

$$L^{(2)}=\frac{F_\pi^2}{4}\mathrm{Tr}D_\mu U\,D^\mu D^\dagger+\frac{F_\pi^2}{4}\mathrm{Tr}m(U+U^\dagger)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6}$$

where

$$D_\mu U\equiv\partial_\mu U+ie[Q,U]A_\mu \tag{7}$$

is the covariant derivative and

$$m=2B_0\begin{bmatrix} m_u & 0 & 0\\ 0 & m_d & 0\\ 0 & 0 & m_s\end{bmatrix},\quad Q=\begin{bmatrix}\frac{2}{3} & 0 & 0\\ 0 & -\frac{1}{3} & 0\\ 0 & 0 & -\frac{1}{3}\end{bmatrix}. \tag{8}$$

The first piece of $L^{(2)}$ contains the meson kinetic energy contribution and is chiral invariant. The second term transforms as $(3_L,\bar{3}_R)+(\bar{3}_L,3_R)$ under chiral rotations and describes the breaking of chiral invariance by the quark masses. Comparing with experimental meson masses yields the normalization[8]

$$2B_0=\frac{2m_K^2}{m_s+m_d}=\frac{2m_\pi^2}{m_u+m_d}=\frac{6m_\eta^2}{m_u+m_d+4m_s}. \tag{9}$$

The last piece of $L^{(2)}$ is simply the free photon Lagrangian.

Even at this level the theory has predictive power—the tree-level evaluation of $L^{(2)}$ yields [at $O(p^2,m^2)$] the familiar Weinberg scattering lengths for $\pi$-$\pi$ scattering,[9]

$$a_0^0=\frac{7m_\pi}{32\pi F_\pi^2},\quad a_0^2=-\frac{m_\pi}{16\pi F_\pi^2}, \tag{10}$$

which are roughly borne out experimentally. However, loop diagrams arising from $L^{(2)}$ produce effects of higher order, $O(p^4,m^2p^2,m^4)$, and contain divergences. These infinites can be eliminated by being absorbed into renormalizing phenomenological chiral couplings of order 4. The most general such Lagrangian has been given by Gasser and Leutwyler:[2]

$$\begin{aligned}L^{(4)}=&L_1[\mathrm{Tr}(D_\mu U\,D^\mu U^\dagger)]^2+L_2(\mathrm{Tr}\,D_\mu U\,D_\nu U^\dagger)^2+L_3\mathrm{Tr}(D_\mu U\,D^\mu U^\dagger)^2+L_4\mathrm{Tr}(D_\mu U\,D^\mu U^\dagger)\mathrm{Tr}m(U+U^\dagger)\\ &+L_5\mathrm{Tr}\,D_\mu U\,D^\mu U^\dagger mU^\dagger+Um+L_6[\mathrm{Tr}m(U+U^\dagger)]^2+L_7[\mathrm{Tr}m(U-U^\dagger)]^2+L_8\mathrm{Tr}(mUmU+mU^\dagger mU^\dagger)\\ &-iL_9\mathrm{Tr}(F^L_{\mu\nu}D^\mu U\,D^\nu U^\dagger+F^R_{\mu\nu}D^\mu U^\dagger D^\nu U)+L_{10}\mathrm{Tr}(F^L_{\mu\nu}UF^{\mu\nu R}U^\dagger)+L_{11}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}+L_{12}\mathrm{Tr}m^2 .\end{aligned} \tag{11}$$

Here $F^L_{\mu\nu},F^R_{\mu\nu}$ are external field-strength tensors defined via

$$\begin{aligned}F^{L,R}_{\mu\nu}&=\partial_\mu F^{L,R}_\nu-\partial_\nu F^{L,R}_\mu-i[F^{L,R}_\mu,F^{L,R}_\nu],\\ F^{L,R}_\mu&=V_\mu\pm A_\mu\end{aligned} \tag{12}$$

and the covariant derivative is defined as

$$D_\mu=\partial_\mu-i[V_{\mu,}]-i\{A_{\mu,}\}. \tag{13}$$

The coefficients $L_1,\dots,L_{12}$ are arbitrary and unphysical (bare) inasmuch as they can be used to absorb divergent loop contributions from the lowest-order chiral Lagrangian $L^{(2)}$. The physical (renormalized) couplings are found to be

$$L_i^r(\mu)=L_i+\frac{\Gamma_i}{32\pi^2}\left[\frac{2}{\epsilon}+\ln 4\pi+1-\gamma\right]\mu^{-\epsilon}, \tag{14}$$

where $\gamma$ is Euler's constant, $\epsilon=d-4$ represents the dimensionality, and

$$\begin{aligned}&\Gamma_1=\tfrac{1}{32},\ \Gamma_2=\tfrac{3}{16},\ \Gamma_3=0,\ \Gamma_4=\tfrac{1}{8},\\ &\Gamma_5=\tfrac{3}{8},\ \Gamma_6=\tfrac{11}{144},\ \Gamma_7=0,\ \Gamma_8=\tfrac{5}{48},\\ &\Gamma_9=\tfrac{1}{4},\ \Gamma_{10}=-\tfrac{1}{4},\ \Gamma_{11}=-\tfrac{1}{8},\ \Gamma_{12}=\tfrac{5}{24}\end{aligned} \tag{15}$$

are constants chosen to cancel the divergences.

In addition to the above terms, the effect of the anomaly must be included. This is contained in the Wess-Zumino-Witten action at order $E^4$ (Ref. 10):

$$S_{\mathrm{WZW}}=\int d^5x\,\epsilon^{ijklm}\mathrm{Tr}(L_iL_jL_kL_lL_m)+\int d^4x\,A_\mu J^\mu+\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu A_\alpha T_\beta \tag{16}$$

with

$$L_\mu=\partial_\mu UU^\dagger,\quad R_\mu=U^\dagger\partial_\mu U\ ,$$
$$J^\mu=\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(QL_\nu L_\alpha L_\beta+QR_\nu R_\alpha R_\beta)\ , \tag{17}$$
$$T_\beta=\mathrm{Tr}(Q^2L_\beta+Q^2R_\beta+\tfrac{1}{2}QUQU^\dagger L_\beta+\tfrac{1}{2}QU^\dagger QUR_\beta)\ .$$

We have given the gauge dependence only for the U(1) photon field. The full non-Abelian anomaly is much more complicated and is not needed for our purposes. The first term in Eq. (16) in this action is written as the integral of a five-dimensional space, whose four-dimensional boundary is our usual space-time. Even though the physics is determined entirely by the four-dimensional boundary, this construction gives a remarkably simple form for the anomaly. The one-loop renormalization of the anomaly action has recently been carried out.[11] Although there is no modification at all to the coefficients given above, there exist induced new terms at order $E^6$. Such terms are themselves purely four dimensional in character and do not modify the anomalous Ward identities. Since we are in this paper working to order $E^4$ we will not consider them further.

Gasser and Leutwyler have analyzed an entire set of electroweak reactions involving pions and kaons,[2]

$$\pi^+\to\pi^+\gamma,\quad \pi^+\to\pi^0e^+\nu_e,\quad \pi^+\to e^+\nu_e\gamma\ ,$$
$$K^+\to\pi^0e^+\nu_e,\quad K^+\to\pi^0\mu^+\nu_\mu,\quad K^0\to\pi^-e^+\nu_e\ , \tag{18}$$
$$K^0\to\pi^-\mu^+\nu_\mu,\quad \text{etc.}\ ,$$

and have shown that such reactions are completely determined in terms of only three of the fourth-order constants: $L_5^r,L_9^r,L_{10}^r$. For example, the charged-pion electromagnetic form factor is calculated as

$$f_\pi(q^2)\underset{q^2\ll m_\pi^2}{\sim}1+q^2\left[\frac{2}{F_\pi^2}L_9^r-\frac{1}{32\pi^2F_\pi^2}\times\left[\frac{1}{2}+\frac{1}{3}\ln\frac{m_\pi^2}{\mu^2}+\frac{1}{6}\ln\frac{m_K^2}{\mu^2}\right]\right]. \tag{19}$$

The experimental value of the pion charge radius[12]

$$\langle r_\pi^2\rangle^{\mathrm{expt}}=0.439\pm0.030\ \mathrm{fm}^2 \tag{20}$$

can then be used in order to produce an empirical value for the chiral parameter $L_9^r$:

$$L_9^r(\mu=m_\eta)=(6.9\pm0.2)\times10^{-3}\ . \tag{21}$$

Similarly the experimental value for the axial structure constant $h_A$ in radiative pion decay, $\pi^+\to e^+\nu_e\gamma$ (Ref. 13),

$$h_A^{\mathrm{expt}}=(8.7\pm0.2)\times10^{-4}\ \mathrm{MeV}^{-1} \tag{22}$$

determines $L_{10}^r$

$$L_{10}^r(\mu=m_\eta)=(-5.2\pm0.3)\times10^{-3} \tag{23}$$

and the measured SU(3) breaking in the pion, kaon decay constants[14]

$$\left.\frac{F_K}{F_\pi}\right|^{\mathrm{expt}}=1.22\pm0.01 \tag{24}$$

provides the size of $L_5^r$:

$$L_5^r(\mu=m_\eta)=(2.2\pm0.5)\times10^{-3}\ . \tag{25}$$

The fact that such chiral coefficients are small, $L_i^r\sim10^{-2}$, is satisfying and is consistent with the theoretical observation that chiral perturbation theory represents an expansion in momentum with a parameter $\Lambda$, which sets the scale of said expansion, given by[15]

$$\Lambda\sim4\pi F_\pi\sim1\ \mathrm{GeV}\ . \tag{26}$$

That is, coefficients of terms in $L^{(2)}$, $L^{(4)}$, $L^{(6)}$, etc., should be of order

$$c^{(2)}:c^{(4)}:c^{(6)}:\cdots\sim1:\Lambda^{-2}:\Lambda^{-4}:\cdots \tag{27}$$

which is consistent with, e.g.,

$$\frac{L_9^r}{F_\pi^2}\sim\frac{0.015}{m_\pi^2}\sim\frac{1}{(1100\ \mathrm{MeV})^2}\ . \tag{28}$$

It is this feature which guarantees the success of chiral perturbation theory at low energies: $s\ll\Lambda^2$. In fact, the analysis made by Gasser and Leutwyler successfully related all the reactions quoted in Eq. (18) in terms of the three parameters $L_5^r$, $L_9^r$, $L_{10}^r$, and the physical particle masses.

Thus far our discussion has constituted a brief review of the Gasser-Leutwyler procedure. The remainder of our paper will involve analysis of previous theoretical attempts to calculate the chiral parameters in chirally symmetric models. We shall first focus our discussion on certain higher-order properties of the pion, and then provide a more general overview.

## III. PHENOMENOLOGICAL UPDATE

Before discussing specific theoretical models, we will first describe the analysis of various pionic properties—the electromagnetic form factor, radiative weak decay, and the polarizability—using the effective Lagrangian framework, and compare the predictions of chiral symmetry with experiment.

The electromagnetic form factor, defined as

$$\langle\pi^+(p_2)|V_\mu^{\mathrm{em}}|\pi^+(p_1)\rangle=f_\pi(q^2)(p_1+p_2)_\mu\ , \tag{29}$$

is characterized in terms of the form factor $f_\pi(q^2)$ which to lowest order is written in the (on-shell) form

$$f_\pi(q^2)=1+\tfrac{1}{6}\langle r_\pi^2\rangle q^2+\cdots\ . \tag{30}$$

Radiative pion beta decay, $\pi^+\to e^+\nu_e\gamma$, is described by the weak electromagnetic matrix element[16]

$$M_{\mu\nu}(q,p)=\int d^4x\, e^{iq\cdot x}\langle 0|T[V_\mu^{\rm em}(x)J_\nu^{\rm wk}(0)]|\pi^+(p)\rangle$$
$$\equiv\sqrt{2}F_\pi(p-q)_\nu(2p-q)_\mu\frac{1}{(p-q)^2-m_\pi^2}(1+\tfrac{1}{6}\langle r_\pi^2\rangle q^2)+\sqrt{2}F_\pi(p-q)_\nu q_\mu\tfrac{1}{6}\langle r_\pi^2\rangle$$
$$-\sqrt{2}F_\pi g_{\mu\nu}+h_A[(p-q)_\mu q_\nu-g_{\mu\nu}(p-q)\cdot q]+h_V\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta-r_A(g_{\mu\nu}q^2-q_\mu q_\nu)\ . \quad (31)$$

Here the first piece is the pion pole contribution, while $h_A$, $r_A$, and $h_V$ are so-called direct or structure-dependent from factors associated with axial-vector and polar-vector compounds of the weak current. We have retained terms proportional to $q^2$ which vanish for physical photons in order to include also $\pi^+\to e^+\nu_e e^+e^-$, which has recently been measured at SIN (Ref. 17). Ordinarily such structure dependence is difficult to measure, being dwarfed by inner bremsstrahlung. However, in this case the nonradiative $\pi^+\to e\nu_e$ process is strongly helicity suppressed[18] allowing experimental access to the direct emission terms. The pion polarizability is given in terms of the Compton-scattering amplitude:[19]

$$A_{\mu\nu}(p_1,q_1,q_2)=\int d^4x\, e^{iq_2\cdot x}\langle\pi^+(p_2)|T[V_\nu^{\rm em}(x)V_\mu^{\rm em}(0)]|\pi^+(p_1)\rangle$$
$$\equiv-\frac{T_\nu(p_1-q_2,p_1)T_\mu(p_2,p_2+q_1)}{(p_1-q_2)^2-m_\pi^2}-\frac{T_\mu(p_1-q_1,p_1)T_\nu(p_2,p_2+q_2)}{(p_1-q_1)^2-m_\pi^2}$$
$$+2g_{\mu\nu}+\tfrac{1}{3}\langle r_\pi^2\rangle(q_1^2g_{\mu\nu}-q_{1\mu}q_{1\nu}+q_2^2g_{\mu\nu}-q_{2\mu}q_{2\nu})+2\gamma_\pi(q_1\cdot q_2 g_{\mu\nu}-q_{1\nu}q_{2\mu})+\cdots\ , \quad (32)$$

where

$$T_\mu(p_2,p_1)=(p_1+p_2)_\mu(1+\tfrac{1}{6}\langle r_\pi^2\rangle q^2)+q_\mu\tfrac{1}{6}\langle r_\pi^2\rangle(p_1^2-p_2^2) \quad (33)$$

is the pion electromagnetic vertex. Connection with the polarizability is made by use of the nonrelativistic expansion[20]

$$\mathrm{Amp}(\gamma\pi\to\gamma\pi)=\hat{\epsilon}_1\cdot\hat{\epsilon}_2\left[-\frac{\alpha}{m_\pi}[1-\tfrac{1}{6}\langle r_\pi^2\rangle(q_1^2+q_2^2)]+\omega_1\omega_2\alpha_E\right]+\hat{\epsilon}_1\times\mathbf{q}_1\cdot\hat{\epsilon}_2\times\mathbf{q}_2\beta_M\ . \quad (34)$$

Here $\alpha$ is the fine-structure constant while $\alpha_E$, $\beta_M$ are the electric and magnetic pion polarizabilities, respectively, defined such that

$$\int d^3x\, H_{\rm int}\propto\tfrac{1}{2}(\alpha_E E^2+\beta_M B^2)\ . \quad (35)$$

Comparison with Eq. (32) immediately reveals the chiral-symmetry requirement

$$\alpha_E=-\beta_M=\gamma_\pi\frac{\alpha}{m_\pi}\ . \quad (36)$$

That $\alpha_E$ must be the negative of $\beta_M$ is also clear from the effective-Lagrangian framework, wherein the lowest-order chiral contribution must have the form

$$L_{\rm eff}\sim e^2F_{\mu\nu}F^{\mu\nu}\mathrm{Tr}(QUQU^\dagger)\propto(E^2-B^2)\ . \quad (37)$$

Before discussing predictions for these basic pion properties, we note that experimental values exist for each,

$$\langle r_\pi^2\rangle|^{\rm expt}=(0.44\pm0.01)\,\mathrm{fm}^2\ \ \text{(Ref. 12)}\ ,$$
$$\left.\frac{h_A}{h_V}\right|^{\rm expt}=0.46\pm0.02\ \ \text{(Ref. 21)}\ ,$$
$$\left.\frac{r_A}{h_V}\right|^{\rm expt}=2.3\pm0.6\ \ \text{(Ref. 17)}\ , \quad (38)$$
$$\alpha_E+\beta_M|^{\rm expt}=(1.4\pm3.1)\times10^{-4}\ \mathrm{fm}^3\ \ \text{(Ref. 22)}\ ,$$
$$\alpha_E|^{\rm expt}=(6.8\pm1.4)\times10^{-4}\ \mathrm{fm}^3\ \ \text{(Ref. 23)}\ ,$$

with which to compare the theoretical analysis. It is also important to note that these *five* experimental quantities are predicted in terms of just *two* of the chiral parameters $L_9$ and $L_{10}$:

$$\langle r_\pi^2\rangle|^{\rm theo}\simeq\frac{12L_9^r}{F_\pi^2},\quad \left.\frac{h_A}{h_V}\right|^{\rm theo}=32\pi^2(L_9^r+L_{10}^r)\ ,$$
$$\left.\frac{r_A}{h_V}\right|^{\rm theo}=32\pi^2L_9^r,\quad \alpha_E+\beta_M|^{\rm theo}=0\ , \quad (39)$$
$$\alpha_E|^{\rm theo}=\frac{4\alpha}{m_\pi F_\pi^2}(L_9^r+L_{10}^r)\ .$$

The effects of pion loops have been worked out for these processes in Ref. 7. We do not include these in our analysis for two reasons. (1) When regularized as in Gasser and Leutwyler's work, the corrections induced by loops in these processes are found to be small. In particular they are smaller both than the experimental uncertainty and than the magnitude of the tree-level coefficients $L_9^r$, $L_{10}^r$. A corollary of this is that the scale dependence is not important either. (2) The theories which we are exploring yield predictions for the tree-level coefficients. Given these comments made in case (1) we feel that the use of the coefficients at the tree level is most illuminating.

Before discussing specific theoretical models for $L_9$, $L_{10}$, we first examine the relationships between these experimental parameters which follow strictly from chiral symmetry. One of these—the reason for the vanishing of $\alpha_E+\beta_M$ in chiral theories—has already been noted. The correction between $r_A/h_V$, $\langle r_\pi^2\rangle$ and between $\alpha_E$, $h_A/h_V$ required in chiral theories can also easily be shown to be results of current-algebra–PCAC (partial conservation of axial-vector current) limits. We have, e.g.,

$$(p-q)^\nu M_{\mu\nu}(p,q)=\int d^4x\, e^{i(p-q)\cdot x}\langle 0|T[\partial^\mu A_\nu^-(x)V_\mu^{\text{em}}(0)]|\pi^+(p)\rangle-\langle 0|A_\mu^-(0)|\pi^+(p)\rangle$$
$$\simeq-\sqrt{2}F_\pi p_\mu+\sqrt{2}F_\pi m_\pi^2\frac{1}{m_\pi^2-(p-q)^2}T_\mu(p-q,p) \tag{40}$$

which yields

$$r_A=\sqrt{2}F_\pi\tfrac{1}{3}\langle r_\pi^2\rangle\ . \tag{41}$$

Also

$$\lim_{p_2\to 0}A_{\mu\nu}(p_1,q_1,q_2)=\frac{i}{\sqrt{2}F_\pi}[M_{\mu\nu}(q_1,p_1)+M_{\nu\mu}(q_2,p_1)] \tag{42}$$

which requires

$$\gamma_\pi=\frac{h_A}{\sqrt{2}F_\pi}\ . \tag{43}$$

The predictions quoted in Eq. (39) then follow from the use of the SU(2) relation[24]

$$h_V=\frac{1}{4\sqrt{2}\pi^2F_\pi} \tag{44}$$

which relates the vector structure constant in radiative pion decay to the $\pi^0\to\gamma\gamma$ amplitude given by the anomaly. The strictures

$$\alpha_E=\frac{\alpha}{8\pi^2m_\pi F_\pi^2}\frac{h_A}{h_V},\quad \frac{r_A}{h_V}=\tfrac{8}{3}\pi^2F_\pi\langle r_\pi^2\rangle \tag{45}$$

then must arise in *any* chirally invariant theory. Previous chiral evaluations which independently calculate $h_A$ *and* the polarizability or $r_A$ *and* the charge radius are redundant.

On the experimental side, besides the requirement $\alpha_E=-\beta_M$ which is well satisfied, we note that[17]

$$\left.\frac{r_A}{h_V}\right|^{\text{expt}}=2.3\pm0.6 \tag{46}$$

is in good agreement with

$$\left.\frac{8\pi^2}{3}F_\pi^2\langle r_\pi^2\rangle\right|^{\text{expt}}\simeq 2.6\ . \tag{47}$$

However,[19]

$$\alpha_E^{\text{expt}}=(6.8\pm1.4)\times10^{-4}\ \text{fm}^3 \tag{48}$$

is more than a factor of 2 larger than

$$\left.\frac{\alpha}{8\pi^2m_\pi F_\pi^2}\frac{h_A}{h_V}\right|^{\text{expt}}=2.8\times10^{-4}\ \text{fm}^3\ . \tag{49}$$

This disturbing discrepancy represents a serious violation of the chiral prediction and we urge an experimental effort to remeasure the polarizability. Indeed the existence of such a large violation would suggest a significant breaking of chiral symmetry required by the validity of QCD. Incidentally, loops are of no help here. Inclusion of final-state $\pi\pi$ and $K\bar{K}$ effects in $\gamma\gamma\pi\pi$ yield[25]

$$\alpha_E=\frac{\alpha}{8\pi^2m_\pi F_\pi^2}\frac{h_A}{h_V}-\frac{\alpha}{m_\pi}\frac{1}{(4\pi F_\pi)^2}\left[\frac{3}{2}+\frac{m_\pi^2}{t}F\left[\frac{t}{m_\pi^2}\right]+\frac{1}{2}\frac{m_K^2}{t}F\left[\frac{t}{m_K^2}\right]\right], \tag{50}$$

where

$$t=(q_1+q_2)^2 \tag{51}$$

is the momentum transfer and, for $x<0$,

$$F(x)=-4\left[\operatorname{arcsinh}\left[\frac{|x|}{4}\right]^{1/2}\right]^2. \tag{52}$$

Note that, in the limit of small $t$,

$$\frac{m^2}{t}F\left[\frac{t}{m^2}\right]\to-1-\frac{t}{12m^2}+\cdots\ , \tag{53}$$

so that

$$\alpha_E-\frac{\alpha}{8\pi^2m_\pi F_\pi^2}\frac{h_A}{h_V}\underset{t\to0}{\longrightarrow}0 \tag{54}$$

consistent with the current-algebra–PCAC constraint. Higher-order contributions are proportional to $t/(4\pi F_\pi)^2\ll1$ and yield more complex angular dependence not seen in the data.

Having examined the experimental state of relations *between* pionic properties required by chiral symmetry, we are prepared to touch base with previous calculations of the absolute size of such effects, which is done in the next section. We shall proceed by relating the various theories to the corresponding low-energy parameters $L_i^{(r)}$. This shows that many of the model calculations of individual reactions are redundant, being merely the recalculation of the same low-energy parameter in different reactions and allows us to understand in a simple fashion the classes of predictions which appear in the literature. In addition we can compare the results with the phenomenological analysis of the previous section in order to select the most realistic of these theoretical models.

## IV. PREDICTIONS AT ORDER $E^{(4)}$

Many theoretical attempts have been made to calculate properties of the pseudoscalar mesons which arise at order $E^{(4)}$, such as the direct radiative pion decay, the pion charge radius, and polarizability which were discussed in the previous section. Different procedures, each of which claims to follow from chiral symmetry, are found to yield quite different predictions, and it is a confusing matter to understand how results of these various "chiral" calculational schemes, e.g., (i) chiral quark model, (ii) chiral field theory, (iii) current-algebra sum rules, (iv) effective Lagrangian, (v) $\sigma$ model, (vi) vector dominance, (vii) leading nonanalytic corrections (chiral logs), etc., are interrelated. Nevertheless, that is our goal in this work.

As emphasized above, *all* chiral-symmetric models, regardless of their origin must agree at $O(L^{(2)})$, i.e., order $p^2$, $m^2$, since the form of $L^{(2)}$ is required in order to reproduce the free pseudoscalar Lagrangian. However, alternative models can have very different results at $O(L^{(4)})$ and these differences can be exploited in order to choose between such models. A simple example is the familiar linear $\sigma$ model, which can be written in the form[26]

$$L=\tfrac{1}{4}\mathrm{tr}\partial_\mu\Sigma\partial^\mu\Sigma^\dagger+\frac{\mu^2}{4}\mathrm{tr}\Sigma^\dagger\Sigma-\frac{\lambda}{16}(\mathrm{tr}\Sigma^\dagger\Sigma)^2\ , \tag{55}$$

where

$$\Sigma\equiv\sigma+i\tau\cdot\pi\ . \tag{56}$$

The vacuum state is

$$\langle\Sigma\rangle=\langle\sigma\rangle=\left[\frac{\mu^2}{\lambda}\right]^{1/2}\equiv v \tag{57}$$

and defining

$$\Sigma=(v+s)\exp\left[i\frac{1}{v}\tau\cdot\pi\right]\equiv(v+s)U \tag{58}$$

we determine

$$L=\tfrac{1}{4}(v+s)^2\mathrm{Tr}\partial_\mu U\partial^\mu U^\dagger+\tfrac{1}{2}(\partial_\mu s\partial^\mu s-2\mu^2s^2)-\lambda vs^3-\frac{\lambda}{4}s^4\ . \tag{59}$$

The theory then contains massless pions described by the nonlinear representation $U$ together with scalar mesons of mass

$$m_s=\sqrt{2}\mu\ . \tag{60}$$

Identifying $F_\pi=v$ we see that the form of $L^{(2)}$ is as required. At an energy scale $E\ll m_s$, such that creation of $s$ particles is not permitted, the scalars still affect the theory through *virtual* processes. To lowest order, this arises from the linear coupling

$$L_{\mathrm{int}}=\frac{v}{2}(\mathrm{Tr}\partial_\mu U\partial^\mu U^\dagger)s \tag{61}$$

which yields, due to the scalar-exchange diagram, a contribution to $L^{(4)}$,

$$L^{(4)}=\frac{F_\pi^2}{4m_s^2}(\mathrm{Tr}\partial_\mu U\partial^\mu U^\dagger)^2\ . \tag{62}$$

In such a model then $L_1$ is nonvanishing but there is no contribution to $L_2$ or $L_3$, and this pattern, were it to be confirmed experimentally, would act as a confirmation of this particular theoretical picture. The linear $\sigma$ model also predicts, however, $L_9=L_{10}=0$ at the tree level in contradiction to the expermental results of Eq. (38). Of course, our presentation here is simplified. A realistic discussion requires also the inclusion of pion and scalar loop effects and one can also rule out the full $\sigma$ model's predictions for $L_1,L_2,L_3$, from the amplitudes in $\pi\pi$ scattering.[21] However, the main point which we wish to make is that *any* such chirally symmetric model can be completely characterized in terms of its order-4 chiral expansion coefficients—the predicted values of these coefficients provide a complete and accurate representation of the predictions of the theory.

The theories applied to various rare pion transitions can be characterized in the following five basic groups.[27]

(i) Chiral quark model. In this approach one considers the pion to be a simple $q\bar{q}$ bound state and evaluates its properties in a simple single quark loop approximation. Calculations have yielded[28]

$$\langle r_\pi^2\rangle=\frac{3}{4\pi^2F_\pi^2},\quad \alpha_E=-\beta_M=\frac{\alpha}{8\pi^2m_\pi F_\pi^2}\ ,\qquad h_A=h_V=\frac{1}{4\pi^2\sqrt{2}F_\pi}\ . \tag{63}$$

(ii) Chiral baryon field theory. In this model one takes the pion as a fundamental field which is coupled to the nucleon in a chirally invariant fashion. Calculations in this case have given[29]

$$\langle r_\pi^2\rangle=\frac{g_A^2}{4\pi^2F_\pi^2},\quad \alpha_E=-\beta_M=\frac{\alpha g_A^2}{24\pi^2m_\pi F_\pi^2}\ ,\qquad \frac{3}{g_A}h_A=h_V=\frac{g_A}{4\pi^2\sqrt{2}F_\pi}\ . \tag{64}$$

(iii) Linear $\sigma$ model. In this approach, as outlined above, one considers the pion to be grouped together with a heavier scalar meson, the $\sigma$, in a chirally invariant fashion. Here we consider the meson sector alone. Often the linear $\sigma$ model is used with fermion fields in addition, but we do not study this case. Calculations then yield[2]

$$\langle r_\pi^2\rangle=\frac{1}{16\pi^2F_\pi^2}\left[\ln\frac{m_\sigma^2}{m_\pi^2}-\frac{11}{6}\right],$$
$$\alpha_E=-\beta_M=-\frac{\alpha}{24\pi^2 m_\pi F_\pi^2},\quad h_A=\frac{-1}{12\pi^2\sqrt{2}F_\pi},\quad h_V=0\ . \tag{65}$$

(iv) Current-algebra–vector dominance: In this picture one uses current-algebra–PCAC techniques to yield exact sum rules, which are then approximated using a vector–axial-vector dominance assumption. Calculations have found, in this approach,[30]

$$\langle r_\pi^2\rangle=\frac{6}{m_\rho^2},\quad \alpha_E=-\beta_M=\frac{\alpha}{m_\pi m_A^2},$$
$$h_A=\sqrt{2}F_\pi\frac{1}{m_A^2},\quad h_V=\frac{1}{4\pi^2\sqrt{2}F_\pi}\ . \tag{66}$$

(v) Leading nonanalytic corrections. Here one keeps only the nonanalytic terms, such as $m_\pi^2\ln m_\pi^2$ or $s\ln s$, which are found when calculating meson loops. The motivation for this is the formal dominance of corrections of order $m_\pi^2\ln m_\pi^2$ over those of order $m_\pi^2$ in the expansion in the energy and/or mass. This leads to

$$\langle r_\pi^2\rangle=\frac{-1}{16\pi^2F_\pi^2}\left[\ln\frac{m_\pi^2}{\mu^2}+\frac{1}{2}\ln\frac{m_K^2}{\mu^2}\right],$$
$$\alpha_E=-\beta_M=0,\quad h_A=0\ . \tag{67}$$

There are additional models but these are sufficient to make our point.

Analysis of the first two models is easiest to perform within the framework of effective-Lagrangian techniques which have been widely used recently in order to attempt to understand the restrictions which QCD places on effective low-energy field theories. We shall outline here the approach due to Balog,[31] but other workers have obtained identical results.[32] We consider the state obtained from the vacuum by means of a local chiral transformation on the quark fields and interpret the energy difference between this state and the vacuum as the low-energy effective action. In this way we find

$$\exp[iW(\Phi)]=\frac{\int Dq\,D\bar{q}\exp\left[i\int d^4x\,\bar{q}\exp\left[i\frac{1}{F_\pi}\lambda\cdot\Phi\gamma_5\right]\nabla\!\!\!/\,\exp\left[i\frac{1}{F_\pi}\lambda\cdot\Phi\gamma_5\right]q\right]}{\int Dq\,D\bar{q}\exp\left[i\int d^4x\,\bar{q}\nabla\!\!\!/\,q\right]}, \tag{68a}$$

where

$$\nabla\!\!\!/=\partial\!\!\!/-i\frac{1}{2}L\!\!\!/(1+\gamma_5)-i\frac{1}{2}R\!\!\!/(1-\gamma_5) \tag{68b}$$

and

$$L_\mu,R_\mu=V_\mu\pm A_\mu \tag{68c}$$

represent external flavor gauge fields. Performing the functional integration using a heat kernal expansion[31] (and keeping up to the $a_2$ coefficient) one finds

$$L^{(4)}=L_{\text{anomaly}}+\frac{N_c}{384\pi^2}\text{Tr}[-4(D_\mu UD^\mu U^\dagger)^2+2D_\mu U^\dagger D_\nu UD_\mu U^\dagger D_\nu U+4(D_\mu D_\mu U^\dagger)(D_\nu D_\nu U)+4U^\dagger L_{\mu\nu}UR_{\mu\nu}$$
$$+8(R_{\mu\nu}D_\mu U^\dagger D_\mu U+L_{\mu\nu}D_\mu UD_\nu U^\dagger)]+\cdots, \tag{69}$$

where $N_c$ is the number of colors, $L_{\text{anomaly}}$ is the anomalous piece of the Lagrangian as given by Witten, and the remaining $O(E^{(4)})$ terms can be put into the form given in Eq. (11) with the identification[31]

$$8L_1=4L_2=-2L_3=L_9=-2L_{10}$$
$$=\frac{N_c}{48\pi^2}=\frac{1}{16\pi^2}\quad\text{for } N_c=3\ . \tag{70}$$

Examination of Eq. (67) suggests that integrating out the quark fields corresponds directly to the simple chiral-quark-model calculations performed in earlier times. This identification is further secured by noting that

$$\frac{h_A}{h_V}=32\pi^2(L_9+L_{10})=1\ ,$$
$$\langle r_\pi^2\rangle=\frac{12L_9}{F_\pi^2}=\frac{3}{4\pi^2F_\pi^2} \tag{71}$$

as given by direct calculation in such models. We observe that the predicted charge radius, $\langle r_\pi^2\rangle^{\text{theo}}\approx0.33$ fm$^2$, is somewhat too small and the ratio $h_A/h_V$ somewhat too large in this chiral quark model.

Some derivations of Eq. (69) claim that gluonic effects have also been taken into account in these coefficients.

We would like to devote a rather technical aside on why this is not the case. This paragraph is intended mainly for theorists who have experience with the heat-kernel expansion. Consider first the functional integral over the fermion variables, prior to the gluonic integration. This result has been calculated in terms of a set of operators of increasing dimensions. These operators can contain gluon and/or chiral fields. Schematically one might give examples of terms in the expansion in the following manner:

$$W_{\text{eff}}(F_{\mu\nu},U)=\alpha_1+\alpha_2F^i_{\mu\nu}F^{i\mu\nu}+\alpha_3(\text{Tr}D_\mu U D^\mu U^\dagger)^2 +\alpha_4F^i_{\mu\nu}F^{i\mu\nu}[\text{Tr}(D_\mu U D^\mu U^\dagger)]^2+\cdots . \quad (72)$$

The $\alpha_3$ and $\alpha_2$ terms are of a dimension (i.e., dim=4) that they could be found in the heat-kernel expansion in the $a_2$ coefficient. The $\alpha_3$ term is of the form given in Eq. (69). In contrast the $\alpha_4$ term has dimension=8 and could only appear in the $a_4$ coefficient, which is not included because it is beyond present calculational capabilities. The $\alpha_4$ operator is given pictorially in Fig. 1(c), involving two gluons plus the chiral field, while the $\alpha_2$ operator is shown in Fig. 1(b). Yet higher-order operators with more gluon fields are also pictured. Next consider the functional integral over the gluonic degrees of freedom. This removes the gluonic operators and we are left with an expansion in the chiral field

$$W_{\text{eff}}(U)=\beta_1+\beta_2[\text{Tr}(\partial_\mu U\partial^\mu U^\dagger)]^2+\cdots . \quad (73)$$

Now the coefficient $\beta_2$ will contain the effect of $\alpha_3$ and also that of $\alpha_4$. Pictorially these give the gluonic corrections pictured in Fig. 2. The result in Eq. (69) is obtained from only the free quark loop Fig. 2(a), as the gluonic portions were contained in the part of the expression which was not calculated. For the anomaly we have the Adler-Bardeen theorem to guarantee that gluonic corrections do not modify the result. However, no such theorem applies to the remaining portion of the effective Lagrangian. In QCD we really need to include the full set of gluonic diagrams. This explanation makes clear why the sophisticated heat-kernel evaluation of Balog and others yields the same coefficients as the simple chiral quark model: They both represent simply a free quark loop calculation.

An identical procedure to that given in Eq. (69) can be used to yield the results of chiral baryon field-theory calculations. In this case, however, $N_c$ must be set equal to

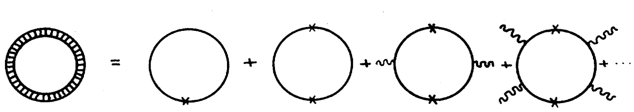


FIG. 1. Schematic picture of some of the contributions to the full fermion determinant obtained by integrating out the quark fields coupled to gluons and to a background chiral field. The × indicates a factor of a chiral field in the resulting action, while a wavy line indicates a factor of a gluon field.

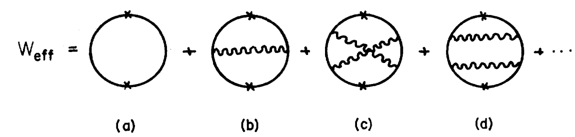


FIG. 2. Schematic picture of some contributions to the effective action obtained after integrating over the gluon fields. (a) is a free quark loop which yields the results obtained from the $a_2$ coefficient in the heat-kernal expansion. (b), (c), (d) are gluonic corrections to this.

unity and an SU(2) charge matrix

$$Q=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad (74)$$

is employed. Also we include the axial-vector coupling constant $g_A$. We find then

$$8L_1=4L_2=-2L_3=L_9=-2L_{10}=\frac{g_A^2}{48\pi^2} . \quad (75)$$

Likewise the coefficient preceding $L_{\text{anomaly}}$ is smaller than in the chiral quark model by a factor of $g_A/3$. It is perhaps surprising then that the $\pi^0\to\gamma\gamma$ decay amplitude is essentially the same in the chiral field theory and chiral quark models. This results obtains, however, because the value of

$$N_c\text{Tr}\lambda_3Q^2 , \quad (76)$$

to which the amplitude is proportional, is equal to unity in both models. The same "miracle" does not occur in evaluation of the polarizability and axial radiative decay since these results involve

$$N_c\text{Tr}\Phi[Q,\Phi] \text{ or } N_c\text{Tr}[Q,\Phi][Q,\Phi] \quad (77)$$

which differ by a factor of $N_c$. Thus, we predict, in such a model,

$$\frac{h_A}{h_V}=\frac{32\pi^2g_A}{3}(L_9+L_{10})=\frac{g_A}{3},\quad \langle r_\pi^2\rangle=\frac{4L_9}{F_\pi^2}=\frac{g_A^2}{4\pi^2F_\pi^2} \quad (78)$$

which agrees with the results of the model calculations. It has often been asserted that the agreement between chiral quark model and chiral field theory calculations cited above is "accidental." This seems clearly to be the case since the chiral coefficients $L_1,L_2,L_3$, which determine $\pi$-$\pi$ scattering, for example, differ by a factor 3. We observe that the predicted pion charge radius is *much* too small. The ratio $h_A/h_V$ is in good agreement with the level measured experimentally. However, this must be regarded as accidental.

The basic structure of the $\sigma$ model has been quoted above. At tree level only the $\sigma$-pole diagram contributes, while loop effects have been evaluated by Gasser and Leutwyler.[7] We observe that the structure in this case is completely ruled out by experiment. Indeed the charge radius is too small. Also the axial-vector to vector structure constant $h_A$ and the pion polarizability $\alpha_E$ are found to have the wrong sign.

In the model which we label as vector dominance, one integrates out the effects of the vector and axial-vector mesons, in a similar fashion to that performed for the scalar meson in Eq. (62) producing an effective chiral Lagrangian of the form Eq. (11) but now with[26,33]

$$L_1=\tfrac{1}{2}L_2=-\tfrac{1}{6}L_3=6\pi\frac{\Gamma_V F_\pi^4}{m_V^5}\,,$$

$$L_9=\frac{F_V G_V}{2M_V^2},\quad L_{10}=\frac{F_A^2}{4M_A^2}-\frac{F_V^2}{4M_V^2}\,. \tag{79}$$

Here $G_V\approx 60$ MeV is the $\rho\pi\pi$ coupling while $F_V$, $F_A$ are the constants of proportionality between vector, axial-vector currents, and the corresponding meson fields. We find

$$F_V\approx 150 \text{ MeV} \tag{80}$$

while $F_A$ is given by the Weinberg sum rule[34]

$$F_V^2=F_A^2+F_\pi^2\,. \tag{81}$$

The charge radius then becomes

$$\langle r_\pi^2\rangle=6\frac{1}{M_V^2}\frac{F_V G_V}{F_\pi^2} \tag{82}$$

while $h_A/h_V$ is given by

$$h_A/h_V=32\pi^2\left[\frac{F_\pi^2}{12}\langle r_\pi^2\rangle+\frac{F_A^2}{4M_A^2}-\frac{F_V^2}{4M_V^2}\right]. \tag{83}$$

These expressions may not appear familiar in this form. However, vector dominance demands $F_V G_V=F_\pi^2$ so that the charge radius assumes its expected form,

$$\langle r_\pi^2\rangle^{\rm VD}\simeq\frac{6}{M_V^2}\approx 0.39 \text{ fm}^2\,, \tag{84}$$

in good agreement with its experimentally measured value. Also, by use of current-algebra–PCAC we can relate $h_A$ to the difference of vector and axial-vector two-point functions:

$$\lim_{p\to 0}M_{\mu\nu}(q,p)=\sqrt{2}F_\pi q_\mu q_\nu\frac{1}{q^2-m_\pi^2}-\sqrt{2}F_\pi g_{\mu\nu}+(\sqrt{2}F_\pi\tfrac{1}{3}\langle r_\pi^2\rangle-h_A)(q_\mu q_\nu-g_{\mu\nu}q^2)$$
$$=\frac{1}{\sqrt{2}F_\pi}\int d^4x\, e^{iq\cdot x}\langle 0|T[2V_\mu^3(x)V_\nu^3(0)-A_\mu^+(x)A_\nu^-(0)]|0\rangle\,. \tag{85}$$

Use of vector–axial-vector dominance for the vacuum expectation value gives

$$\lim_{p\to 0}M_{\mu\nu}(q,p)\simeq\frac{-\sqrt{2}}{F_\pi}\left[F_V^2\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{m_V^2}\right]-F_A^2\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{m_A^2}\right]-F_\pi^2 q_\mu q_\nu\frac{1}{q^2-m_\pi^2}\right]. \tag{86}$$

We find then from the coefficients of

$$\text{(i) } g_{\mu\nu}:\ F_V^2-F_A^2=F_\pi^2\,, \tag{87}$$

which is the Weinberg sum rule, and of

$$\text{(ii) } q_\mu q_\nu:\ h_A=\sqrt{2}F_\pi\left[\tfrac{1}{3}\langle r_\pi^2\rangle-\left[\frac{F_V^2}{m_V^2}-\frac{F_A^2}{m_A^2}\right]\frac{1}{F_\pi^2}\right], \tag{88}$$

which corresponds to the result given in Eq. (83) and is a sum rule first given by Das, Mathur, and Okubo.[35] Using the vector-dominance approximation and the Weinberg sum rule we find then

$$h_A=\sqrt{2}F_\pi\left[\frac{2}{m_V^2}-\frac{F_V^2}{F_\pi^2}\left[\frac{1}{m_V^2}-\frac{1}{m_A^2}\right]-\frac{1}{m_A^2}\right]. \tag{89}$$

Using the approximate result[36] $F_V^2/F_\pi^2\approx 2$ we find

$$h_A=\sqrt{2}F_\pi\frac{1}{m_A^2} \tag{90}$$

or

$$\frac{h_A}{h_V}=8\pi^2\frac{F_\pi^2}{m_A^2} \tag{91}$$

which, for an axial meson with the reasonable mass value $m_A\sim 1260$ MeV yields[37]

$$\left.\frac{h_A}{h_V}\right|^{\rm theo}\approx 0.43 \tag{92}$$

in good agreement with experiment.

Finally we include an approximation based on the loop structure of chiral perturbation theory. In an expansion in the energy, a term of order $m_\pi^2\ln m_\pi^2/\mu^2$ or $q^2\ln m_\pi^2/\mu^2$ is technically larger than one of order $m_\pi^2$ or $q^2$, if $\mu$ is chosen at some fixed hadronic scale, i.e., $\mu\sim m_\rho$ or 1 GeV. Such terms arise when meson loop effects are calculated. An approximation which is widely used is to keep only these nonanalytic contributions and to disregard all of the polynomical terms. This is effectively the same as setting $L_i=0$. How well does this work in practice? A typical example is the pion charge radius, where chiral logs would predict

TABLE I. Tabulated are various $O(L^{(4)})$ properties of the pion as predicted in various chiral models and as measured experimentally. Note that only the vector-dominance picture is uniformly successful.

| | Quark loop | Chiral field theory | Vector dominance | Linear $\sigma$[a] | Chiral logs | Expt |
|---|---|---|---|---|---|---|
| $\langle r_\pi^2\rangle$ (fm$^2$) | 0.33 | 0.17 | 0.39 | 0.05 | 0.12 | 0.44±0.01 |
| $h_V(m_\pi^{-1})$ | 0.026 | 0.033 | 0.026 | 0.026[a] | 0.026 | $0.029^{+0.019}_{-0.014}$ |
| $h_A/h_V$ | 1 | 0.41 | 0.43 | −0.33[a] | 0 | 0.46±0.02 |
| $r_A/h_V$ | 1.9 | 1.0 | 2.3 | 0.3 | 0.7 | 2.3±0.6 |
| $\alpha_E+\beta_M$ ($10^{-4}$ fm$^3$) | 0 | 0 | 0 | 0 | 0 | 1.4±3.1 |
| $\alpha_E$ ($10^{-4}$ fm$^2$) | 6.1 | 2.5 | 2.6 | −2.0 | 0 | 6.8±1.4 |

[a]In the linear $\sigma$ model, we have used $m_\sigma=0.7$ GeV and have used Eq. (44) for $h_V$ even though the anomaly is strictly not present in the mesonic sector of the $\sigma$ model.

$$\langle r_\pi^2\rangle=\frac{-1}{16\pi^2F_\pi^2}\left[\ln\frac{m_\pi^2}{\mu^2}+\frac{1}{2}\ln\frac{m_K^2}{\mu^2}\right]\approx\begin{cases}0.12\ \text{fm}^2, & \mu=1\ \text{GeV},\\ 0.06\ \text{fm}^2, & \mu=\frac{1}{2}\ \text{GeV},\end{cases}\tag{93}$$

which is very much smaller than the experimental result. From our previous discussion it is clear that this must fail for $r_A/h_V$ also. In the case of $h_A/h_V$ and $\alpha_E$, the leading-log approximation predicts

$$h_A=0,\quad \alpha_E=0\tag{94}$$

again in conflict with the data. It should also be mentioned that the leading nonanalytic corrections to $\pi\pi$ scattering have been studied[37] and are also in disagreement with the data, being too small and not effective in the right channels. Overall we conclude that the leading nonanalytic term approximation has very little in common with the real world. Rather the reverse seems true; i.e., the tree-level coefficients $L_i$ are dominant and the effects of chiral logs are small.

Our results for chiral calculations of pionic properties are summarized in Table I. It is clear that only the vector–axial-vector dominance picture gives overall a satisfactory representation of experimental results (with the exception of the pion polarizability which urgently needs to be remeasured, as noted above). This point is made even more vividly by comparing the calculated chiral expansion parameters with values determined empirically by Gasser and Leutwyler. Strictly speaking, this comparison is somewhat ambiguous since the empirically determined values are *not* fundamental constants but rather are determined at a given mass scale $\mu$, because of the renormalization-point dependence arising from pion loop corrections. Nevertheless, as noted above, the scale dependence and residual effects from loops are not large and one can use the overall agreement as a gauge of how well a given theoretical picture is able to represent experimental reality. A summary is given in Table II. We observe that both the chiral quark model and vector dominance give an approximate picture of experimental reality. However, the vector-dominance calculation is in much better agreement with the results of pion radiative decay and the pion charge radius, as already noted. Chiral field theory predictions are consistently a factor of 3 too small, associated with the lack of color degrees of freedom, while the linear $\sigma$ model or chiral logs simply do not provide an adequate representation of nature.

## V. CONCLUSION

We have noted that higher-order effects in models possessing chiral symmetry can be expressed in terms of a small number of phenomenological parameters $L_1,\ldots,L_{10}$. In particular we examined a range of pion interactions, involving the charge radius, radiative decay, and polarizability, and found that all could be understood in terms of only $L_9$ and $L_{10}$, requiring a relationship between some of these experimental quantities in chiral theories. At the present time a discrepancy exists between the measured and predicted pion polarizability, and remeasurement of $\alpha_E$ is strongly suggested. Such chiral parameters also offer a concise and useful way to characterize theoretical calculations of high-order chiral effects. In particular, we noted that previous quark loop calculations can be characterized by the use of a recently calculated effective chiral Lagrangian. A similar Lagrangian is found to represent previous calculations in so-called chiral field theory. The former is found to offer a semiquantitative but hardly precise description of ex-

TABLE II. Listed are values of the chiral expansion parameters $L_i$ as calculated in various chiral models and as measured experimentally. All should be multiplied by a factor of $10^{-3}$. Note that only the vector-domance picture is uniformly successful.

| | Quark loop | Chiral field theory | Vector dominance | Linear $\sigma$ | Empirical |
|---|---|---|---|---|---|
| $L_1+\frac{L_3}{2}$ | −0.8 | −0.4 | −2.1 | −0.5 | −2.3±0.5 |
| $L_2$ | 1.6 | 0.8 | +2.1 | 1.5 | 2.0±0.5 |
| $L_9$ | 6.3 | 3.3 | 7.3 | 0.9 | 6.9±0.2 |
| $L_{10}$ | −3.2 | −1.7 | −5.8 | −2.0 | −5.2±0.3 |

perimental results, while the latter is too small by a factor of 3. Satisfactory agreement *is* found, however, in a picture based upon vector dominance.

Very often the linear $\sigma$ model or the leading nonanalytic approximation are used to study some consequences of chiral symmetry. This is tempting because they offer simple and easily manageable theories. However, it is clear from the comparison with data that they bear little relation to reality. They are not a good representation of low-energy QCD, and consequences derived from them are suspect.

The pattern found here agrees well with the results of an analysis of $\pi\pi$ scattering. There also,[26] the vector-meson picture provides an excellent description of the low-energy constants involved, in that case $L_1$ and $L_2$. The scale dependence and effects of loops are greater in that analysis, but the comparison was made to a tree-level fit to the data. The chiral quark model predicts both $L_1$ and $L_2$ too small, with the ratio $L_1/L_2$ differing by a factor of 2 from the fit. In analogy with the work of the present paper, the chiral field theory model results would be a factor of 3 below those of the quark loop. It is perhaps not a surprise that vector-dominance calculations of the low-energy constants work so well. After all, vector dominance has long been a successful phenomenological tool, even if its fundamental origin is not completely clear. What is interesting about recent works on the origins of effective Lagrangians is that they extend vector-dominance ideas to the rigorous techniques of chiral symmetry.

Further tests of this picture are also suggested. Indeed for any of the higher-order properties of the pion discussed above, there exist corresponding parameters describing the structure of the charged kaon, and only some of these have been measured. The polarizability of the neutral pion and kaon are also predicted unambiguously in chiral models in terms of (finite) meson loop contributions. Also worthy of study is a more direct connection of the vector-dominance–chiral-Lagrangian union with QCD.

## ACKNOWLEDGMENTS

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