Direct experimental reconstruction of the *pp* elastic scattering amplitudes between 447 and 579 MeV

R. Hausammann,* E. Heer, R. Hess, C. Lechanoine-Leluc, W. R. Leo,† Y. Onel,‡ and D. Rapin
Département de Physique Nucléaire et Corpusculaire, Université de Genève, CH-1211 Genève 4, Switzerland
(Received 31 October 1988)

A direct experimental reconstruction of the five complex *pp* elastic-scattering amplitudes has been performed at 447, 497, 517, 539, and 579 MeV. The reconstruction is done over the c.m. angles from 38° to 90° and is based on either 11 or 15 spin observables depending on the angular range. The reconstructed amplitudes are presented and compared to phase-shift analysis. A smooth energy behavior is observed for the amplitudes.

I. INTRODUCTION

We report here on a direct, model-independent, experimental reconstruction of the *pp* elastic-scattering matrix M separately at five energies and 14 c.m. angles between 38° and 90° from 11 to 15 different spin observables measured at SIN [now the Paul-Scherrer-Institute (PSI)] with a polarized beam and target. These observables consist of the spin correlation parameters A_{00nn} (Ref. 1) A_{00kk}, A_{00ss}, and A_{00sk} (Ref. 2) between $38°_{\text{c.m.}}$ and $90°_{\text{c.m.}}$, the polarization parameter P_{n000} and the two-spin polarization and three-spin tensors D_{n0n0}, K_{n00n}, $D_{s'0s0}$, $D_{s'0k0}$, $M_{s'0sn}$, and $M_{s'okn}$ (Refs. 3 and 4), measured about $90°_{\text{c.m.}}$ from $34°_{\text{c.m.}}$ to $118°_{\text{c.m.}}$. The 579-MeV reconstruction, between $66°_{\text{c.m.}}$ and $90°_{\text{c.m.}}$, has already been published[5] and constitutes the first direct amplitude reconstruction ever made for *pp* scattering. This paper extends the angular domain of the 579-MeV data down to $38°_{\text{c.m.}}$ and gives the final values obtained after more refined studies of possible systematic uncertainties as explained in detail in Ref. 4. It also presents the same direct reconstruction at four other energies: namely, 447, 497, 517, and 539 MeV. It is also meant to detail the formalism and the fitting procedures used in the analysis (see Ref. 6 for full information).

II. FORMALISM

We have followed the formalism developed in Ref. 7, since it is extremely well developed and detailed: all observables are explicitly evaluated and given therein along with the transformations to equivalent formalisms. The *pp* elastic-scattering matrix, assuming parity and time-reversal invariance, is given by

$$\begin{aligned} M(E,\theta)=\tfrac{1}{2}\{&(a+b)+(a-b)(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})\otimes(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})\\ &+(c+d)(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{m}})\otimes(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})\\ &+(c-d)(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{l}})\otimes(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{l}})\\ &+e[(\boldsymbol{\sigma}_1\otimes 1_2+1_1\otimes\boldsymbol{\sigma}_2)\cdot\hat{\mathbf{n}}]\}\ . \end{aligned} \tag{1}$$

Here the amplitudes a, b, c, d, and e are complex functions of two variables, i.e., the center-of-mass-system (c.m.s.) energy E and scattering angle θ. The c.m.s. basis vectors are

$$\hat{\boldsymbol{l}}=\frac{\hat{\mathbf{k}}_f+\hat{\mathbf{k}}_i}{|\hat{\mathbf{k}}_f+\hat{\mathbf{k}}_i|},\quad \hat{\mathbf{m}}=\frac{\hat{\mathbf{k}}_f-\hat{\mathbf{k}}_i}{|\hat{\mathbf{k}}_f-\hat{\mathbf{k}}_i|},\quad \hat{\mathbf{n}}=\frac{\hat{\mathbf{k}}_i\times\hat{\mathbf{k}}_f}{|\hat{\mathbf{k}}_i\times\hat{\mathbf{k}}_f|}\ , \tag{2}$$

where $\hat{\mathbf{k}}_i$ and $\hat{\mathbf{k}}_f$ are unit vectors in the direction of the incident and scattered particle momenta in the c.m.s (see Fig. 1). The spin matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ (the Pauli matrices) act on the first- and the second-nucleon wave functions, respectively.

Equation (1) is the most general expression of the elastic-scattering matrix for two identical spin-$\frac{1}{2}$ particles subject to invariance under space rotations, parity, and time reversal. In writing Eq. (1), we have assumed that the particles are identical which is strictly valid for *pp* or *nn* scattering. For *np* scattering, this assumes isotopic invariance of the nucleon-nucleon interaction. The scattering matrix for the elastic scattering of two nonidentical particles would contain an additional sixth term. For $\bar{p}p$ scattering, C conjugation plays the role of the Pauli principle in nucleon-nucleon scattering. Thus, if P, T, and C are conserved separately only the five amplitudes a, b, c, d, and e contribute.

Any observables (measured in the laboratory) can be expressed in terms of these five complex amplitudes as detailed in Ref. 7. When discussing experiments in the laboratory system (l.s.), we shall use the unit vectors

$$\hat{\mathbf{k}},\ \hat{\mathbf{k}}',\ \text{and}\ \hat{\mathbf{k}}'' \tag{3}$$

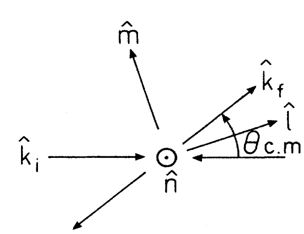

FIG. 1. Center-of-mass frame.

©1989 The American Physical Society

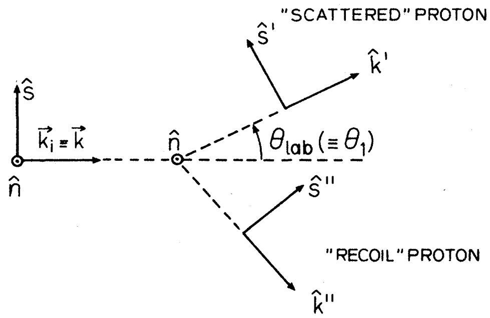

FIG. 2. Laboratory frames attached to the incident, scattered, and recoil protons.

which are in the directions of the initial, scattered, and recoil particle momenta in the l.s. ($\hat{\mathbf{k}} \equiv \hat{\mathbf{k}}_i$). Further we use the transverse vectors

$$\hat{\mathbf{s}} = \hat{\mathbf{n}} \times \hat{\mathbf{k}}, \quad \hat{\mathbf{s}}' = \hat{\mathbf{n}} \times \hat{\mathbf{k}}', \quad \hat{\mathbf{s}}'' = \hat{\mathbf{n}} \times \hat{\mathbf{k}}'' , \tag{4}$$

where $\hat{\mathbf{n}}$ is defined in Eq. (2). This is illustrated in Fig. 2. In this formalism any observable is written with four indices, X_{pqik}, which refer to the polarization directions of the scattered (p), recoil (q), beam (i), and target (k) protons, respectively.

III. WHAT IS A COMPLETE EXPERIMENT?

In 1957 Puzikov *et al.*[8] introduced the concept of the complete experiment for the case of *NN* scattering. They proposed a certain ensemble of observables to be measured which they called complete if it contained sufficient information for a complete and exhaustive description of the interaction. This is to be put in contrast with single observables, such as the differential cross section or the polarization parameter, which only provide very specific information about the process considered. Theorists introduced[9] the scattering matrix which is unfortunately not directly accessible to experiment but convenient to describe the entire interaction in all its aspects. Therefore, a complete experiment may be defined as a set of observables which allows a direct and unambiguous reconstruction of the scattering matrix.

What is the minimum number of observables needed for a set to be complete? There exists extensive literature (see references in Ref. 10 about this problem). These rather mathematical investigations find a certain final result in Ref. 10, where a completely general prescription for necessary and sufficient conditions for reactions with arbitrary spin is derived. If n is the number of independent amplitudes, it states that a complete knowledge of the scattering matrix up to an overall phase requires only $(2n-1)$ real functions, since there are $(n-1)^2$ independent nonlinear relations between the set of n^2 observables. Once the $(2n-1)$ measurements are done, the amplitudes are extracted by solving the set of $(2n-1)$ simultaneous quadratic equations in the amplitudes.

The above is in principle a complete scheme to determine the scattering matrix from experiment. Applied to the example of *pp* scattering which is described by five complex amplitudes, there are 25 linearly independent experiments which are related by 16 independent quadratic equations (see Ref. 7). Therefore, a minimum set of nine well-chosen experiments is sufficient to extract the amplitudes.

In practice, however, the above results are of rather academic interest since the situation is more complicated for the following reasons. (1) The set of available experiments is to a large extent determined by the experimental facilities. It may even happen that the sets accessible to experiments are not complete. (2) No attention has been given to the actual numerical values of the observables nor to their experimental errors, as they play a crucial role in the resolution of ambiguities. (3) Analytical solutions for the amplitudes are often carried out in the c.m. frame. But due to relativistic effects a certain number of c.m. experiments may correspond to a larger number of laboratory experiments.

For these reasons, the minimum number of experiments has to be larger than $(2n-1)$. Different approaches have been used (see Refs. 11–13) to search for complete sets of observables that are experimentally practical to measure. In the last approach, the method applied is based on a Monte Carlo simulation of possible experimental values for the observables, distributed around predicted phase-shift-analysis (PSA) values with assumed realistic experimental errors. In this way the stability of the solution for the amplitude reconstruction is tested and predictions of the errors on the amplitudes are made assuming a 1% precision on the measured asymmetries. The conclusion was that the following set of 13 observables was complete: $d\sigma/d\Omega$, P_{n000}, A_{00nn}, D_{n0n0}, K_{n00n}, $D_{s'0s0}$, $D_{s'0k0}$, $M_{s'0sn}$, $M_{s'0kn}$, $K_{0s''s0}$, $K_{0s''k0}$, $N_{0s''sn}$, $N_{0s''kn}$. It has the important practical advantage, that only a vertically polarized target is needed.

This set is in fact a subset of the one used in this amplitude reconstruction which consists of 16 observables. The additional parameters are the two-spin correlation parameters A_{00ss}, A_{00kk}, A_{00sk}. The set used consists of

$$\begin{aligned} &d\sigma/d\Omega,\ P_{n000},\ A_{00nn},\ A_{00ss},\ A_{00kk},\ A_{00sk}\ , \\ &\quad D_{n0n0},\ K_{n00n},\ D_{s'0s0},\ D_{s'0k0},\ M_{s'0sn}\ , \\ &\quad M_{s'0kn},\ K_{0s''s0},\ K_{0s''k0},\ N_{0s''sn},\ N_{0s''kn}\ . \end{aligned} \tag{5}$$

This set was shown to have various advantageous features. (1) It disposes of an elegant and simple analytical solution, even an ideal one when relativistic effects and the magnetic field of the target are neglected. (2) It contains equally distributed information for all the five amplitudes to be reconstructed with about the same precision. (3) It forms a largely overdetermined system of equations for the five amplitudes. Therefore, it allows a fit of normalization parameters which provide interesting estimates of systematic uncertainties on the measured observables (see Ref. 2). (4) This set is also complete for the reconstruction of six amplitudes including the T-violating one, if one distinguishes P_{n000} ($\equiv P_{0n00}$) from A_{00n0} ($\equiv A_{000n}$). This allows one to set an interesting upper limit for T violation (see Refs. 6 and 14).

Further studies have been undertaken in search of other complete observable sets. In view of the experimental

TABLE I. Relations between scattered parameters and their recoil counterparts.

$D_{k'0k0}(\theta)=K_{0k''k0}(\pi-\theta)$	$P(\theta)=-P(\pi-\theta)$
$M_{s'0sn}(\theta)=-N_{0s''sn}(\pi-\theta)$	$D_{n0n0}(\theta)=K_{n00n}(\pi-\theta)$
$M_{s'0kn}(\theta)=N_{0s''kn}(\pi-\theta)$	$D_{s'0s0}(\theta)=K_{0s''s0}(\pi-\theta)$
$M_{k'0sn}(\theta)=N_{0k''sn}(\pi-\theta)$	$D_{k'0s0}(\theta)=-K_{0k''s0}(\pi-\theta)$
$M_{k'0kn}(\theta)=-N_{0k''kn}(\pi-\theta)$	$D_{s'0k0}(\theta)=-K_{0s''k0}(\pi-\theta)$

problems a polarized target brings along, the question arose as to whether or not the measurement of three-spin tensors should be dropped in favor of a better precision on the two-spin transfer parameters. Tests with a set of 12 observables, containing parameters only up to two spins, gave the interesting result that a four-times better statistics can almost be balanced by the addition of even poorly measured three-spin tensors. In practice, the overdetermination of the system was shown to be of much greater importance than a high precision which is, as the experiment proved, anyway limited by systematic uncertainties. In light of the crucial role the three-spin parameters play, we finally decided to carry out this experiment with a polarized target.

IV. AMPLITUDE DECOMPOSITION

The Pauli principle requires the following symmetry properties for the five amplitudes:

$$\begin{aligned} &a(\theta)=-a(\pi-\theta), \quad b(\theta)=-c(\pi-\theta)\ , \\ &c(\theta)=-b(\pi-\theta), \quad d(\theta)=d(\pi-\theta)\ , \\ &e(\theta)=e(\pi-\theta)\ . \end{aligned} \tag{6}$$

This tells us that we need to know the amplitudes only up to $90°_{\text{c.m.}}$. In this light, measurements made above $90°_{\text{c.m.}}$ may be reinterpreted as additional information below $90°_{\text{c.m.}}$. Thus the observables determined for the "scattered" proton at $(\pi-\theta)$ can be equated to equivalent parameters for the "recoil" proton at θ. The corresponding symmetries are listed in Table I.

In this decomposition, several factors must be considered: (1) relativistic effect and (2) the effect of magnetic field of the polarized proton target (PPT) in data where the scattered proton polarization is analyzed in a polarimeter. The treatment of these effects is discussed below.

A. Relativistic effect

As detailed in Ref. 7, all observables can be expressed as functions of the five complex amplitudes, but with polarization defined along $\hat{l}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{m}}$ in the c.m. system. Since our measured parameters are defined in the l.s., one must first transform them to the c.m.s. This is the only instance where relativistic effects have to be taken into account. Note the formalism developed so far is nonrelativistic.

The relativistic rotation angles are

$$\begin{aligned} &\Omega_1=\theta-2\theta_1=2\alpha, \quad \text{i.e.,}\ \alpha=\frac{\theta}{2}-\theta_1\ , \\ &\Omega_2=-\pi+\theta+2\theta_2=-\pi+2\beta, \quad \text{i.e.,}\ \beta=\frac{\theta}{2}+\theta_2\ , \end{aligned} \tag{7}$$

for the scattered and recoil particles, respectively, where θ is the c.m.s. scattering angle, and θ_1 and θ_2 are the l.s. scattering and recoil angles. The nonrelativistic case corresponds to

$$\alpha=0, \quad \beta=\frac{\pi}{2}\ . \tag{8}$$

All angles and vectors involved for the scattered proton are illustrated in Fig. 3. One sees that the final polarization vector in the l.s., k'_R, is seen from the c.m.s. as being rotated around $\hat{\mathbf{n}}$ by the additional Wigner angle Ω_1. Therefore, the relativistic correction to be applied to the laboratory components attached to the scattered particle is expressed by the rotation around $\hat{\mathbf{n}}$:

$$\begin{aligned} &\hat{\mathbf{k}}'_R=\hat{l}\cos\alpha+\hat{\mathbf{m}}\sin\alpha\ , \\ &\hat{\mathbf{s}}'_R=-\hat{l}\sin\alpha+\hat{\mathbf{m}}\cos\alpha, \quad \hat{\mathbf{n}}_R=\hat{\mathbf{n}}\ . \end{aligned} \tag{9}$$

In an analogous way, the correction for the recoil proton is given by

$$\hat{\mathbf{k}}''_R=-\hat{l}\cos\beta-\hat{\mathbf{m}}\sin\beta, \quad \hat{\mathbf{s}}''_R=\hat{l}\sin\beta-\hat{\mathbf{m}}\cos\beta\ . \tag{10}$$

In this way one may express all laboratory parameters defined along $(\hat{\mathbf{s}},\hat{\mathbf{n}},\hat{\mathbf{k}})$, $(\hat{\mathbf{s}}',\hat{\mathbf{n}},\hat{\mathbf{k}}')$, and $(\hat{\mathbf{s}}'',\hat{\mathbf{n}},\hat{\mathbf{k}}'')$ in terms of the c.m. components $(\hat{l},\hat{\mathbf{n}},\hat{\mathbf{m}})$ which is the system in which M is expressed. In Table II all used observables are listed.

B. Effect of the PPT magnetic field

This effect can be summaried as follows: at the carbon scatterer in the polarimeter, where the scattered particle polarization is analyzed, only transverse-polarization components are determined. Because of the magnetic field which precesses the proton spin, the transverse components at the carbon scatterer are no longer the same as at the interaction vertex in the PPT. It has been shown[4] that the magnetic field effect can be treated as a rotation around the vertical axis $\hat{\mathbf{n}}$ by an angle ω. Since the rotation axis of the spin precession ω is the same as for the

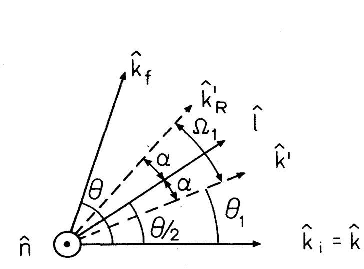

FIG. 3. Wigner rotation of outgoing polarization.

TABLE II. Observables as functions of the amplitudes.

$I_{0000}=\sigma=(|a|^2+|b|^2+|c|^2+|d|^2+|e|^2)/2$
$\sigma A_{00nn}=(|a|^2-|b|^2-|c|^2+|d|^2+|e|^2)/2$
$\sigma D_{n0n0}=\sigma D_{0n0n}=(|a|^2+|b|^2-|c|^2-|d|^2+|e|^2)/2$
$\sigma K_{n00n}=\sigma K_{0nn0}=(|a|^2-|b|^2+|c|^2-|d|^2+|e|^2)/2$
$\sigma P_{n000}=\sigma P_{0n00}=\sigma A_{00n0}=\sigma A_{000n}=\mathrm{Re}(a^*e)$
$\sigma D_{s'0s0}=\cos(\alpha+\theta/2)\mathrm{Re}(a^*b)+\cos(\alpha-\theta/2)\mathrm{Re}(c^*d)-\sin(\alpha+\theta/2)\mathrm{Im}(b^*e)$
$\sigma D_{s'0k0}=-\sin(\alpha+\theta/2)\mathrm{Re}(a^*b)+\sin(\alpha-\theta/2)\mathrm{Re}(c^*d)-\cos(\alpha+\theta/2)\mathrm{Im}(b^*e)$
$\sigma M_{s'0sn}=\sin(\alpha+\theta/2)\mathrm{Im}(a^*b)+\sin(\alpha-\theta/2)\mathrm{Im}(c^*d)+\cos(\alpha+\theta/2)\mathrm{Re}(b^*e)$
$\sigma M_{s'0kn}=\cos(\alpha+\theta/2)\mathrm{Im}(a^*b)-\cos(\alpha-\theta/2)\mathrm{Im}(c^*d)-\sin(\alpha+\theta/2)\mathrm{Re}(b^*e)$
$\sigma K_{0s''s0}=-\cos(\beta+\theta/2)\mathrm{Re}(a^*c)-\cos(\beta-\theta/2)\mathrm{Re}(b^*d)+\sin(\beta+\theta/2)\mathrm{Im}(c^*e)$
$\sigma K_{0s''k0}=\sin(\beta+\theta/2)\mathrm{Re}(a^*c)-\sin(\beta-\theta/2)\mathrm{Re}(b^*d)+\cos(\beta+\theta/2)\mathrm{Im}(c^*e)$
$\sigma N_{0s''sn}=-\sin(\beta+\theta/2)\mathrm{Im}(a^*c)-\sin(\beta-\theta/2)\mathrm{Im}(b^*d)-\cos(\beta+\theta/2)\mathrm{Re}(c^*e)$
$\sigma N_{0s''kn}=-\cos(\beta+\theta/2)\mathrm{Im}(a^*c)+\cos(\beta-\theta/2)\mathrm{Im}(b^*d)+\sin(\beta+\theta/2)\mathrm{Re}(c^*e)$
$\sigma A_{00ss}=\cos(\theta)\mathrm{Re}(a^*d)+\mathrm{Re}(b^*c)-\sin(\theta)\mathrm{Im}(d^*e)$
$\sigma A_{00sk}=\sigma A_{00ks}=-\sin\theta\,\mathrm{Re}(a^*d)-\cos\theta\,\mathrm{Im}(d^*e)$
$\sigma A_{00kk}=-\cos(\theta)\mathrm{Re}(a^*d)+\mathrm{Re}(b^*c)+\sin(\theta)\mathrm{Im}(d^*e)$
with $\alpha=\theta/2-\theta_1,\quad \beta=\theta/2+\theta_2$

Wigner rotation [see Eqs. (9) and (10)], ω can simply be added to the relativistic correction α. By replacing

$$s'\to\omega \text{ for the index, } \alpha\to\alpha+\omega \text{ for the angle ,} \tag{11}$$

the same formalism as in Table II can be kept. In analogy to Eq. (11), the formalism for the recoil parameters in Table II is also kept by replacing

$$s''\to\omega \text{ for the index, } \beta\to\beta-\omega \text{ for the angle .} \tag{12}$$

In a similar way, the same symmetry relations as in Table

TABLE III. Moduli and phases relative to e of reconstructed amplitudes for pp elastic scattering at 579 MeV. The differential cross sections used as normalization are also given.

$\theta_{c.m.}$ (deg)	$\lvert a\rvert$ (mb/sr)$^{1/2}$	$\lvert b\rvert$	$\lvert c\rvert$	$\lvert d\rvert$	$\lvert e\rvert$
38	1.550±0.071	1.507±0.027	0.503±0.119	0.961±0.049	1.959±0.055
42	1.536±0.080	1.434±0.029	0.357±0.153	0.909±0.048	1.944±0.056
46	1.340±0.097	1.412±0.045	0.527±0.100	0.769±0.042	1.998±0.029
50	1.284±0.063	1.332±0.026	0.228±0.134	0.762±0.027	2.033±0.038
54	1.158±0.154	1.289±0.036	0.317±0.137	0.704±0.033	2.039±0.038
58	1.022±0.085	1.193±0.071	0.507±0.118	0.652±0.075	2.052±0.019
62	0.849±0.051	1.167±0.032	0.420±0.050	0.691±0.034	2.064±0.013
66	0.786±0.045	1.114±0.024	0.549±0.039	0.652±0.032	2.021±0.024
70	0.596±0.038	1.029±0.023	0.548±0.033	0.671±0.029	2.054±0.018
74	0.482±0.038	0.978±0.024	0.542±0.034	0.655±0.032	2.054±0.016
78	0.438±0.044	0.966±0.026	0.619±0.031	0.648±0.026	1.999±0.010
82	0.238±0.034	0.870±0.024	0.670±0.027	0.620±0.030	2.030±0.012
86	0.111±0.021	0.814±0.024	0.675±0.023	0.605±0.025	2.043±0.008
90	0.000	0.773±0.020	0.773±0.020	0.590±0.030	2.024±0.009

$\theta_{c.m.}$ (deg)	Φ_a (rad)	Φ_b	Φ_c	Φ_d	$d\sigma/d\Omega$ (mb/sr)
38	0.430±0.062	0.538±0.055	5.320±0.148	4.490±0.120	4.845
42	0.493±0.057	0.650±0.058	5.358±0.186	4.633±0.119	4.574
46	0.621±0.051	1.004±0.065	5.613±0.139	4.353±0.147	4.325
50	0.626±0.048	1.055±0.061	5.349±0.621	4.672±0.101	4.093
54	0.697±0.075	1.255±0.123	5.331±0.582	4.510±0.222	3.878
58	0.710±0.051	1.328±0.061	5.613±0.368	4.146±0.127	3.680
62	0.647±0.059	1.463±0.061	5.322±0.150	4.256±0.096	3.499
66	0.705±0.058	1.589±0.070	5.526±0.079	4.310±0.107	3.336
70	0.579±0.087	1.752±0.065	5.391±0.085	4.258±0.104	3.190
74	0.624±0.101	1.787±0.065	5.396±0.090	4.212±0.114	3.065
78	0.879±0.079	1.948±0.063	5.479±0.065	4.287±0.105	2.962
82	0.675±0.179	1.955±0.075	5.381±0.064	4.214±0.115	2.884
86	0.450±0.375	1.982±0.072	5.322±0.053	4.337±0.118	2.836
90	0.000	2.173±0.041	5.315±0.041	4.237±0.106	2.819

I are obtained:

$$K_{n00n}(\theta)=D_{n0n0}(\pi-\theta) \quad \text{not affected by } \omega ,$$

and

$$\begin{aligned} K_{0\omega s0}(\theta)&=D_{\omega 0s0}(\pi-\theta) , \\ K_{0\omega k0}(\theta)&=-D_{\omega 0k0}(\pi-\theta) , \\ N_{0\omega sn}(\theta)&=-M_{\omega 0sn}(\pi-\theta) , \\ N_{0\omega kn}(\theta)&=M_{\omega 0kn}(\pi-\theta) . \end{aligned} \tag{13}$$

C. Measured sets

P is antisymmetric around $90°_{\text{c.m.}}$ and its measurement above $90°_{\text{c.m.}}$ does not bring any new information. The redefinition of parameters measured above $90°_{\text{c.m.}}$ results in the fact that we actually determined 15 polarization observables from $62°_{\text{c.m.}}$ to $90°_{\text{c.m.}}$. Below $62°_{\text{c.m.}}$, we did not have the recoil parameters at our disposition. Since here we could not obtain D_{n0n0} from K_{n00n}, this was obtained directly from the beam polarization data along $\hat{\mathbf{n}}$ ($\equiv\hat{\mathbf{Y}}$).

These sets were not yet complete, however, as the scale of the amplitudes must be determined. This is given by the differential cross section (see Table II). For this, values from the Saclay-Geneva PSA (Ref. 15) which presents a suitable world average of the very abundant existing data were taken. These numerical values are listed along with the results (Tables III–VII). The observables available for the amplitude reconstruction were then the following.

(1) Between $38°_{\text{c.m.}}$ and $58°_{\text{c.m.}}$ (11 spin observables): "small-angle region"

$$P_{n000},\ A_{00nn},\ D_{n0n0},\ K_{n00n},\ A_{00ss},\ A_{00kk} ,\quad A_{00sk},\ D_{\omega 0s0},\ D_{\omega 0k0},\ M_{\omega 0sn},\ M_{\omega okn} . \tag{14}$$

(2) Between $62°_{\text{c.m.}}$ and $90°_{\text{c.m.}}$ (15 spin observables): "large-angle region"

$$P_{n000},\ A_{00nn},\ D_{n0n0},\ K_{n00n},\ A_{00ss},\ A_{00kk} ,\quad A_{00sk},\ D_{\omega 0s0},\ D_{\omega 0k0},\ M_{\omega 0sn},\ M_{\omega okn} ,\quad K_{0\omega s0},\ K_{0\omega k0},\ N_{0\omega sn},\ N_{0\omega kn} . \tag{15}$$

TABLE IV. Same as Table III but at 539 MeV.

$\theta_{\text{c.m.}}$ (deg)	$\|a\|$ (mb/sr)$^{1/2}$	$\|b\|$	$\|c\|$	$\|d\|$	$\|e\|$
38	1.394±0.117	1.450±0.034	0.274±0.158	1.015±0.047	2.075±0.060
42	1.311±0.075	1.375±0.026	0.453±0.097	0.946±0.043	2.091±0.049
46	1.258±0.074	1.290±0.032	0.417±0.108	0.892±0.039	2.124±0.038
50	1.035±0.054	1.245±0.029	0.579±0.070	0.853±0.043	2.167±0.031
54	0.968±0.049	1.183±0.032	0.608±0.069	0.819±0.042	2.159±0.027
58	0.905±0.048	1.146±0.030	0.532±0.069	0.794±0.042	2.161±0.027
62	0.793±0.038	1.126±0.021	0.563±0.042	0.728±0.030	2.161±0.019
66	0.706±0.033	1.086±0.019	0.638±0.034	0.718±0.028	2.133±0.016
70	0.580±0.028	1.056±0.017	0.666±0.028	0.680±0.026	2.139±0.013
74	0.418±0.023	1.042±0.019	0.712±0.026	0.643±0.024	2.140±0.008
78	0.372±0.030	0.984±0.016	0.758±0.022	0.616±0.025	2.140±0.011
82	0.227±0.029	0.960±0.018	0.776±0.022	0.623±0.026	2.143±0.009
86	0.096±0.017	0.919±0.017	0.846±0.017	0.585±0.024	2.145±0.008
90	0.000±0.000	0.847±0.017	0.847±0.000	0.657±0.029	2.152±0.008

$\theta_{\text{c.m.}}$ (deg)	Φ_a (rad)	Φ_b	Φ_c	Φ_d	$d\sigma/d\Omega$ (mb/sr)
38	0.446±0.089	0.484±0.058	5.326±0.589	4.341±0.174	4.728
42	0.491±0.072	0.609±0.050	5.319±0.277	4.431±0.173	4.542
46	0.571±0.065	0.859±0.056	5.538±0.284	4.589±0.160	4.364
50	0.461±0.085	0.955±0.055	5.489±0.205	4.411±0.153	4.191
54	0.580±0.064	1.242±0.055	5.598±0.222	4.440±0.161	4.021
58	0.667±0.063	1.417±0.063	5.580±0.223	4.320±0.131	3.858
62	0.654±0.056	1.568±0.056	5.633±0.093	4.488±0.108	3.708
66	0.678±0.051	1.673±0.049	5.656±0.062	4.478±0.114	3.576
70	0.636±0.060	1.700±0.045	5.499±0.055	4.410±0.097	3.466
74	0.546±0.084	1.783±0.039	5.520±0.049	4.219±0.080	3.381
78	0.768±0.081	1.931±0.041	5.494±0.046	4.325±0.089	3.319
82	0.712±0.143	2.007±0.041	5.331±0.052	4.256±0.097	3.279
86	0.494±0.325	2.128±0.039	5.377±0.032	4.490±0.088	3.256
90	0.000±0.000	2.081±0.032	5.222±0.000	4.137±0.081	3.249

V. ANALYTICAL SOLUTIONS

The importance of having an analytical solution for the system of equations was twofold: (1) as an algebraic tool in examining the completeness of the observable set when errors are neglected; (2) the analytical solution represents an approximate estimate for the amplitudes and thus serves as a starting point in the numerical least-squares minimization procedure which estimates the amplitudes taking into account overdetermination and experimental errors. In this way the convergence to a single minimum ensured the uniqueness of the solution.

The full and detailed analytical reconstruction can be found in Ref. 6 and in the Appendix. Both sets of measurements given by Eqs. (14) and (15) are shown, on the level of the analytical solution, to be almost identical. In view of this, the recoil parameters do not seem to be of great importance. In practice, however, the overdetermination given by the four additional parameters make the system behave in a very different way (see Sec. VII).

VI. DETERMINATION OF THE AMPLITUDES

In order to be independent of the undetermined overall phase, we defined the amplitude e as purely real. This choice is arbitrary, but is determined by the fact that the absolute value of e stays large over the entire measured angular domain. We have used a polar representation of the amplitudes: namely,

$$A=|A|\exp(i\phi_A) \quad (16)$$

with $A=a,b,c,d$, and ϕ_A the phase of A relative to e. In this way the undetermined common phase is set in the phase of e. Only the modulus $|e|$ which we will simply call e is then determined. This is true at each angle.

The overall phase (which multiplies all the amplitudes) is not experimentally measurable. This is clearly seen from the equations given in Table II: any change in the common phase would not be seen on the observables as the amplitudes always appear as a modulus square or a product of conjugates.

The amplitudes were estimated by a numerical least-squares minimization procedure using the computer program MINUIT (Ref. 16). The theoretical expressions for the observables, listed in Table II and modified for ω, were directly fitted to their corresponding experimental values at each separate angle, taking into account their errors, but neglecting their correlations which were found to be very small. For the double- and triple-spin parameters, only the statistical errors were taken. The systematic uncertainties will be discussed in a more natural way in

TABLE V. Same as Table III but at 517 MeV.

$\theta_{\text{c.m.}}$ (deg)	$\lvert a\rvert$ (mb/sr)$^{1/2}$	$\lvert b\rvert$	$\lvert c\rvert$	$\lvert d\rvert$	$\lvert e\rvert$
38	1.371±0.068	1.377±0.032	0.501±0.085	0.963±0.033	2.038±0.044
42	1.310±0.106	1.324±0.026	0.393±0.123	0.959±0.043	2.068±0.051
46	1.245±0.062	1.244±0.027	0.387±0.095	0.915±0.030	2.113±0.038
50	1.088±0.050	1.252±0.025	0.479±0.067	0.874±0.028	2.129±0.027
54	0.968±0.053	1.211±0.031	0.575±0.056	0.801±0.027	2.152±0.020
58	0.851±0.040	1.157±0.026	0.596±0.056	0.804±0.035	2.164±0.023
62	0.757±0.035	1.134±0.025	0.635±0.041	0.710±0.030	2.178±0.017
66	0.670±0.031	1.081±0.022	0.662±0.037	0.727±0.027	2.172±0.014
70	0.521±0.023	1.074±0.018	0.734±0.028	0.683±0.026	2.167±0.013
74	0.433±0.024	1.036±0.017	0.746±0.026	0.702±0.024	2.165±0.011
78	0.326±0.024	1.032±0.017	0.844±0.024	0.624±0.026	2.152±0.011
82	0.220±0.025	1.018±0.017	0.921±0.022	0.616±0.026	2.129±0.011
86	0.103±0.018	0.942±0.018	0.901±0.018	0.656±0.023	2.162±0.008
90	0.000±0.000	0.941±0.018	0.941±0.000	0.680±0.028	2.137±0.008

$\theta_{\text{c.m.}}$ (deg)	ϕ_a (rad)	ϕ_b	ϕ_c	ϕ_d	$d\sigma/d\Omega$ (mb/sr)
38	0.488±0.063	0.390±0.041	5.509±0.168	4.584±0.105	4.552
42	0.553±0.081	0.578±0.055	5.738±0.340	4.646±0.155	4.410
46	0.627±0.056	0.718±0.045	5.767±0.211	4.684±0.117	4.273
50	0.565±0.060	0.878±0.043	5.614±0.136	4.552±0.116	4.138
54	0.645±0.060	1.116±0.052	5.635±0.122	4.593±0.135	4.004
58	0.631±0.057	1.274±0.048	5.583±0.163	4.521±0.114	3.875
62	0.644±0.055	1.442±0.053	5.471±0.091	4.559±0.109	3.755
66	0.660±0.054	1.581±0.051	5.517±0.065	4.369±0.090	3.650
70	0.534±0.068	1.637±0.048	5.353±0.057	4.407±0.093	3.564
74	0.615±0.074	1.819±0.044	5.454±0.050	4.392±0.084	3.498
78	0.617±0.098	1.862±0.042	5.430±0.044	4.380±0.090	3.452
82	0.660±0.142	2.020±0.044	5.396±0.043	4.567±0.117	3.422
86	0.567±0.280	2.012±0.044	5.272±0.033	4.306±0.075	3.406
90	0.000±0.000	2.138±0.032	5.280±0.000	4.225±0.085	3.401

Sec. VII. For the spin-correlation data, the total errors including both statistical and systematic errors were entered, as for these data no experimental check of the target polarization was possible (see Refs. 3, 4, and 6).

A few small corrections to the exact number of measured observables used in the fit have to be given: (1) at 447 MeV, D_{n0n0} exists only between $34°_{\mathrm{c.m.}}$ and $62°_{\mathrm{c.m.}}$, and between $74°_{\mathrm{c.m.}}$ and $102°_{\mathrm{c.m.}}$, (2) at 497 MeV, no A_{00nn} data were measured, so interpolated values between A_{00nn} measurements at 515 and 470 MeV (Ref. 1) were used for angles below $62°_{\mathrm{c.m.}}$ (see Sec. VII), (3) at 579 MeV (Ref. 4), D_{n0n0} was measured only between $34°_{\mathrm{c.m.}}$ and $62°_{\mathrm{c.m.}}$.

The particular role of the differential cross section also caused certain problems. As it would not make sense introducing an error for the cross section which only defines the scale of the amplitudes, the error propagation to the moduli of the amplitudes being trivial, these cross sections were taken out of the fit. By doing this, however, we then had too many parameters to be fitted. This was remedied by fixing one parameter e, which was found to be the largest and best determined. We summarize the procedure applied for the reconstruction of the amplitudes.

Step 1. Analytical reconstruction which gave a first raw estimate of the amplitudes: a_0, b_0, c_0, d_0, e_0 (see Sec. V and the Appendix).

Step 2. Fix e to e_0.

Step 3. Start minimization procedure at a_0, b_0, c_0, d_0. Solution at the minimum: a_1, b_1, c_1, d_1.

Step 4. Renormalize the moduli to the differential cross section by the factor

$$r=[2\sigma/(|a_1|^2+|b_1|^2+|c_1|^2+|d_1|^2+|e_0|^2)]^{1/2} \tag{17}$$

which then gave the final solution for the amplitudes:

$$\begin{aligned} &a_f=|a_1|r\exp(i\phi_{a_1}),\quad b_f=|b_1|r\exp(i\phi_{b_1})\ ,\\ &c_f=|c_1|r\exp(i\phi_{c_1}),\quad d_f=|d_1|r\exp(i\phi_{d_1})\ ,\\ &e_f=e_0 r\ . \end{aligned} \tag{18}$$

The error on the normalization factor r was calculated from the errors on the fitted amplitudes including their correlations. The errors on the renormalized moduli were then recalculated, taking into account the error on r as well as the correlation of the moduli with r. Furthermore, the correlations of e_f with other moduli via r were computed.

TABLE VI. Same as Table III but at 497 MeV

$\theta_{\mathrm{c.m.}}$ (deg)	$\lvert a\rvert$ (mb/sr)$^{1/2}$	$\lvert b\rvert$	$\lvert c\rvert$	$\lvert d\rvert$	$\lvert e\rvert$
38	1.300±0.074	1.324±0.048	0.320±0.071	1.050±0.026	2.055±0.056
42	1.240±0.100	1.223±0.042	0.366±0.107	1.051±0.050	2.090±0.016
46	1.158±0.075	1.147±0.031	0.320±0.151	1.030±0.030	2.145±0.041
50	1.083±0.107	1.099±0.015	0.318±0.139	0.970±0.036	2.185±0.032
54	0.935±0.053	1.031±0.030	0.533±0.077	0.900±0.034	2.223±0.028
58	0.852±0.048	1.044±0.028	0.652±0.060	0.888±0.032	2.174±0.025
62	0.799±0.044	1.014±0.027	0.674±0.052	0.845±0.030	2.172±0.023
66	0.617±0.032	1.041±0.027	0.737±0.042	0.814±0.032	2.170±0.019
70	0.537±0.030	1.025±0.023	0.737±0.036	0.771±0.029	2.182±0.016
74	0.415±0.028	1.048±0.023	0.789±0.032	0.721±0.031	2.172±0.015
78	0.413±0.036	1.017±0.028	0.855±0.035	0.685±0.029	2.156±0.009
82	0.237±0.031	1.012±0.023	0.900±0.028	0.686±0.027	2.156±0.012
86	0.128±0.032	0.981±0.020	0.922±0.021	0.668±0.027	2.171±0.011
90	0.000±0.000	0.944±0.019	0.944±0.000	0.621±0.028	2.193±0.009

$\theta_{\mathrm{c.m.}}$ (deg)	ϕ_a (rad)	ϕ_b	ϕ_c	ϕ_d	$d\sigma/d\Omega$ (mb/sr)
38	0.498±0.064	0.595±0.061	7.200±0.430	4.486±0.134	4.436
42	0.561±0.068	0.560±0.062	5.981±0.389	4.590±0.140	4.322
46	0.602±0.074	0.751±0.063	5.860±0.517	4.516±0.156	4.211
50	0.665±0.060	1.013±0.068	5.792±0.337	4.533±0.147	4.099
54	0.651±0.064	1.178±0.074	5.734±0.215	4.496±0.137	3.987
58	0.709±0.059	1.199±0.069	5.714±0.176	4.559±0.137	3.879
62	0.752±0.053	1.456±0.070	5.467±0.104	4.532±0.104	3.777
66	0.583±0.072	1.448±0.054	5.480±0.069	4.447±0.092	3.689
70	0.624±0.071	1.698±0.047	5.431±0.066	4.492±0.095	3.618
74	0.608±0.090	1.750±0.046	5.388±0.056	4.322±0.088	3.564
78	0.810±0.078	1.825±0.045	5.464±0.051	4.439±0.105	3.527
82	0.764±0.136	1.888±0.046	5.311±0.052	4.513±0.108	3.505
86	0.955±0.190	1.998±0.043	5.291±0.036	4.391±0.084	3.493
90	0.000±0.000	2.093±0.034	5.234±0.000	4.477±0.115	3.489

At $90^\circ_{\text{c.m.}}$, where the Pauli principle requires that $a=0$, $b=-c$, only b, d, and e were determined using the same method.

VII. RESULTS AND DISCUSSION

Figures 4 and 5 display the results of the amplitude analysis. The numerical values are listed in Tables III–VII separately for each energy. The errors given are purely statistical. The solid lines represent the fit obtained by the Saclay-Geneva PSA (Ref. 15) at fixed energy using the same data completed with points from a small-angle experiment.[17] Tables III–VII also give the values for the differential cross section to which these amplitudes have been normalized.

These reconstructed values constitute a complete and exhaustive description of *pp* elastic scattering which now enables any experiment at these energies and angles to be predicted. This should be put in contrast to the measurement of a single observable which only provides very particular information about the process. Figures 4 and 5 also forcefully demonstrate the importance of both the spin-independent and spin-dependent parts of the *pp* interaction. The angular dependence of the amplitudes is, as already noticed for the parameters, very smooth. We are still in the region where only the nuclear part of the interaction is important. For smaller angles (below $10^\circ_{\text{c.m.}}$), the Coulomb interaction sets in and gives rise to violent changes.

It is interesting to look at the phase-shift behavior,[15] as drawn in Fig. 6. One notices that in our energy domain (indicated by the two vertical lines) the dominant phase shifts are the *P*'s ($^3P_0$, $^3P_1$, $^3P_2$), and not the $^1D_2$ phase. All these different phase shifts combine to construct smooth energy-dependent amplitudes.

The reconstruction has been applied to the two sets of data defined in Eq. (15) (from $90^\circ_{\text{c.m.}}$ to $62^\circ_{\text{c.m.}}$), and in Eq. (14) (from $58^\circ_{\text{c.m.}}$ to $38^\circ_{\text{c.m.}}$). The fit performed above $58^\circ_{\text{c.m.}}$ showed no problem, due to the overdetermination of the system. The statistical precision is equally distributed over all the amplitudes, since the set of observables contained sufficient information on all of them. Towards $90^\circ_{\text{c.m.}}$, where $|a|$ goes to zero, ϕ_a becomes undetermined. A χ^2 per degree of freedom of 1.21, 1.29, 1.64, 1.50, and 1.12, averaged over the eight angles, was found at 579, 539, 517, 497, and 447 MeV, respectively. This confirms the consistency of the data which originate from different and independent experiments. The system was so largely overdetermined that one could easily remove certain observables from the fit in order to undertake further con-

TABLE VII. Same as Table III but at 447 MeV.

$\theta_{\text{c.m.}}$ (deg)	$\lvert a\rvert$ (mb/sr)$^{1/2}$	$\lvert b\rvert$	$\lvert c\rvert$	$\lvert d\rvert$	$\lvert e\rvert$
38	1.058±0.047	1.186±0.031	0.348±0.077	1.169±0.031	2.085±0.026
42	1.002±0.041	1.104±0.027	0.433±0.067	1.100±0.028	2.149±0.024
46	1.146±0.067	1.092±0.024	0.419±0.067	1.031±0.024	2.091±0.028
50	0.918±0.037	1.043±0.024	0.589±0.054	0.985±0.025	2.176±0.022
54	0.862±0.037	1.026±0.024	0.602±0.051	0.963±0.021	2.182±0.020
58	0.804±0.034	1.023±0.030	0.636±0.058	0.922±0.035	2.182±0.022
62	0.770±0.033	1.100±0.022	0.777±0.040	0.836±0.030	2.115±0.019
66	0.556±0.031	1.014±0.026	0.700±0.044	0.864±0.033	2.212±0.018
70	0.497±0.028	1.052±0.020	0.784±0.031	0.787±0.026	2.190±0.014
74	0.390±0.027	1.006±0.020	0.838±0.027	0.781±0.025	2.201±0.012
78	0.318±0.033	0.985±0.021	0.885±0.025	0.773±0.027	2.199±0.012
82	0.218±0.034	1.007±0.019	0.901±0.023	0.766±0.023	2.192±0.009
86	0.079±0.014	1.007±0.018	0.966±0.019	0.715±0.025	2.190±0.009
90	0.000±0.000	0.956±0.018	0.956±0.000	0.758±0.026	2.203±0.008

$\theta_{\text{c.m.}}$ (deg)	Φ_a (rad)	Φ_b	Φ_c	Φ_d	$d\sigma/d\Omega$ (mb/sr)
38	0.317±0.085	0.196±0.041	5.521±0.243	4.199±0.098	4.181
42	0.381±0.074	0.330±0.038	5.330±0.150	4.252±0.078	4.120
46	0.646±0.055	0.627±0.044	5.792±0.195	4.505±0.109	4.058
50	0.515±0.060	0.750±0.038	5.311±0.132	4.321±0.099	3.992
54	0.598±0.055	1.046±0.042	5.529±0.131	4.406±0.111	3.923
58	0.601±0.054	1.125±0.041	5.350±0.177	4.308±0.104	3.854
62	0.696±0.046	1.228±0.045	5.571±0.094	4.566±0.104	3.790
66	0.506±0.090	1.501±0.060	5.292±0.121	4.265±0.103	3.733
70	0.603±0.077	1.542±0.048	5.311±0.078	4.385±0.093	3.691
74	0.585±0.098	1.750±0.051	5.363±0.072	4.480±0.121	3.661
78	0.709±0.117	1.805±0.053	5.261±0.069	4.396±0.111	3.643
82	0.805±0.151	1.829±0.048	5.215±0.067	4.345±0.105	3.633
86	0.303±0.457	1.841±0.049	5.166±0.044	4.214±0.084	3.629
90	0.000±0.000	1.870±0.042	5.011±0.000	4.215±0.100	3.628

sistency checks. For example, the horizontal two-spin correlations A_{00pq} which form a subgroup of measurements independent of the others, were taken out of the fit. A comparable χ^2 and the same result was found for the amplitudes, with, of course, somewhat larger errors. The excellent properties of the set given in Eq. (15) for an amplitude reconstruction are thus confirmed under realistic conditions.

The reconstruction based on observables from the set defined in Eq. (14) from $58°_{\text{c.m.}}$ to $38°_{\text{c.m.}}$ showed a different behavior due to the fact that it was reduced by the four recoil parameters. Nevertheless, examining the reconstructed amplitudes one sees that it also succeeded in extending the reconstruction down to the region where the recoil parameters were absent. It even reproduces the amplitudes a, b, d, and e with a comparable precision, except for ϕ_d. For ϕ_c, however, there is a certain loss of information. The precision on ϕ_c which is determined via b and d, becomes worse towards $58°_{\text{c.m.}}$, since $|b|$ and $|d|$ diminish. At angles where $|c|$ also happens to be small, ϕ_c shows large errors. Here further measurements with a target polarization in the scattering plane could help. For instance, $M_{s'0ns}$ and $M_{s'0nk}$ contain products of c with e, the largest and best determined amplitude. But one must not forget about the additional complications such an experiment would bring, keeping in mind all in-

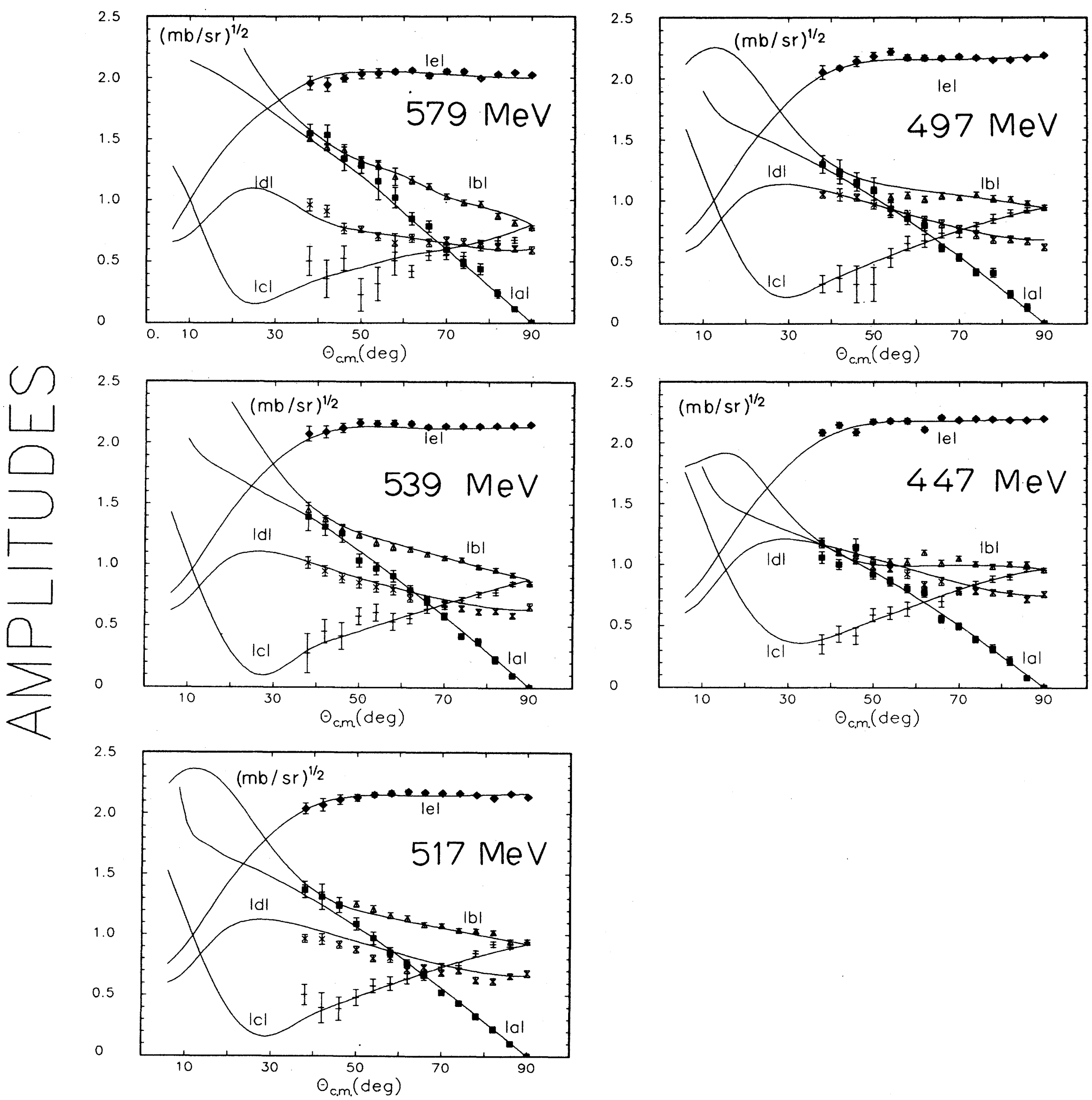

FIG. 4. Absolute values of the five complex amplitudes separately for each energy (579, 539, 517, 497, and 447 MeV). The solid lines represent the Saclay-Geneva PSA amplitudes as obtained from the same experimental data.

stances where our field direction facilitated the analysis. At $54°_{\rm c.m.}$ even an ambiguity for ϕ_c was observed at 579 MeV and at 539 MeV: another local minimum was found around 3 and 2 rad, respectively, solutions which could be rejected imposing continuity. At 497 MeV, where no A_{00nn} measurements were available, ϕ_c value was determined with very large error ($\simeq 1.3$ rad) at angles where $|c|$ was small. It was, therefore, necessary to introduce A_{00nn} values interpolated between measurements at 470 and 515 MeV. (At angles larger than $58°_{\rm c.m.}$, as the system is overdetermined, no such problem existed.) It must be pointed out that the system of equations is not ambiguous algebraically. Therefore, the ambiguity is uniquely an effect of the experimental errors. This demonstrates their importance for a unique amplitude reconstruction. Taking only statistical errors also for the two-spin correlations at critical angles for ϕ_c, the situation was slightly improved, increasing, of course, the χ^2 a bit. As a conclusion one may say that the set defined by Eq. (14) is at the limit of completeness. The χ^2, averaged over the six angles below $62°_{\rm c.m.}$, was 1.34, 1.22, 0.8, 1.35, and 0.85 at 579, 539, 517, 497, and 447 MeV, respectively.

It is worth mentioning that amplitude reconstructions have been published on other sets of data, at 800 MeV (Ref. 18) on the LAMPF data, at 6 GeV/c (Ref. 19) on the Argonne data, and more recently at 11 momenta between 834 and 2696 MeV (Ref. 20) by the nucleon-nucleon group at Saclay.

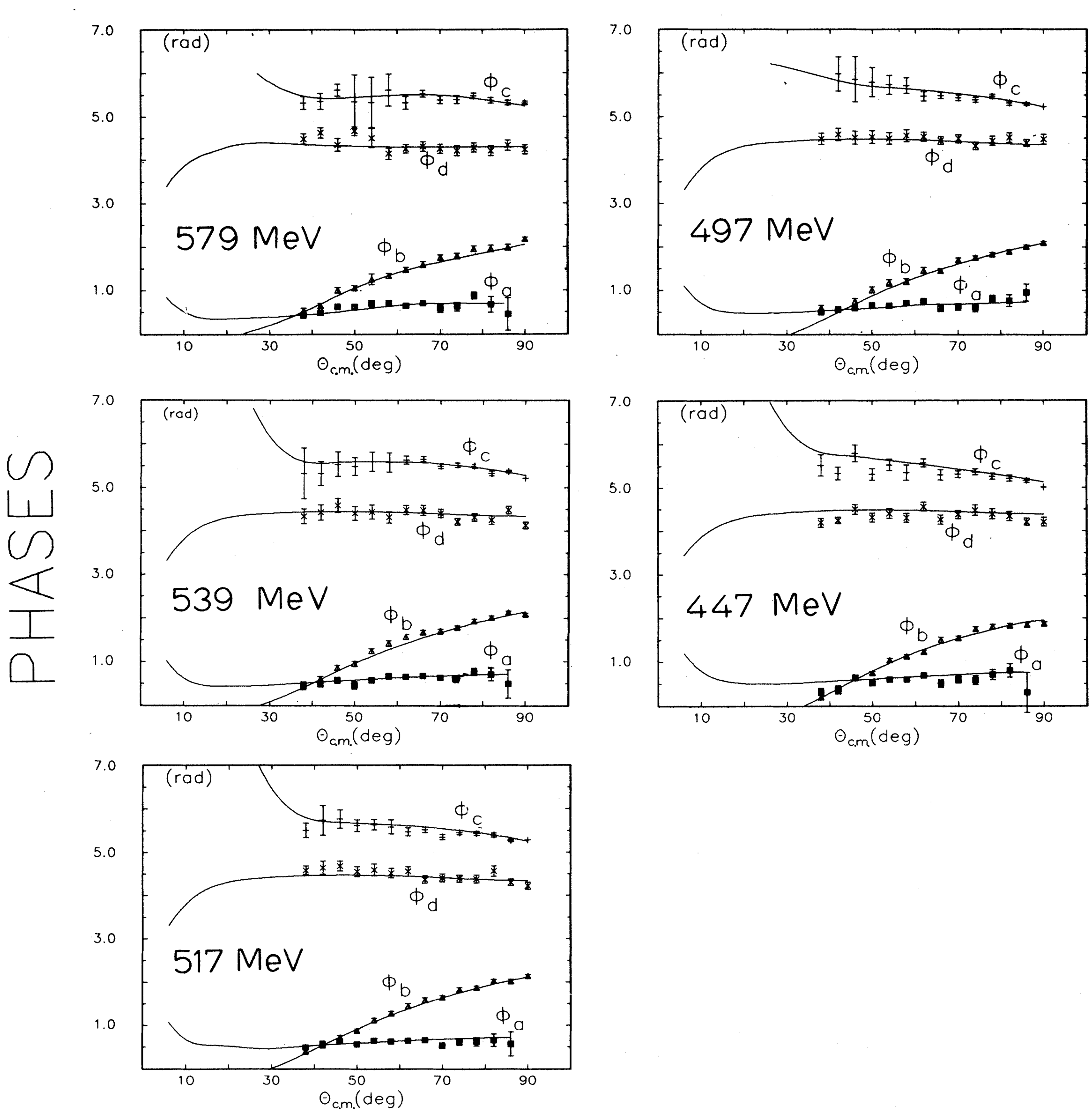

FIG. 5. Phases relative to e of the five complex amplitudes separately for each energy (579, 539, 517, 497, and 447 MeV). The solid lines represent the Saclay-Geneva PSA amplitudes as obtained from the same experimental data.

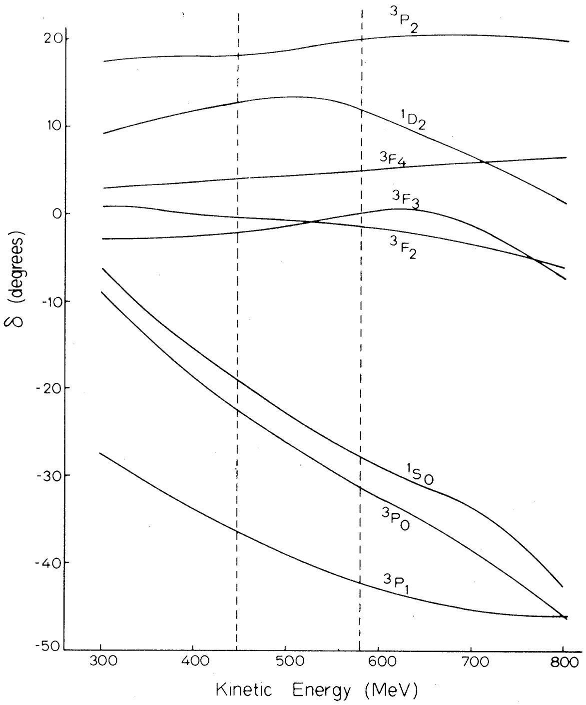

FIG. 6. Dominant *pp* phase shifts (Ref. 15) as a function of kinetic energy.

The Saclay-Geneva PSA fit[15] on essentially this same data resulted in a comparable χ^2. This indicates that the model-dependent assumptions made for the PSA are adequate for describing the experimental data (see Figs. 4 and 5). In this view, the PSA fit represents a valuable angular smoothing of our points measured independently at separate angles.

The influence of the systematic uncertainties on the fitted amplitudes was studied in great detail at 579 MeV (Ref. 6). The tight determination of the complete system, in particular for the angles from $62^\circ_{\text{c.m.}}$ to $90^\circ_{\text{c.m.}}$, caused a slow nonlinear increase in the errors on the amplitudes as a function of the uncertainties on the parameters, unlike usual error propagation. This made the overall effect surprisingly small. The systematic uncertainties of the amplitudes remained within the statistical errors, with the exception of d, where the systematic error grew to about 0.1 $(\text{mb/sr})^{1/2}$ above $62^\circ_{\text{c.m.}}$. The χ^2 per degree of freedom was about 2 for the worst interference of all the uncertainties on the observables.

VIII. CONCLUSION

The successful reconstruction of the *pp* elastic-scattering amplitudes at 14 c.m. angles between $38^\circ_{\text{c.m.}}$ and $90^\circ_{\text{c.m.}}$ has been made possible by the measurements of 11 spin-polarization parameters, P, D_{n0n0}, K_{n00n}, $D_{s'0s0}$, $D_{s'0k0}$, $M_{s'0sn}$, $M_{s'0kn}$, A_{00nn}, A_{00kk}, A_{00sk}, A_{00ss}. For angles above $58^\circ_{\text{c.m.}}$, four additional parameters $K_{0s''s0}$, $K_{0s''k0}$, $N_{0s''sn}$, $N_{0s''kn}$ were obtained taking advantage of the (anti)symmetry relation around $90^\circ_{\text{c.m.}}$. Below $62^\circ_{\text{c.m.}}$, the amplitude reconstruction was found to be at the limit of completeness. Nevertheless, it extended the experimental determination of the scattering matrix down to $38^\circ_{\text{c.m.}}$ with almost the same precision as that at $62^\circ_{\text{c.m.}}$ and above. At $90^\circ_{\text{c.m.}}$, however, a unique reconstruction would not have been possible without the three-spin parameters. Smooth energy dependence of the reconstructed amplitudes was observed between 447 and 579 MeV.

ACKNOWLEDGMENTS

We would like to thank all our colleagues who have helped us at a different stage of the data taking, as well as the technical staff of the University of Geneva. We express our gratitude to Professor J. P. Blaser, the SIN director. During the course of this research, support and discussions with Dr. F. Lehar and his group at Saclay were greatly appreciated. This work was supported by the Swiss National Science Foundation. Part of the analysis was performed at the Computer Center of the Geneva Cantonal Hospital. The assistance of its director, Professor J. R. Scherrer, as well as its computing staff was greatly appreciated.

APPENDIX: ANALYTICAL SOLUTIONS FOR THE AMPLITUDES

We present and discuss the analytical solutions for the two different angular domains in which different number of observables were measured as given in Eqs. (14) and (15). Since the overall phase of the scattering matrix M cannot be determined, we chose e to be real and positive (see Sec. VI). This choice will be helpful for the evaluation of the bilinear products in e. The number of these products happens to be the largest in our system. For convenience we put $\sigma=1$, since it only scales the magnitudes of the amplitudes. We will also use the notation

$$a_r=\text{Re}(a),\quad a_i=\text{Im}(a)\ ,$$

where a is one of the five amplitudes.

1. Set I [Eq. (15) used within $90^\circ_{\text{c.m.}}$ and $62^\circ_{\text{c.m.}}$]

The evaluation of the A_{00pq} gives

$$A_{00ss}+A_{00kk}=2\,\text{Re}(b^*c)\ ,$$

$$\begin{aligned}A_{00sk}\sin\theta+(A_{00kk}-A_{00ss})\cos\theta/2\\ =-\text{Re}(a^*d)=-(a_rd_r+a_id_i)=C\ ,\end{aligned}\tag{A1}$$

$$\begin{aligned}-A_{00sk}\cos\theta+(A_{00kk}-A_{00ss})\sin\theta/2\\ =\text{Im}d^*e=-d_ie=F\ ,\end{aligned}$$

which directly provides d_i. The polarization P gives a_r:

$$P=\text{Re}(a^*e)=a_re\ .\tag{A2}$$

With

$$|d|^2=(1+A_{00nn}-D_{n0n0}-K_{n00n})/2=G\tag{A3}$$

one finds

$$d_r = \pm[G - (F/e)^2]^{1/2} \tag{A4}$$

which is ambiguous in sign. We thus have a_r, d_r, and d_i as functions of e. With the help of C, we find

$$a_i = -(C + a_r d_r)/d_i = Ce/F \pm (P/F)[G - (F/e)^2]^{1/2} . \tag{A5}$$

By putting a_i^2 and a_r^2 into the expression

$$|a|^2 = (1 + A_{00nn} + D_{n0n0} + K_{n00n})/2 - e^2 = I - e^2 \tag{A6}$$

we get an equation in e which, after quadrature, turns out to be of second order in (e^2) only, simplifying the number of possible solutions:

$$c_0 + c_1(e^2) + c_2(e^2)^2 = 0 \tag{A7}$$

with

$$c_0 = (F^2 I - P^2 G)^2 + 4(CPF)^2 ,$$

$$c_1 = -2(IF^2 - P^2 G)(F^2 + C^2) - 4C^2 P^2 G ,$$

$$c_2 = (F^2 + C^2)^2 .$$

This equation yields at most two real positive solutions in e^2 and therefore in e since we have chosen e to be real and positive. So far we have determined a, d, and e up to a twofold ambiguity: d_r in sign and e. For the determination of b and c, one may proceed as follows: D's and M's are combined to give the expression

$$\begin{aligned} \sin(2\alpha)a^* b + i\cos(2\alpha)be &= \sin(\alpha - \theta/2)(D_{s'0s0} + iM_{s'0kn}) \\ &\quad - \cos(\alpha - \theta/2)(D_{s'0k0} - iM_{s'0sn}) \\ &= E , \end{aligned} \tag{A8}$$

which provides b, since we know a:

$$b = E/[a^* \sin(2\alpha) + ie\cos(2\alpha)] . \tag{A9}$$

In an analogous way, one can extract c combining K's and N's:

$$c = \frac{\cos(\beta - \theta/2)(K_{0s''k0} - iN_{0s''sn}) - \sin(\beta - \theta/2)(K_{0s''s0} + iN_{0s''kn})}{a^* \sin(2\beta) + ie\cos(2\beta)} . \tag{A10}$$

At this point we have determined all the amplitudes up to the twofold ambiguity. This can be lifted by using the remaining six equations:

$$\begin{aligned} A_{00ss} + A_{00kk} &= 2\,\mathrm{Re}(b^* c) , \\ (1 - A_{00nn} - D_{n0n0} + K_{n00n})/2 &= |c|^2 , \end{aligned} \tag{A11}$$

plus four other independent linear combinations in D, M, K, and N.

Looking at the situation where the relativistic corrections and the magnetic field of the target are neglected ($\alpha = 0, \beta = \pi$), we almost find the minimum number of experiments as a subset of 10 (or 11 if one would not consider the second one as a single measurement, although it is possible in principle):

$$\sigma,\ (A_{00ss} + A_{00kk}),\ D\text{'s},\ K\text{'s},\ M\text{'s},\ N\text{'s} . \tag{A12}$$

Combining equations for the D's and M's in the same way as (A8) with $\alpha = 0$ yields

$$be = W . \tag{A13}$$

A different combination gives

$$\begin{aligned} a^* b + d^* e &= \cos(\theta/2)(D_{s'0s0} + iM_{s'0kn}) \\ &\quad - \sin(\theta/2)(D_{s'0k0} + iM_{s'0sn}) = X . \end{aligned} \tag{A14}$$

In the same way, for the recoil K's and N's,

$$\begin{aligned} ce &= \cos(\theta/2)(N_{0s''kn} - iK_{0s''s0}) \\ &\quad + \sin(\theta/2)(N_{0s''sn} + iK_{0s''k0}) = Y \end{aligned} \tag{A15}$$

and

$$\begin{aligned} a^* c - d^* b &= \sin(\theta/2)(K_{0s''s0} + iN_{0s''kn}) \\ &\quad + \cos(\theta/2)(K_{0s''k0} - iN_{0s''sn}) = Z . \end{aligned} \tag{A16}$$

These expressions provide the immediate simple solution

$$\begin{aligned} &b = W/e, \quad c = Y/e , \\ &a^* = e(WX + e^2 Z)/(W^2 + Ye^2) , \\ &d^* = e(XY - ZW)/(W^2 + Ye^2) , \end{aligned} \tag{A17}$$

which is unique as long as $e \neq 0$. But the second observable term in (A12) determines e unambiguously as a simple square root:

$$A_{00ss} + A_{00kk} = 2\,\mathrm{Re}(b^* c) = 2\,\mathrm{Re}(W^* Y)/e^2 \tag{A18}$$

and, therefore,

$$e = +[2\,\mathrm{Re}(W^* Y)/(A_{00ss} + A_{00kk})]^{1/2}$$

with our convention of e real positive. If the denominator happens to be zero, one of the other six experiments not used so far would do as well.

Next we briefly examine the case at $90^\circ_{\mathrm{c.m.}}$ where the Pauli principle demands

$$a = 0, \quad b = -c . \tag{A19}$$

Here one need only determine b, d, and e. Let us try using the two-spin parameters only, as suggested in Ref. 21. P does not contain any information since $P = 0$. e is then immediately given by

$$e = +[(1 + A_{00nn} + D_{n0n0} + K_{n00n})/2]^{1/2} . \tag{A20}$$

For d we proceed as in (A1) and (A4):

$$d_i = -F/e, \quad d_r = \pm[G-(F/e)^2]^{1/2} , \tag{A21}$$

with a sign ambiguity. From the D's which contain the same information as the K's at $90°_{\mathrm{c.m.}}$, we extract

$$\mathrm{Re}(c^*d) = -\mathrm{Re}(b^*d) = -(b_r d_r + b_i d_i)$$

and (A22)

$$\mathrm{Im}(b^*e) = -\mathrm{Im}(c^*e) = b_i e$$

from whence we get b_i. In order to find b_r, we must pass through (A22) and thus transfer the same sign ambiguity from d_r to b_r. The A_{00pq}'s still provide the quantity

$$\mathrm{Re}(b^*c) = -\mathrm{Re}(b^*b) = -(b_r^2 + b_i^2) \tag{A23}$$

which is quadratic and brings in a further sign ambiguity.

Looking at all the other two-spin parameters, there is no new information available other than terms quadratic in the moduli that cannot help resolve sign ambiguities. Therefore, at $90°_{\mathrm{c.m.}}$ a unique amplitude reconstruction requires three-spin tensors.

2. Set II [Eq. (14) used below $62°_{\mathrm{c.m.}}$]

Here, we do not have the recoil parameters K and N at our disposal. However, the only instance where they were used in the solution presented in point 1 was for the determination of c in (A10). Knowing a, b, d, and e, we may as well extract d^*c from the equations for D and M only, leaving the more complicated expression

$$\begin{aligned} d^*c =& \cos(\alpha-\theta/2)D_{s'0s0} + \sin(\alpha-\theta/2)D_{s'0k0} \\ & -\cos(2\alpha)\mathrm{Re}(a^*b) + \sin(2\alpha)\mathrm{Im}(b^*e) \\ & -i[\sin(\alpha-\theta/2)M_{s'0sn} - \cos(\alpha-\theta/2)M_{s'0kn} \\ & + \cos(2\alpha)\mathrm{Im}(a^*b) - \sin(2\alpha)\mathrm{Re}(b^*e)] . \end{aligned} \tag{A24}$$

So the analytical solution for this set of observables is identical with that of point 1 except for the determination of c. The number of ambiguities is the same as well. For their resolution, however, there remains only two equations. They correspond to the two given in (A11). Indeed, on the level of analytical solution, the two sets discussed in this appendix are almost identical. In view of this, the recoil parameters do not seem to be of great importance. In practice, however, the overdetermination given by the four additional parameters make the system behave in a very different way.

---

*Present address: LeCroy, route du Nant-d'Avril 101, 1217 Meyrin, Switzerland.
†Present address: OFAC, rue Pedro-Meylan 7, 1208 Geneva, Switzerland.
‡Present address: Dept. of Physics, University of Iowa, Iowa City, IA 52242.

[1]D. Besset et al., Nucl. Phys. **A345**, 435 (1980).
[2]E. Aprile et al., Phys. Rev. D **28**, 21 (1983).
[3]E. Aprile et al., Phys. Rev. D **34**, 2566 (1986).
[4]E. Aprile et al., Phys. Rev. D **27**, 2600 (1983).
[5]E. Aprile et al., Phys. Rev. Lett. **46**, 1047 (1981).
[6]R. Hausammann, Ph.D. thesis, University of Geneva, Report No. 2038, 1982.
[7]J. Bystricky et al., J. Phys. (Paris) **39**, 1 (1978).
[8]L. Puzikov et al., Nucl. Phys. **3**, 436 (1957).
[9]W. Heisenberg, Z. Phys. **120**, 513 (1942).
[10]G. R. Goldstein et al., Lett. Nuovo Cimento **11**, 137 (1974).
[11]W. de Boer and J. Soffer, Nucl. Instrum. Methods **136**, 331 (1976).
[12]P. W. Johnson et al., Phys. Rev. D **15**, 1895 (1977).
[13]D. Besset et al., Nucl. Instrum. Methods **148**, 129 (1978).
[14]E. Aprile et al., Phys. Rev. Lett. **47**, 1360 (1981).
[15]J. Bystricky et al., J. Phys. (Paris) **48**, 199 (1987).
[16]F. James and M. Roos, CERN Computer Library Report No. D506, 1977.
[17]D. Besset et al., Phys. Rev. D **21**, 580 (1980).
[18]M. J. Moravcsik et al., Phys. Rev. D **31**, 1577 (1985).
[19]I. P. Auer et al., Phys. Rev. D **32**, 1609 (1985).
[20]C. D. Lac et al., in *Higher Energy Spin Physics*, proceedings of the VIII International Symposium, Minneapolis, Minnesota, 1988, edited by K. Heller (AIP Conf. Proc. No. 187) (AIP, New York, in press).
[21]C. R. Schumacher and H. A. Bethe, Phys. Rev. **121**, 1534 (1961).