One-jet inclusive cross section at order α_s^3. Gluons only

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The next-to-leading-order jet cross section is calculated for the simplified case in which there are only gluons. Numerical results of this calculation have been presented elsewhere. Here, we describe the calculational method in some detail, deriving all of the formulas that are necessary for the numerical evaluation of the cross section. The calculation is based on the matrix elements of Ellis and Sexton. In a future paper, we will extend the calculation to include quarks as well as gluons. When quarks are included, the algebraic complexity of the problem increases substantially. However, the method to be applied is essentially the same as in the gluon-only case, since there are no new singularities in the amplitudes that are not present in the gluon-only amplitudes.

I. ORGANIZATION

A. Introduction

One of the dramatic features of the recent data[1] from hadron colliders, both at CERN and Fermilab, is the obvious appearance of hadron jets[2] as a characteristic feature of a sizable fraction of the final states. These jets are an essential tool for organizing and analyzing the data and are potentially important as a test of our quantitative understanding of the underlying strong-interaction theory, QCD. This is particularly true if one wants to look[3] for a breakdown of the standard model due, for example, to the possible composite structure of the particles now thought to be elementary. One would like to analyze the scattering of these elementary partons at the largest p_T scale possible. The signal for hard parton-parton scattering is jet production and the most straightforward jet cross section is that for the inclusive production of a jet.

Unfortunately, there remain important ambiguities which limit our ability to perform detailed quantitative studies with the observed jet cross sections. One source of ambiguity is the experimental error in the measurement of the jet energy in a calorimeter. Another source is the uncertainty in the parton distribution functions,[4] which will be improved only by further deeply inelastic lepton scattering data and, for the gluons, by the sort of jet analysis discussed here. This paper addresses another class of ambiguities. These ambiguities are related to perturbation theory beyond the Born level and to the fact that a jet is *not* intrinsically well defined. At the Born level, one looks at a cross section for parton scattering, and assumes that each outgoing parton materializes into a narrow jet of particles. However, for reasons of color, energy-momentum conservation, and quantum-mechanical interference, a jet of hadrons *cannot* be the residue of a *single* parton. One first sees the difficulties in a calculation at one order beyond the Born level. Here, one finds that a careful definition of jet measurement is necessary.[3] The differences in jet definitions are presumably responsible for at least some of the approximately 50% difference between the reported[1] jet cross sections from UA1 and UA2. Further issues related to higher orders of perturbation theory are the choice of the renormalization/factorization[5] scale μ^2 and the value of the so-called "K factor" (characterizing the uncertainty in magnitude of the cross section due to higher-order contributions). Thus we can improve the situation by performing a complete calculation at one order beyond the Born approximation (i.e., at order α_s^3), leading to a theoretical uncertainty smaller than the current experimental error.

In earlier theoretical studies,[6] only incomplete QCD matrix elements at order α_s^3 were available. Recently the full order-α_s^3 matrix elements in $4-2\epsilon$ dimensions have been calculated.[7] In the present paper we describe in some detail a calculation[8] of the inclusive jet cross section using these full matrix elements, applied to the simplified case of gluons only. Results from this calculation have been presented elsewhere.[9] We find that the uncertainty associated with the choice of the renormalization scale μ^2 is reduced compared to the Born cross section, while a significant dependence on the cone size R used in the jet definition appears. In a subsequent publication[10] we will present the results of a full analysis of $p\bar{p}$ collisions including quarks.

An analysis based on the Ellis-Sexton matrix elements and focused mainly on single-particle inclusive production has been given by Aversa, Chiapetta, Greco, and Guillet.[11] These authors also calculate a jet cross section, but only in the limit in which the jet cone size R is much smaller than 1.

In this paper we describe the calculation of the jet cross section, deriving all of the formulas that are necessary for the numerical evaluation of the cross section. The essential problem is to carefully treat the collinear and soft singularities in the matrix elements. These singularities lead to integrals that are divergent when $\epsilon \to 0$, but the divergences cancel between real and virtual graphs when the physical jet cross section is calculated. Thus one must isolate the divergent contributions and perform the divergent integrations analytically, while leaving only finite integrations to be performed numerically. An important feature of the calculation is that, in contrast with earlier theoretical studies[6] and to the calculation of Aversa, Chiapetta, Greco, and Guillet,[11] we do *not* assume that the cone size R that appears in the jet definition is much smaller than 1 rad. This is important because the experimental groups, with good reason, use cone sizes in the range $\frac{1}{2} < R \leq 1$, and there is no reason to believe, without performing the calculation, that the small-angle approximation $R \ll 1$ is good anywhere in this range of R.

We are currently engaged in a calculation of the jet cross section with quarks and antiquarks included. In this calculation, there is some added complication compared to the gluon only case because there are so many processes to consider ($gg \to ggg, gg \to q\bar{q}g, qg \to qgg, \ldots$). It is important to realize, however, that there is no new physics when quarks are included: the singularity structure of the amplitudes is the same (or sometimes less singular) as in the gluon-only case. Thus the method described in this paper will work for quarks also.

The quantity that we wish to calculate is the inclusive cross section

$$\frac{d\sigma}{dy_J\, d\phi_J\, dp_J}, \qquad (1.1)$$

for production of a jet with rapidity y_J, azimuthal angle ϕ_J, and transverse energy p_J plus anything. Throughout this paper, we use p with an appropriate subscript to denote the magnitude of the transverse momentum of a gluon—that is, its transverse energy. In the case that two gluons make up the jet, p_J denotes the sum of the magnitudes of the transverse momenta of the gluons—that is, the total transverse energy of the jet. The rapidity y of a particle is defined as $\frac{1}{2}\ln[(E+p_z)/(E-p_z)]$. In the analysis of jet experiments, one would neglect particle masses compared to their transverse momenta. Then the particle rapidity is equal to its pseudorapidity, $-\ln[\tan(\theta/2)]$. In our calculation, only massless quarks and gluons appear, so there is no distinction between the rapidity and pseudorapidity of a particle. The mass of a jet consisting of several particles is not always negligible, so one must specify a definition of the jet rapidity. This definition is included in our jet definition below.

At the Born level, the cross section is computed using graphs such as that shown in Fig. 1(a). Strictly speaking, the cross section at this level is not a jet cross section but is rather just the parton cross section. At higher orders in perturbation theory, the parton cross section is infinite unless a finite jet size is introduced. At the Born level, the parton cross section is strongly μ^2 dependent, and it has no jet definition dependence to match that in the experimental cross section. Only at truly enormous values of p_J, where the size of a jet due to perturbative effects is small compared to the experimental resolution, is the Born cross section a reliable estimate of the jet cross section. Because of the ambiguities in the Born cross section, the so-called "K factor" is also poorly defined for the jet cross section—in contrast with the situation for lepton pair production (the Drell-Yan process), where the Born level process is essentially a QED process and is fairly well defined independent of α_s and jets.

At order α_s^3, graphs such as those in Fig. 1(b) are allowed and there is an explicit dependence on the jet definition. In the calculation, one must decide when two partons count as two jets and when they count as one. The calculation at this order allows us to account for the power of the "experimental microscope" to resolve one parton into two. It is exactly this careful treatment of the finite size of the jet which renders the jet cross section finite at all orders in perturbation theory, in analogy to what happens for similar quantities in e^+e^- physics.[12] Also, when two partons do count as one jet, one must define the resulting jet axis and jet transverse energy. In the experimental measurement, the differences between jet definition algorithms are now expected to matter, in that they can change the measured cross section at the same level as the α_s^3 corrections in the theory.

Let us consider for a moment the criteria which characterize a "good" jet definition. In general it has the following properties: (1) it is simple to implement in an experimental analysis; (2) it is simple to implement in the theoretical calculation; (3) it is defined at any order of perturbation theory; (4) it yields finite cross sections at any order of perturbation theory; and (5) it yields a cross section that is insensitive to hadronization.

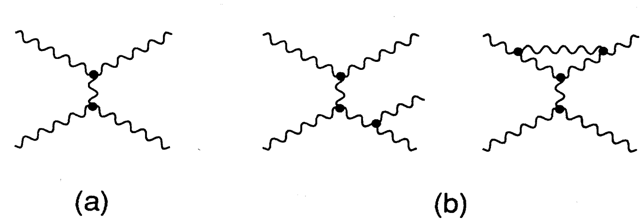

FIG. 1. (a) A Feynman diagram at order α_s^2; (b) subset of Feynman diagrams at order α_s^3.

The definition we use is as follows.[13] Let the calorimeter consist of cells i, in which transverse energies p_i are measured. Define a jet cone of radius R in y-ϕ space, centered on a cone axis (y_c, ϕ_c):

$$(y - y_c)^2 + (\phi - \phi_c)^2 < R^2. \qquad (1.2)$$

The cone radius R can be anything we like, subject to the restriction

$$R < \pi/3. \qquad (1.3)$$

(Certain complications would occur in the calculation if this bound were not imposed.) The transverse energy p_J of the jet is then

$$p_J = \sum_{i \text{ in cone}} p_i \,. \qquad (1.4)$$

The jet axis is defined by the weighted averages

$$y_J = \frac{1}{p_J} \sum_{i \text{ in cone}} p_i \, y_i, \quad \phi_J = \frac{1}{p_J} \sum_{i \text{ in cone}} p_i \, \phi_i \,. \qquad (1.5)$$

Finally, the cone axis (y_c, ϕ_c) must agree with the jet axis (y_J, ϕ_J) determined by Eq. (1.5).

With this definition, a single isolated parton with parameters (p, y, ϕ) will be "reconstructed" as a jet with these same parameters. Two partons with parameters (p_1, y_1, ϕ_1) and (p_2, y_2, ϕ_2) may be combined into a single jet. When two partons are combined into one jet, the jet transverse energy is

$$p_J = p_1 + p_2 \,, \qquad (1.6)$$

while the jet angles are

$$y_J = \frac{1}{p_J}(p_1 y_1 + p_2 y_2), \qquad (1.7)$$

$$\phi_J = \frac{1}{p_J}(p_1 \phi_1 + p_2 \phi_2). \qquad (1.8)$$

To determine if the two partons are to be combined, we see if the two partons fit in a cone of radius R about the jet axis. The condition that parton 1 fits into the cone is [denoting a two-dimensional vector $\Omega = (y, \phi)$]

$$|\Omega_1 - \Omega_J|^2 = (y_1 - y_J)^2 + (\phi_1 - \phi_J)^2 < R^2,$$

or

$$\frac{p_2}{p_1 + p_2} |\Omega_1 - \Omega_2| < R. \qquad (1.9)$$

The condition that parton 2 fit in the cone is

$$\frac{p_1}{p_1 + p_2} |\Omega_1 - \Omega_2| < R. \qquad (1.10)$$

Thus the combined condition is

$$|\Omega_1 - \Omega_2| < \frac{p_1 + p_2}{\max(p_1, p_2)} R. \qquad (1.11)$$

If the parton angles satisfy this condition, then we count one combined jet as specified above, but not the two smaller jets.

Note that in the case

$$R < |\Omega_1 - \Omega_2| < \frac{p_1 + p_2}{\max(p_1, p_2)} R \,, \qquad (1.12)$$

the two partons might with some logic count also as separate jets, but they do *not* in our calculation. In this respect, we differ from the jet definition of Furman,[6] which is also used by Aversa, Chiapetta, Greco, and Guillet.[11] In the Furman definition, when Eq. (1.12) is obeyed one has three jet choices that contribute to the inclusive jet cross section: $\{1\}, \{2\}, \{1 \text{ and } 2\}$. In our jet definition, the only choice retained is $\{1 \text{ and } 2\}$.

It is probably also informative at this point to briefly compare the jet definition we are using to the jet definitions being implemented by currently running experiments as described in the literature.[14] The basic jet structure can be pictured as a set of calorimeter cells in which energy has been detected. The energy measured in a specific cell is often characterized in terms of the transverse energy E_T (denoted as p_J above), defined by projecting the measured energy onto the transverse plane (i.e., by multiplying by a factor of $\sin\theta$ corresponding to the angular location of that specific calorimeter cell). The jet-reconstruction algorithm of the UA2 Collaboration involves the clustering together of all *contiguous* cells with individual energies in excess of 400 MeV. The clusters (of cells) formed in this way are then tested for multiple local energy maxima within a single cluster. Two local maxima separated by an energy "valley" of depth greater than 5 GeV are split apart into two clusters. The resulting sample of clusters are labeled as jets. The UA1 algorithm involves the concept of an "initiator" cell. Among cells with $E_T > 2.5$ GeV, the largest E_T cell initiates the first jet. In decreasing order in E_T each subsequent cell (with $E_T > 2.5$ GeV) is either included in the nearest (in y, ϕ) existing jet, if it is within a cone of R=1 of that jet's initiator direction, or it initiates a new jet. The remaining cells (with $E_T < 2.5$ GeV) are included in the nearest jet if the cell's momentum transverse to the jet direction is < 1 GeV/c and the angular separation from the jet is less than 45°. The CDF group, with somewhat smaller cell size, uses the following algorithm. Contiguous cells (towers) with $E_T > 1.0$ GeV are associated together as preclusters (initiators). For each precluster with $E_T > 2.0$ GeV an E_T weighted centroid in y, ϕ is computed and all cells with $E_T > 0.2$ GeV within a cone of $R = 0.6$ about the centroid form a cluster. The process of recalculating the centroid of the new cluster and redefining the cluster within $R = 0.6$ of the new centroid is iterated until the cluster list is stable. Clusters which overlap are merged if they share more than 50% of their energy. The common cells of unmerged clusters are assigned to the nearest cluster. The resulting clusters are the jets. The final jet E_T is defined as the scalar sum of the cell energies in the jet times $\sin\theta$ of the jet centroid. Clearly the latter two algorithms are closer to that used in the present theoretical analysis. In the present analysis we will not discuss how the differences in the various jet algorithms affect the magnitude of the jet cross section. Nor we will discuss the related issues of fragmentation effects, underlying event contributions, or perturbative contributions at even higher orders.

The jet cross section calculated in this paper is expressed as the convolution of a hard-scattering cross section with parton distribution functions. At the order of perturbation theory that we use, one must make a choice of definitions of the parton distribution functions. We have chosen the modified minimal-subtraction ($\overline{\rm MS}$) definition, in which the parton distribution functions are defined as the proton matrix elements of certain simple operators.[15] Graphs for the operators are ultraviolet divergent, and the divergences are removed by the $\overline{\rm MS}$ renormalization. This definition leads to a simple prescription for subtracting the divergences corresponding to collinear parton emission from the incoming lines in the jet calculation. This prescription is applied in Eq. (4.56). A general discussion of the relation of the calculational prescription to the definition of the parton distribution functions can be found in the literature.[16]

The organization of this paper is as follows. In Sec. I B we discuss the $gg \to gg$ hard process, which includes the virtual corrections to the Born amplitude. In Sec. I C we discuss the organization of the calculation for the $gg \to ggg$ hard process. The rest of the main body of the paper follows the singularity structure of the amplitude. In Sec. II we discuss the singularity in which the softest gluon is very soft or is collinear to the gluon in the jet direction. Most of the key ideas of the paper are introduced in this section. In Sec. III we discuss the singularity in which the softest gluon is very soft or is collinear to the gluon opposite to the jet direction. In Sec. IV we discuss the singularity in which the softest gluon is very soft or is collinear to one of the incoming gluons. In Sec. V we demonstrate the cancellation of the $\epsilon \to 0$ divergences. Finally, in Sec. VI, we state some conclusions. There are two appendixes, one concerning the definition of the jet axis in $2-2\epsilon$ transverse dimensions, the other concerning the analytical calculation of the integrals that arise when the softest gluon becomes very soft.

B. The 2→2 hard process

First consider two gluons, A and B, scattering to two gluons, 1 and 2. Let us suppose that gluon 2 makes the observed jet. We shall use light-cone coordinates in which vectors are given by components $p^\mu = (p^+, p^-, \mathbf{p})$, with $p^\pm = (p^0 \pm p^3)/\sqrt{2}$ and $\mathbf{p} = (p^1, p^2)$. Throughout this paper, we shall take momentum components in the hadron-hadron center-of-mass frame, with the z axis aligned with the beam direction. With this notation, we denote the outgoing gluon momenta by

$$\begin{aligned} p_1^\mu &= \left(\frac{1}{\sqrt{2}} p_J e^{y_1}, \frac{1}{\sqrt{2}} p_J e^{-y_1}, -\mathbf{p}_J\right), \\ p_2^\mu &= \left(\frac{1}{\sqrt{2}} p_J e^{y_J}, \frac{1}{\sqrt{2}} p_J e^{-y_J}, \mathbf{p}_J\right), \end{aligned} \tag{1.13}$$

where p_k denotes the absolute value of the transverse part $\mathbf{p}_k$ of p_k^μ. We have used the fact, which follows from transverse-momentum conservation, that

$$\mathbf{p}_1 = -\mathbf{p}_2. \tag{1.14}$$

Note that y_1 is not fixed. We will express the other variables in the problem in terms of the observed jet variables, the hadron-hadron center-of-mass energy s, and also y_1, which we will retain as an integration variable.

Let us now look at the momenta of the incoming partons. They may be written in terms of momentum fraction variables x_A, x_B as

$$p_A^\mu = (x_A\sqrt{s/2}, 0, \mathbf{0}), \quad p_B^\mu = (0, x_B\sqrt{s/2}, \mathbf{0}). \tag{1.15}$$

Let us define another set of momentum fraction variables X_A, X_B as functions of y_1, y_J, and p_J so that $X_A\sqrt{s/2}$ is the plus component of the momentum of the system formed by gluon 1 and the jet and $X_B\sqrt{s/2}$ is the minus component of the momentum of this system. Thus

$$X_A = \frac{p_J}{\sqrt{s}}\,(e^{y_1} + e^{y_J}), \quad X_B = \frac{p_J}{\sqrt{s}}\,(e^{-y_1} + e^{-y_J}). \tag{1.16}$$

In the $2 \to 2$ hard scattering,which we are now discussing, momentum conservation implies that X_A and X_B are the momentum fractions of the respective incoming gluons:

$$x_A = X_A, \quad x_B = X_B. \tag{1.17}$$

However, this relation is modified when there is an additional gluon in the final state.

We also introduce variables $\hat{S}, \hat{T}, \hat{U}$ formed from y_1, y_J, and p_J in such a way that, when we are discussing the $2 \to 2$ cross section, these are the Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$ of the elementary gluon scattering. Thus [cf. Eq. (1.19) below],

$$\begin{aligned} \hat{S} &= 2p_J^2[1 + \cosh(y_J - y_1)], \\ \hat{T} &= -p_J^2(1 + e^{y_J - y_1}), \\ \hat{U} &= -p_J^2(1 + e^{y_1 - y_J}). \end{aligned} \tag{1.18}$$

(We use upper-case letters for $\hat{S}, \hat{T}, \hat{U}$ in order to avoid confusion with, for instance, the center-of-mass energy squared $\hat{s}$ of the two incoming gluons in the $2 \to 3$ process.)

It is useful to have a catalog of all of the invariants:

$$\begin{aligned} &p_A \cdot p_B = \tfrac{1}{2}\hat{S} = p_J^2[1 + \cosh(y_J - y_1)], \\ &p_1 \cdot p_A = -\tfrac{1}{2}\hat{T} = \tfrac{1}{2}p_J^2(1 + e^{y_J - y_1}), \\ &p_2 \cdot p_A = -\tfrac{1}{2}\hat{U} = \tfrac{1}{2}p_J^2(1 + e^{y_1 - y_J}), \\ &p_1 \cdot p_2 = p_A \cdot p_B, \quad p_1 \cdot p_B = p_2 \cdot p_A, \\ &p_2 \cdot p_B = p_1 \cdot p_A. \end{aligned} \tag{1.19}$$

Let us now express the contribution to the one-jet inclusive cross section in terms of the invariant matrix element for the 2 → 2 process. We denote the invariant matrix element squared and summed over final spins and averaged over initial spins in $4-2\epsilon$ dimensions by $\langle|\mathcal{M}|^2\rangle$. We denote the distribution functions for the incoming partons by $f_A(x_A)$ and $f_B(x_B)$. Then the cross section in $4-2\epsilon$ dimensions is

$$d\sigma = \int dx_A f_A(x_A) \int dx_B f_B(x_B) \frac{1}{2x_A x_B s}\, \mu^{2\epsilon}\frac{dy_1\, d\mathbf{p}_1}{2(2\pi)^{3-2\epsilon}}\, \mu^{2\epsilon}\frac{dy_2\, d\mathbf{p}_2}{2(2\pi)^{3-2\epsilon}}\, \langle|\mathcal{M}|^2\rangle\, \mu^{-2\epsilon}(2\pi)^{4-2\epsilon}\delta^{4-2\epsilon}(p_1+p_2-p_A-p_B)$$
$$= \int dx_A f_A(x_A) \int dx_B f_B(x_B)\, \frac{\mu^{2\epsilon}}{8x_A x_B\, s(2\pi)^{2-2\epsilon}}\, \langle|\mathcal{M}|^2\rangle\, dy_1\, dy_2\, d\mathbf{p}_2 \delta(p_A^+ - p_1^+ - p_2^+)\delta(p_B^- - p_1^- - p_2^-), \qquad (1.20)$$

where, as usual, the dimensionful parameter μ is introduced in order to maintain a dimensionless coupling g.

The delta functions for conservation of the plus and minus components of momentum can be used to perform the x_A and x_B integrations. We use

$$\begin{aligned}\delta(p_A^+ - p_1^+ - p_2^+) &= \delta(x_A\sqrt{s/2} - p_1^+ - p_2^+)\\ &= \sqrt{2/s}\; \delta(x_A - X_A),\\ \delta(p_B^- - p_1^- - p_2^-) &= \delta(x_B\sqrt{s/2} - p_1^- - p_2^-)\\ &= \sqrt{2/s}\; \delta(x_B - X_B).\end{aligned} \qquad (1.21)$$

For the remaining transverse-momentum integration we use

$$d\mathbf{p}_2 = p_2^{1-2\epsilon} dp_2\; d^{1-2\epsilon}\phi_2, \qquad (1.22)$$

where $d^{1-2\epsilon}\phi$ denotes integration over a $(1-2\epsilon)$-dimensional sphere. We identify the p_2^μ variables with the jet variables. We thus obtain [with $s = \hat{S}/(X_A X_B)$]

$$\frac{d\sigma}{dp_J\, dy_J\, d^{1-2\epsilon}\phi_J} = \frac{p_J}{(4\pi)^2}\left(\frac{p_J^2}{4\pi^2\mu^2}\right)^{-\epsilon} \times \int dy_1\; X_A f_A(X_A)\; X_B f_B(X_B) \times \frac{1}{\hat{S}^2}\, \langle|\mathcal{M}|^2\rangle. \qquad (1.23)$$

This result will enable us to calculate the contribution from the 2 → 2 hard process to the one-jet inclusive cross section from $\mathcal{M}$. At the Born level, we just need one numerical integration. At the next order, the matrix element will have $1/\epsilon$ and $1/\epsilon^2$ terms. We will have to extract these terms in the cross section $d\sigma/dp_J dy_J$ and show that they cancel. Only then can we set $\epsilon = 0$ and perform the numerical integration.

We denote the cross section of the 2 → 2 subprocess as $I^{(2\to2)}$. It can be written as an integral over y_1 of a function $G^{(2\to2)}(y_1, p_J, y_J, \phi_J)$,

$$I^{(2\to2)} = \frac{d\sigma}{dp_J\, dy_J\, d^{1-2\epsilon}\phi_J} = \int dy_1 G^{(2\to2)}(y_1, p_J, y_J, \phi_J). \qquad (1.24)$$

The function $G^{(2\to2)}$ is simply related to the quantities calculated by Ellis and Sexton as follows:

$$G^{(2\to2)}(y_1, p_J, y_J, \phi_J) = C(\epsilon)L(X_A, X_B) \times \left[d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon) + \frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q_{\rm ES}^2}\right)^{\epsilon}\Gamma_K(\epsilon)[D_S(\hat{S},\hat{T},\hat{U};\epsilon) + D_{\rm NS}(\hat{S},\hat{T},\hat{U})]\right]. \qquad (1.25)$$

Here

$$C(\epsilon) = \left(\frac{p_J^2}{4\pi^2\mu^2}\right)^{-\epsilon} \frac{\alpha_s^2 p_J}{4s^2(1-\epsilon)^2 V^2}, \qquad (1.26)$$

where

$$V = N^2 - 1, \qquad (1.27)$$

with $N = 3$ being the number of colors. The function L describes the parton luminosity:

$$L(X_A, X_B) = \frac{f_{g/A}(X_A)\; f_{g/B}(X_B)}{X_A X_B}. \qquad (1.28)$$

The function $\Gamma_K(\epsilon)$ denotes the product of gamma functions:

$$\Gamma_K(\epsilon) = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} = 1 + O(\epsilon). \qquad (1.29)$$

Finally, $Q_{\rm ES}^2$ is an arbitrary scale parameter introduced by Ellis and Sexton, called simply Q^2 in their paper. The cross section is independent of $Q_{\rm ES}^2$.

The invariant function $d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)$ is

$$\begin{aligned} d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon) &= 16VN^2\,(1-\epsilon)^2\left(3 - \frac{\hat{U}\hat{T}}{\hat{S}^2} - \frac{\hat{U}\hat{S}}{\hat{T}^2} - \frac{\hat{S}\hat{T}}{\hat{U}^2}\right)\\ &= 4VN^2\,(1-\epsilon)^2\,\frac{\left(\hat{S}^2+\hat{T}^2+\hat{U}^2\right)\left(\hat{S}^4+\hat{T}^4+\hat{U}^4\right)}{\hat{S}^2\hat{T}^2\hat{U}^2}. \end{aligned} \qquad (1.30)$$

The first form is the form given by Ellis and Sexton. The second form is equivalent (using $\hat{S}+\hat{T}+\hat{U}=0$) and is more convenient for algebraic manipulations.

In the next-order piece, $D_{\rm S}$ denotes the terms that are singular as $\epsilon \to 0$ (together with some simple nonsingular terms) and $D_{\rm NS}$ denotes the remaining finite terms. In general, the function $D_{\rm S}$ has the form

$$D_{\rm S}(\hat{S},\hat{T},\hat{U};\epsilon) = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\; D_{\rm S1} + D_{\rm S2},$$

$$D_{\rm S1} = -\frac{4N}{\epsilon^2} + \frac{8T_R - 22N}{3\epsilon} + \frac{20T_R - 67N}{9} + N\pi^2 + \frac{11N - 4T_R}{3}\ln\left(\frac{\mu^2}{Q_{\rm ES}^2}\right), \quad (1.31)$$

$$D_{\rm S2} = \frac{16VN^3}{\epsilon}\left[\ln\left(\frac{\hat{S}}{Q_{\rm ES}^2}\right) f_{\rm S}(\hat{S},\hat{T},\hat{U}) + \ln\left(\frac{-\hat{T}}{Q_{\rm ES}^2}\right) f_{\rm S}(\hat{T},\hat{U},\hat{S}) + \ln\left(\frac{-\hat{U}}{Q_{\rm ES}^2}\right) f_{\rm S}(\hat{U},\hat{S},\hat{T})\right],$$

where

$$f_{\rm S}(\hat{S},\hat{T},\hat{U}) = 3 - \frac{2\hat{T}\hat{U}}{\hat{S}^2} + \frac{\hat{T}^4+\hat{U}^4}{\hat{T}^2\hat{U}^2} = \frac{(\hat{T}^2+\hat{U}^2)(\hat{S}^4+\hat{T}^4+\hat{U}^4)}{2\hat{S}^2\hat{T}^2\hat{U}^2}. \quad (1.32)$$

Here T_R is half the number of flavors, which is zero in our present calculation, in which we use QCD with gluons but not quarks. The first form of $f_{\rm S}$ in Eq. (1.32) is the form given by Ellis and Sexton. The second form is equivalent (using $\hat{S}+\hat{T}+\hat{U}=0$) and is more convenient for algebraic manipulations. Later in the calculation, we will make use of an identity relating $f_{\rm S}(\hat{S},\hat{T},\hat{U})$ to $d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)$:

$$4VN^2\,(1-\epsilon)^2[f_{\rm S}(\hat{S},\hat{T},\hat{U}) + f_{\rm S}(\hat{T},\hat{U},\hat{S}) + f_{\rm S}(\hat{U},\hat{S},\hat{T})] = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon). \quad (1.33)$$

In this paper, we use the symbol μ to denote both the renormalization scale, which arises from the removal of ultraviolet divergences, and the factorization scale, which arises from the removal of collinear divergences. If one wishes to distinguish the two scales as $\mu_{\rm UV}$ and $\mu_{\rm coll}$, then the μ in $D_{\rm S1}$ above is $\mu_{\rm UV}$.

The nonsingular part is

$$D_{\rm NS}(\hat{S},\hat{T},\hat{U}) = 4VN^2[\,f_{\rm NS}(\hat{S},\hat{T},\hat{U}) + f_{\rm NS}(\hat{T},\hat{U},\hat{S}) + f_{\rm NS}(\hat{U},\hat{S},\hat{T})], \quad (1.34)$$

where

$$f_{\rm NS}(\hat{S},\hat{T},\hat{U}) = N\Bigg\{\left(\frac{2(\hat{T}^2+\hat{U}^2)}{\hat{T}\hat{U}}\right)\left[\ln^2\!\left(\frac{|\hat{S}|}{Q_{\rm ES}^2}\right) - \pi^2\,\theta(\hat{S}>0)\right] + \left(\frac{4\hat{S}(\hat{T}^3+\hat{U}^3)}{\hat{T}^2\hat{U}^2} - 6\right)\ln\!\left(\frac{|\hat{T}|}{Q_{\rm ES}^2}\right)\ln\!\left(\frac{|\hat{U}|}{Q_{\rm ES}^2}\right) + \left[\frac{4}{3}\frac{\hat{T}\hat{U}}{\hat{S}^2} - \frac{14}{3}\frac{\hat{T}^2+\hat{U}^2}{\hat{T}\hat{U}} - 14 - 8\left(\frac{\hat{T}^2}{\hat{U}^2}+\frac{\hat{U}^2}{\hat{T}^2}\right)\right]\ln\!\left(\frac{|\hat{S}|}{Q_{\rm ES}^2}\right) - 1 - \pi^2\Bigg\}. \quad (1.35)$$

C. Organizing the 2→3 calculation

We now consider the case of three final-state gluons, numbered 1,2,3. We shall follow the same notation used for the $2\to 2$ hard process and define

$$p_k^\mu = \left(\frac{1}{\sqrt{2}}p_k e^{y_k}, \frac{1}{\sqrt{2}}p_k e^{-y_k}, \mathbf{p}_k\right) \quad (1.36)$$

for $k = 1,2,3$.

We must integrate over the momenta of the three final-state gluons. However, we can make the calculation more efficient if we integrate over each event topology only once. This amounts to defining which of the three identical gluons is to be labeled with each index 1,2,3. We shall define gluon 3 to be the gluon with the smallest transverse momentum:

$$p_3 < p_1, p_2. \quad (1.37)$$

In addition, we shall differentiate between gluons 1 and 2 by saying that the azimuthal angle of gluon 2 is nearer to that of the jet axis than is the azimuthal angle of gluon 1:

$$|\phi_2 - \phi_J| < |\phi_1 - \phi_J|. \quad (1.38)$$

Let us consider the implications of this choice on which singularities can occur in the integration region. The singularities are (a) gluon 3 soft, (b) gluon 3 collinear with gluon 1, (c) gluon 3 collinear with gluon 2, (d) gluon 3 collinear with gluon A, and (e) gluon 3 collinear with gluon B. Other singularities do not occur with these definitions. Transverse-momentum conservation does not allow two gluons to be soft or collinear to one of the beam gluons, A or B, at the same time. Gluon 1 or gluon 2 cannot be soft while gluon 3 is hard because of condition (1.37). Similarly, gluon 1 or gluon 2 cannot be collinear to one of the beam gluons while gluon 3 is hard because the transverse-momentum of gluon 1 or 2 would have to go to zero, contradicting condition (1.37). Finally, gluon 1 cannot be collinear with gluon 2 because then transverse-momentum conservation would imply that gluon 3 had the largest transverse momentum, contradicting (1.37).

Let us now consider the implications of the conditions (1.37) and (1.38) for the question of which gluons can be in the jet. The possibilities are (i) gluon 3 is the jet,

(ii) gluon 2 is the jet, and (iii) gluons 2 and 3 are the jet. Other possibilities do not occur. Gluon 1 cannot be the jet because of condition (1.38). Gluons 1 and 3 cannot be the jet because of condition (1.38): transverse momentum conservation would imply that gluon 2 was in the opposite half-circle from the jet axis, while gluon 1 is in the same half-circle. [Recall from Eq. (1.3) that we assume $R < \pi/3$.]

Finally, gluons 1 and 2 cannot be the jet. To see why not, we suppose that gluons 1 and 2 *do* constitute the jet and show that the jet cone size would then have to be bigger than the maximum allowed cone size: $R_{\max} = \pi/3$.

The argument is illustrated in Fig. 2. We suppose that gluons 1 and 2 constitute the jet. Let θ_k be the absolute value of the angle between $\mathbf{p}_k$ and $-\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2$, for $k = 1, 2$, as indicated in Fig. 2(a). Then conservation of transverse momentum gives

$$p_1 \sin(\theta_1) = p_2 \sin(\theta_2), \tag{1.39}$$

$$p_1 \cos(\theta_1) + p_2 \cos(\theta_2) = p_3. \tag{1.40}$$

Eliminating p_2 from these equations gives

$$p_1 \left(\cos(\theta_1) + \frac{\sin(\theta_1)}{\sin(\theta_2)} \cos(\theta_2) \right) = p_3 \tag{1.41}$$

or

$$\sin(\theta_1 + \theta_2) = \sin(\theta_2)\, \frac{p_3}{p_1}. \tag{1.42}$$

We use these equations to map a region from which (θ_1, θ_2) is excluded, as illustrated in Fig. 2(b). The first boundary arises from the requirement (1.37) that p_3 be less than p_1. Using Eq. (1.42), this gives

$$\sin(\theta_1 + \theta_2) < \sin(\theta_2) \tag{1.43}$$

or

$$\frac{\pi}{2} - \theta_2 < \theta_1 + \theta_2 - \frac{\pi}{2}. \tag{1.44}$$

Thus

$$\theta_1 + 2\theta_2 > \pi. \tag{1.45}$$

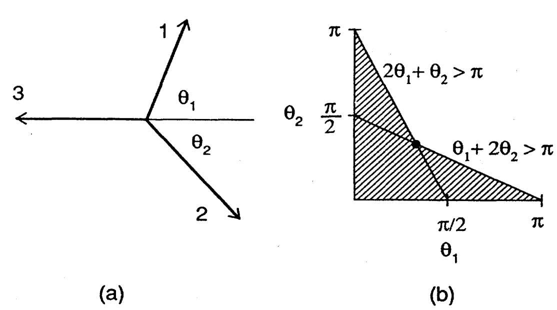

FIG. 2. Demonstration that gluons 1 and 2 cannot be the jet.

Similar reasoning with the roles of gluons 1 and 2 interchanged give

$$2\theta_1 + \theta_2 > \pi. \tag{1.46}$$

The two excluded regions are indicated by shading in Fig. 2(b). By adding the two inequalities (1.45) and (1.46), we obtain

$$\theta_1 + \theta_2 > \frac{2\pi}{3}, \tag{1.47}$$

where the minimum value of $\theta_1 + \theta_2$ is realized at the black dot in the figure. Since gluon 1 and gluon 2 are supposed to constitute the jet, we have

$$\theta_1 + \theta_2 < 2R. \tag{1.48}$$

This contradicts the condition $R < \pi/3$ that we have imposed. We conclude that, contrary to our hypothesis, gluons 1 and 2 cannot constitute the jet.

We now discuss the decomposition of the matrix element $\langle |\mathcal{M}|^2 \rangle$ for the $2 \to 3$ process. As we have seen, the only singularities that we have to worry about are those connected with gluon 3 becoming soft or collinear with one of the other gluons. Each term in $\langle |\mathcal{M}|^2 \rangle$ as given by Ellis and Sexton contains singular factors of the form

$$\frac{1}{p_n \cdot p_3\; p_m \cdot p_3}, \tag{1.49}$$

where $n, m = A, B, 1, 2$ with $n \neq m$. Denoting the coefficient of this denominator, after extracting some common factors, as f_{nm}, we have

$$\langle |\mathcal{M}|^2 \rangle_{2\to 3} = \frac{16\pi^3 \alpha_s^3}{(1-\epsilon)^2 V^2} \sum_{\substack{n,m \\ n<m}} \frac{f_{nm}}{p_n \cdot p_3\; p_m \cdot p_3}. \tag{1.50}$$

Strictly speaking, the f_{nm} are not uniquely defined by Eq. (1.50) away from $p_3^\mu = 0$. For instance, one could add $p_1 \cdot p_3$ to $f_{1,2}$ and subtract $p_A \cdot p_3$ from $f_{A,2}$ without changing $\langle |\mathcal{M}|^2 \rangle$. However, this lack of uniqueness does not affect the result of our calculation. We have performed the decomposition (1.50) following the highly symmetric form for $\langle |\mathcal{M}|^2 \rangle$ given by Ellis and Sexton, in which the denominator structure of Eq. (1.50) is explicit.

Each denominator factor in Eq. (1.50) can be rewritten in the form

$$\frac{1}{p_n \cdot p_3\; p_m \cdot p_3} = \frac{1}{(p_n + p_m) \cdot p_3\; p_n \cdot p_3} + \frac{1}{(p_n + p_m) \cdot p_3\; p_m \cdot p_3}. \tag{1.51}$$

Since $(p_n + p_m)^\mu$ is a timelike vector, the factor $1/(p_n + p_m) \cdot p_3$ gives a singularity only when p_3 is soft. In this way we decompose $\langle |\mathcal{M}|^2 \rangle$ into

$$\langle |\mathcal{M}|^2 \rangle_{2\to 3} = \langle |\mathcal{M}|^2 \rangle_A + \langle |\mathcal{M}|^2 \rangle_B + \langle |\mathcal{M}|^2 \rangle_1 + \langle |\mathcal{M}|^2 \rangle_2, \tag{1.52}$$

where (denoting $f_{nm} = f_{mn}$ for $n > m$)

$$\langle|\mathcal{M}|^2\rangle_n = \frac{16\pi^3\alpha_s^3}{(1-\epsilon)^2V^2}\,\frac{1}{p_n\cdot p_3}\sum_{\substack{m\\ m\neq n}}\frac{f_{nm}}{(p_n+p_m)\cdot p_3}. \tag{1.53}$$

We see that $\langle|\mathcal{M}|^2\rangle_n$ contains a $1/p_n\cdot p_3$ collinear singularity and a soft singularity for gluon 3. Each of these four terms is treated separately in the calculation.

In much of our computation, we write the reduced matrix elements $\langle|\mathcal{M}|^2\rangle_n$ in terms of functions f_n in a form that displays both the $p_n\cdot p_3\to 0$ singularity and the $p_3^\mu\to 0$ singularity in $\langle|\mathcal{M}|^2\rangle_n$,

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle_A &= \frac{16\pi^3\alpha_s^3}{(1-\epsilon)^2V^2}\,\frac{p_{AB}}{p_{B3}}\,\frac{f_A}{p_{A3}},\\ \langle|\mathcal{M}|^2\rangle_B &= \frac{16\pi^3\alpha_s^3}{(1-\epsilon)^2V^2}\,\frac{p_{AB}}{p_{A3}}\,\frac{f_B}{p_{B3}},\\ \langle|\mathcal{M}|^2\rangle_1 &= \frac{16\pi^3\alpha_s^3}{(1-\epsilon)^2V^2}\,\frac{p_1}{p_3}\,\frac{f_1}{p_{13}},\\ \langle|\mathcal{M}|^2\rangle_2 &= \frac{16\pi^3\alpha_s^3}{(1-\epsilon)^2V^2}\,\frac{p_2}{p_3}\,\frac{f_2}{p_{23}},\end{aligned} \tag{1.54}$$

where p_{ij} denotes the dot product

$$p_{ij} = p_{i\mu}p_j^\mu. \tag{1.55}$$

[To avoid confusion, we note that this definition gives f_A a zero when p_3^μ becomes parallel to p_B. Similarly, f_B has a zero when p_3^μ becomes parallel to p_A and f_1 and f_2 have zeros when the transverse components of p_3^μ becomes zero with nonzero p^+ or p^-. We also note that these functions f_A and f_B are not related to the parton distribution functions $f_A(x)$ and $f_B(x)$. The distinction between these two sets of functions should be clear from the context.]

We have computed the functions f_A, f_B, f_1,and f_2 using MACSYMA, starting from the matrix elements given by Ellis and Sexton. These functions are needed in our calculation only with $\epsilon=0$ or with $\epsilon\neq 0$ but with the kinematic variables evaluated at certain singular points. The results that we will use are as follows.

Full expressions at $\epsilon = 0$:

$$\begin{aligned}f_A(\epsilon=0) = \frac{2N^3V}{p_{AB}}\sum_{\substack{n,m\\ n<m}}p_{nm}^4\biggl(&\frac{1}{p_{A1}\,p_{B1}\,p_{B2}\,(p_{A3}+p_{23})/p_{B3}}+\frac{1}{p_{A2}\,p_{B1}\,p_{B2}\,(p_{A3}+p_{13})/p_{B3}}\\ &+\frac{1}{p_{12}\,p_{A1}\,p_{B2}\,(p_{A3}+p_{B3})/p_{B3}}+\frac{1}{p_{12}\,p_{AB}\,p_{B2}\,(p_{A3}+p_{13})/p_{B3}}\\ &+\frac{1}{p_{12}\,p_{A2}\,p_{B1}\,(p_{A3}+p_{B3})/p_{B3}}+\frac{1}{p_{12}\,p_{AB}\,p_{B1}\,(p_{A3}+p_{23})/p_{B3}}\biggr),\end{aligned} \tag{1.56}$$

$$f_B(\epsilon=0) = f_A(\epsilon=0)\quad\text{with } A\leftrightarrow B, \tag{1.57}$$

$$\begin{aligned}f_1(\epsilon=0) = \frac{2N^3V}{p_1}\sum_{\substack{n,m\\ n<m}}p_{nm}^4\biggl(&\frac{1}{p_{A1}\,p_{A2}\,p_{B2}\,(p_{B3}+p_{13})/p_3}+\frac{1}{p_{A2}\,p_{B1}\,p_{B2}\,(p_{A3}+p_{13})/p_3}\\ &+\frac{1}{p_{12}\,p_{AB}\,p_{B2}\,(p_{A3}+p_{13})/p_3}+\frac{1}{p_{A1}\,p_{AB}\,p_{B2}\,(p_{13}+p_{23})/p_3}\\ &+\frac{1}{p_{12}\,p_{A2}\,p_{AB}\,(p_{B3}+p_{13})/p_3}+\frac{1}{p_{A2}\,p_{AB}\,p_{B1}\,(p_{13}+p_{23})/p_3}\biggr),\end{aligned} \tag{1.58}$$

$$f_2(\epsilon=0) = f_1(\epsilon=0)\quad\text{with } 1\leftrightarrow 2. \tag{1.59}$$

Collinear limits: Let gluon 2 be the jet, $p_2^\mu = p_J^\mu$ where we denote

$$p_J^\mu = (p_Je^{y_J}/\sqrt{2}, p_Je^{-y_J}/\sqrt{2}, p_J\cos\phi_J, p_J\sin\phi_J). \tag{1.60}$$

Let gluon 3 be collinear with the incoming gluon from hadron A, $p_3^\mu = (1-z)p_A^\mu$. Then

$$f_A = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\,\frac{1-z}{z}\,\tilde{P}_{gg}(z), \tag{1.61}$$

where $\tilde{P}_{gg}(z)$ is the Altarelli-Parisi kernel for $g\to g$, but without the usual regulation for $z\to 1$. Specifically

$$\tilde{P}_{gg}(z) = 2N\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right), \tag{1.62}$$

whereas

$$\begin{aligned}P_{gg}(z) &= 2N\left(\frac{1-z}{z}+\frac{z}{(1-z)_+}+z(1-z)\right)\\ &\quad+\tfrac{1}{2}\beta_0\,\delta(1-z),\end{aligned} \tag{1.63}$$

$$\beta_0 = \frac{11N-4T_R}{3}.$$

(Again, T_R is half the number of flavors, which is zero in our gluon-only calculation.) Similarly, when $p_2^\mu = p_J^\mu$ and $p_3^\mu = (1-z)p_B^\mu$ we find

$$f_B = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\ \frac{1-z}{z}\ \tilde{P}_{gg}(z). \tag{1.64}$$

Let gluon 2 be the jet, $p_2^\mu = p_J^\mu$, and let gluon 3 be collinear with gluon 1, $p_3^\mu = [(1-z)/z]p_1^\mu$. Then

$$f_1 = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\ \frac{1-z}{z}\ \tilde{P}_{gg}(z). \tag{1.65}$$

Finally, let gluons 2 and 3 be collinear and together constitute the jet, $p_3^\mu = (1-z)p_J^\mu$, $p_2^\mu = zp_J^\mu$. Then

$$f_2 = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\ \frac{1-z}{z}\ \tilde{P}_{gg}(z). \tag{1.66}$$

Soft limits: Let gluon 2 be the jet, $p_2^\mu = p_J^\mu$, and let $p_3^\mu = 0$. Then the functions f_{nm} defined in Eq. (1.50) are given in terms of the function f_S defined in Eq. (1.32) by

$$\begin{aligned} f_{AB} = f_{12} &= 4VN^3(1-\epsilon)^2\ \hat{S}\ f_S(\hat{S},\hat{T},\hat{U}),\\ f_{A1} = f_{B2} &= 4VN^3(1-\epsilon)^2\ (-\hat{T})\ f_S(\hat{T},\hat{U},\hat{S}),\\ f_{A2} = f_{B1} &= 4VN^3(1-\epsilon)^2\ (-\hat{U})\ f_S(\hat{U},\hat{S},\hat{T}). \end{aligned} \tag{1.67}$$

II. TERM 2

A. Kinematics

In this section we are concerned with the term $\langle|\mathcal{M}|^2\rangle_2$ in the decomposition (1.53) of the invariant matrix element. We are thus interested in the integration region in which gluons 2 and 3 may become collinear, or in which gluon 3 may become soft.

There are nine variables needed to describe the momenta of the final-state gluons, and these are restricted by two conservation equations after the other two conservation equations are used to determine x_A and x_B. Thus seven variables remain as integration variables. (Of these, three will be eliminated later by the delta functions that define the jet variables p_J, y_J, ϕ_J.) We take the seven variables to be

$$y_1; p_2, y_2, \phi_2; p_3, y_3, \phi_3. \tag{2.1}$$

Here, as before, p_k denotes the magnitude of the transverse part $\mathbf{p}_k$ of p_k^μ.

We shall need to express p_1 and ϕ_1 in terms of these seven variables. We have $p_1^2 = (\mathbf{p}_2 + \mathbf{p}_3)^2$ so

$$p_1 = [p_2^2 + p_3^2 + 2p_2p_3\cos(\phi_2 - \phi_3)]^{1/2}. \tag{2.2}$$

For the evaluation of invariants involving ϕ_1, we use transverse-momentum conservation:

$$\begin{aligned} -\mathbf{p}_1\cdot\mathbf{p}_2 &= (\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{p}_2 = p_2^2 + p_2p_3\cos(\phi_2-\phi_3),\\ -\mathbf{p}_1\cdot\mathbf{p}_3 &= (\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{p}_3 = p_3^2 + p_2p_3\cos(\phi_2-\phi_3). \end{aligned} \tag{2.3}$$

Of particular importance is condition (1.38), $|\phi_2 - \phi_J| < |\phi_1 - \phi_J|$, or $\cos(\phi_2-\phi_J) > \cos(\phi_1 - \phi_J)$. To express this in terms of the chosen variables, we write

$$\frac{\mathbf{p}_J\cdot\mathbf{p}_2}{p_Jp_2} > \frac{\mathbf{p}_J\cdot\mathbf{p}_1}{p_Jp_1},\quad \frac{\mathbf{p}_J\cdot\mathbf{p}_2}{p_Jp_2} > -\frac{\mathbf{p}_J\cdot(\mathbf{p}_2+\mathbf{p}_3)}{p_Jp_1},$$

$$\cos(\phi_2-\phi_J) > -\frac{p_2\cos(\phi_2-\phi_J) + p_3\cos(\phi_3-\phi_J)}{p_1}.$$

Thus the required condition is

$$(p_1+p_2)\cos(\phi_2-\phi_J) + p_3\cos(\phi_3-\phi_J) > 0. \tag{2.4}$$

The other conditions, $p_3 < p_2$ and $p_3 < p_1$, are expressed in terms of the integration variables, once we have Eq. (2.2).

We can determine x_A and x_B from the final-state momenta using momentum conservation, as before:

$$x_A\sqrt{s/2} = p_1^+ + p_2^+ + p_3^+, \tag{2.5}$$

so

$$x_A = \frac{1}{\sqrt{s}}(p_1e^{y_1} + p_2e^{y_2} + p_3e^{y_3}), \tag{2.6}$$

and similarly

$$x_B = \frac{1}{\sqrt{s}}(p_1e^{-y_1} + p_2e^{-y_2} + p_3e^{-y_3}). \tag{2.7}$$

In the numerical integration program, the invariants are be expressed in terms of the integration variables using the previous results and Eqs. (1.15) and (1.36). Using the notation of Eq. (1.55), the invariants are

$$\begin{aligned} &p_{AB} = x_A\ x_B\ s/2,\quad p_{A1} = x_A\sqrt{s}\ p_1\ e^{-y_1}/2,\\ &p_{A2} = x_A\sqrt{s}\ p_2\ e^{-y_2}/2,\quad p_{A3} = x_A\sqrt{s}\ p_3\ e^{-y_3}/2,\\ &p_{B1} = x_B\sqrt{s}\ p_1\ e^{+y_1}/2,\quad p_{B2} = x_B\sqrt{s}\ p_2\ e^{+y_2}/2,\\ &p_{B3} = x_B\sqrt{s}\ p_3\ e^{+y_3}/2,\\ &p_{12} = p_1p_2\cosh(y_1-y_2) + p_2^2 + p_2p_3\cos(\phi_2-\phi_3),\\ &p_{13} = p_1p_3\cosh(y_1-y_3) + p_3^2 + p_2p_3\cos(\phi_2-\phi_3),\\ &p_{23} = p_2p_3\cosh(y_2-y_3) - p_2p_3\cos(\phi_2-\phi_3). \end{aligned} \tag{2.8}$$

We will be especially interested in the invariant $p_2\cdot p_3$. We have expressed it in terms of the variables we have chosen as

$$p_2\cdot p_3 = p_2p_3[\cosh(y_2-y_3) - \cos(\phi_2-\phi_3)]. \tag{2.9}$$

Now let us consider the relation between the cross section and the invariant matrix element. We have

$$\begin{aligned} d\sigma_2 = \int dx_A\ f_A(x_A)\int dx_B\ f_B(x_B)\frac{1}{2\hat{s}}\ &\mu^{2\epsilon}\frac{dy_1\ d^{2-2\epsilon}\mathbf{p}_1}{2(2\pi)^{3-2\epsilon}}\ \mu^{2\epsilon}\frac{dy_2\ d^{2-2\epsilon}\mathbf{p}_2}{2(2\pi)^{3-2\epsilon}}\ \mu^{2\epsilon}\frac{dy_3\ d^{2-2\epsilon}\mathbf{p}_3}{2(2\pi)^{3-2\epsilon}}\ \langle|\mathcal{M}|^2\rangle_2\\ &\times\mu^{-2\epsilon}(2\pi)^{4-2\epsilon}\delta^{4-2\epsilon}(p_1+p_2+p_3-p_A-p_B)\theta(p_3<p_1)\ \theta(p_3<p_2)\\ &\times\theta((p_1+p_2)\cos(\phi_2-\phi_J) + p_3\cos(\phi_3-\phi_J) > 0). \end{aligned} \tag{2.10}$$

Here the theta functions enforce our chosen division of the integration region, as discussed above.

We now use the transverse-momentum delta function to perform the $\mathbf{p}_1$ integration, the plus momentum delta function to perform the x_A integration, and the minus momentum delta function to perform the x_B integration. Following the results of the previous section, this gives the replacements

$$\int d\mathbf{p}_1 \delta(\mathbf{p}_1 + \cdots) \to 1, \quad \int dx_A \delta(p_1^+ + \cdots) \to \sqrt{2/s}, \quad \int dx_B \delta(p_1^- + \cdots) \to \sqrt{2/s}.$$

We write the integrations over the transverse momenta of gluons 2 and 3 as

$$d\mathbf{p}_2 \to p_2^{1-2\epsilon} dp_2 \; d^{1-2\epsilon}\phi_2, \quad d\mathbf{p}_3 \to p_3^{1-2\epsilon} dp_3 \; d^{1-2\epsilon}\phi_3.$$

These replacements give

$$d\sigma_2 = dy_1 \; dy_2 \; dy_3 \; dp_2 \; dp_3 \; d^{1-2\epsilon}\phi_2 \; d^{1-2\epsilon}\phi_3 \; x_A f_A(x_A) \; x_B f_B(x_B) \times \frac{p_2 p_3}{8(2\pi)^5 \hat{s}^2} \left(\frac{p_2 p_3}{(2\pi)^2 \mu^2}\right)^{-2\epsilon} \langle |\mathcal{M}|^2 \rangle_2$$
$$\times \theta(p_3 < p_1) \; \theta(p_3 < p_2) \; \theta((p_1 + p_2)\cos(\phi_2 - \phi_J) + p_3 \cos(\phi_3 - \phi_J) > 0). \tag{2.11}$$

In what follows we shall denote the angular variables y and ϕ for each particle by a single $(2-2\epsilon)$-dimensional vector variable $\mathbf{\Omega} = (y, \phi)$, where, we recall, ϕ denotes a point on a $(1-2\epsilon)$-dimensional sphere. We shall have to be careful of the definitions in a nonphysical number of dimensions. Although we use a vector notation for the angular variables, this notation is to be interpreted in a spherically symmetric sense in the case of ϕ. Thus we use the notation $|\mathbf{\Omega}_a - \mathbf{\Omega}_b|$ to denote $[(y_a - y_b)^2 + (\phi_{ab})^2]^{1/2}$, where ϕ_{ab} is the arc length between ϕ_a and ϕ_b. Similarly, $\cos(\phi_a - \phi_b)$ denotes $\cos(\phi_{ab})$. The linear combination $(1-x)\mathbf{\Omega}_A + x\mathbf{\Omega}_B = \mathbf{\Omega}_c$ with $0 < x < 1$ denotes the variable $\mathbf{\Omega}_c = (y_c, \phi_c)$ where $y_c = (1-x)y_A + x y_B$ and ϕ_c lies a fraction x of the way along the arc joining ϕ_A and ϕ_B. Other manipulations in the following are to be given a similar rotationally invariant interpretation. See Appendix A for the precise definitions.

We adopt a notation that displays the $1/p_2 \cdot p_3$ singularity in $\langle |\mathcal{M}|^2 \rangle_2$, while summarizing everything else as a function F_2. We define F_2 by

$$(p_3/\mu)^{-2\epsilon} \frac{F_2(y_1; p_2, \mathbf{\Omega}_2; p_3; \mathbf{\Omega}_3)}{p_3[\cosh(y_2 - y_3) - \cos(\phi_2 - \phi_3)]} = x_A f_A(x_A) \; x_B f_B(x_B) \frac{p_2 p_3}{8(2\pi)^5 \hat{s}^2} \left(\frac{p_2 p_3}{(2\pi)^2 \mu^2}\right)^{-2\epsilon} \langle |\mathcal{M}|^2 \rangle_2$$
$$\times \theta(p_3 < p_1) \; \theta(p_3 < p_2) \; \theta((p_1 + p_2)\cos(\phi_2 - \phi_J)$$
$$+ p_3 \cos(\phi_3 - \phi_J) > 0), \tag{2.12}$$

where $\hat{s} = x_A x_B \, s$.

Using Eq. (1.54), we can express F_2 in terms of f_2:

$$F_2 = H_2 f_2, \tag{2.13}$$

where

$$H_2 = \left(\frac{p_2}{p_J}\right)^{1-2\epsilon} (4\pi^2)^\epsilon \, \frac{C(\epsilon)}{2\pi} \, L(x_A, x_B) \, \tilde{\Theta} \, \frac{\alpha_s}{2\pi} \, . \tag{2.14}$$

Here, $C(\epsilon)$ and $L(x_A, x_B)$ are defined as in (1.26) and (1.28) and $\tilde{\Theta}$ is the product of theta functions appearing in (2.12),

$$\tilde{\Theta} = \theta(p_3 < p_1) \; \theta(p_3 < p_2) \; \theta \; ((p_1 + p_2)\cos(\phi_2 - \phi_J)$$
$$+ p_3 \cos(\phi_3 - \phi_J) > 0). \tag{2.15}$$

B. Jet definition

With the notation established so far, we can write the cross section in the form

$$d\sigma_2 = dy_1 \; dp_2 \; \frac{dp_3}{p_3} \; d^{2-2\epsilon}\mathbf{\Omega}_2 \; d^{2-2\epsilon}\mathbf{\Omega}_3 \left(\frac{p_3}{\mu}\right)^{-2\epsilon}$$
$$\times \frac{F_2(y_1; p_2, \mathbf{\Omega}_2; p_3, \mathbf{\Omega}_3)}{\cosh(y_2 - y_3) - \cos(\phi_2 - \phi_3)}. \tag{2.16}$$

Now we have to insert the jet definition. Consider first the possibility that gluons 2 and 3 together constitute the jet, which happens if both $\mathbf{\Omega}_2$ and $\mathbf{\Omega}_3$ are within an angle R of the jet axis $\mathbf{\Omega}_J = (y_J, \phi_J)$. Then, the jet axis is related to the parton momenta by the definition

$$\mathbf{\Omega}_J = \frac{p_2\mathbf{\Omega}_2 + p_3\mathbf{\Omega}_3}{p_2 + p_3}. \tag{2.17}$$

The jet transverse momentum is then defined to be

$$p_J = p_2 + p_3. \tag{2.18}$$

The condition that $\mathbf{\Omega}_2$ is within an angle R of the jet axis is

$$R > |\mathbf{\Omega}_2 - \mathbf{\Omega}_J| = \frac{p_3|\mathbf{\Omega}_2 - \mathbf{\Omega}_3|}{p_J}. \tag{2.19}$$

The condition that $\mathbf{\Omega}_3$ is within an angle R of the jet axis is

$$R > |\mathbf{\Omega}_3 - \mathbf{\Omega}_J| = \frac{p_2|\mathbf{\Omega}_2 - \mathbf{\Omega}_3|}{p_J}. \tag{2.20}$$

Since $p_3 < p_2$, this latter condition is the more restrictive. Thus the jet condition is

$$|\mathbf{\Omega}_2 - \mathbf{\Omega}_3| < \frac{p_2 + p_3}{p_2} R. \tag{2.21}$$

Let us also consider the possibility that gluon 2 by itself is the jet: $\mathbf{\Omega}_J = \mathbf{\Omega}_2$. Then, according to the jet definition, we have to demand that gluons 2 and 3 cannot form a legitimate jet:

$$|\mathbf{\Omega}_2 - \mathbf{\Omega}_3| > \frac{p_2 + p_3}{p_2} R. \tag{2.22}$$

Finally, we consider the possibility that gluon 3 by itself is the jet: $\mathbf{\Omega}_J = \mathbf{\Omega}_3$. Then we have to demand that gluons 2 and 3 cannot form a legitimate jet, as specified by the condition (2.22).

We thus have, for the contribution to the jet cross section,

$$\begin{aligned}\frac{d\sigma_2}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = \int dy_1 \int dp_2 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\mathbf{\Omega}_2 \int d^{2-2\epsilon}\mathbf{\Omega}_3 \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{F_2(y_1;p_2,\mathbf{\Omega}_2;p_3,\mathbf{\Omega}_3)}{\cosh(y_2-y_3)-\cos(\phi_2-\phi_3)} \\ \times\Bigg[\delta(p_2+p_3-p_J)\;\delta^{2-2\epsilon}\left(\frac{1}{p_J}(p_2\mathbf{\Omega}_2+p_3\mathbf{\Omega}_3)-\mathbf{\Omega}_J\right) \\ \times\theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_3| < \frac{p_2+p_3}{p_2}R\right) \\ +\delta(p_2-p_J)\;\delta^{2-2\epsilon}(\mathbf{\Omega}_2-\mathbf{\Omega}_J)\;\theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_3| > \frac{p_2+p_3}{p_2}R\right) \\ +\delta(p_3-p_J)\;\delta^{2-2\epsilon}(\mathbf{\Omega}_3-\mathbf{\Omega}_J)\;\theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_3| > \frac{p_2+p_3}{p_2}R\right)\Bigg].\end{aligned} \tag{2.23}$$

C. Extraction of the singular contribution

We now manipulate the expression in (2.23) in order to make the cancellation of the collinear singularity manifest. Let us define sum and difference variables for the angular integrations:

$$\bar{\mathbf{\Omega}} = \mathbf{\Omega}_3 - \mathbf{\Omega}_2, \quad \mathbf{\Omega} = \frac{1}{p_2+p_3}(p_2\mathbf{\Omega}_2 + p_3\mathbf{\Omega}_3). \tag{2.24}$$

Then

$$\mathbf{\Omega}_2 = \mathbf{\Omega} - \frac{p_3}{p_2+p_3}\,\bar{\mathbf{\Omega}}, \quad \mathbf{\Omega}_3 = \mathbf{\Omega} + \frac{p_2}{p_2+p_3}\,\bar{\mathbf{\Omega}}. \tag{2.25}$$

We use the first of the jet-defining delta functions to perform the p_2 integration in the first two terms of (2.23) and the p_3 integration in the third term.

In the first term of (2.23), we change variables to $\bar{\mathbf{\Omega}}$ and $\mathbf{\Omega}$. The Jacobian for this transformation is 1. This would follow trivially from (2.24) if there were exactly two transverse dimensions. In $2-2\epsilon$ dimensions, some care is needed to specify exactly what the definitions (2.24) mean and then to evaluate the Jacobian. This analysis is given in Appendix A, where we find that the Jacobian remains 1, even in $2-2\epsilon$ dimensions. We complete the manipulation of the first term by using the delta function that specifies the jet axis to perform the $\mathbf{\Omega}$ integration.

In the second term of (2.23), we simply use the angular delta function to perform the $\mathbf{\Omega}_2$ integration, then change variables from $\mathbf{\Omega}_3$ to $\bar{\mathbf{\Omega}}$ for the remaining integral. Similarly, in the third term, we use the delta function to perform the $\mathbf{\Omega}_3$ integration, then change variables from $\mathbf{\Omega}_2$ to $\bar{\mathbf{\Omega}}$ for the remaining integral.

These operations give

$$\frac{d\sigma_2}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})}$$
$$\times\left[F_2\left(y_1; p_J - p_3, \boldsymbol{\Omega}_J - \frac{p_3}{p_J}\bar{\boldsymbol{\Omega}}; p_3, \boldsymbol{\Omega}_J + \frac{p_J - p_3}{p_J}\bar{\boldsymbol{\Omega}}\right)\ \theta\!\left(|\bar{\boldsymbol{\Omega}}| < \frac{p_J}{p_J - p_3}R\right)\right.$$
$$\left. + F_2(y_1; p_J, \boldsymbol{\Omega}_J; p_3, \boldsymbol{\Omega}_J + \bar{\boldsymbol{\Omega}})\ \theta\!\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J + p_3}{p_J}R\right)\right]$$
$$+ \int dy_1 \int dp_2 \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}}\ \frac{1}{p_J}\left(\frac{p_J}{\mu}\right)^{-2\epsilon} \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})}$$
$$\times F_2(y_1; p_2, \boldsymbol{\Omega}_J - \bar{\boldsymbol{\Omega}}; p_J, \boldsymbol{\Omega}_J)\ \theta\!\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J + p_2}{p_2}R\right). \qquad (2.26)$$

Let us first rewrite this as a nonsingular contribution with theta functions plus a singular contribution in which the theta functions have been eliminated:

$$\frac{d\sigma_2}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = I_{2,\mathrm{S}} + I_{2,\mathrm{NS}}, \qquad (2.27)$$

where

$$I_{2,\mathrm{S}} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})} F_2\left(y_1; p_J - p_3, \boldsymbol{\Omega}_J - \frac{p_3}{p_J}\bar{\boldsymbol{\Omega}}; p_3, \boldsymbol{\Omega}_J + \frac{p_J - p_3}{p_J}\bar{\boldsymbol{\Omega}}\right) \qquad (2.28)$$

and

$$I_{2,\mathrm{NS}} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})}$$
$$\times\left[-F_2\left(y_1; p_J - p_3, \boldsymbol{\Omega}_J - \frac{p_3}{p_J}\bar{\boldsymbol{\Omega}}; p_3, \boldsymbol{\Omega}_J + \frac{p_J - p_3}{p_J}\bar{\boldsymbol{\Omega}}\right) \theta\!\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J}{p_J - p_3}R\right)\right.$$
$$\left. + F_2(y_1; p_J, \boldsymbol{\Omega}_J; p_3, \boldsymbol{\Omega}_J + \bar{\boldsymbol{\Omega}})\ \theta\!\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J + p_3}{p_J}R\right)\right]$$
$$+ \int dy_1 \int dp_2 \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}}\ \frac{1}{p_J}\left(\frac{p_J}{\mu}\right)^{-2\epsilon} \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})}$$
$$\times F_2(y_1; p_2, \boldsymbol{\Omega}_J - \bar{\boldsymbol{\Omega}}; p_J, \boldsymbol{\Omega}_J)\ \theta\!\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J + p_2}{p_2}R\right). \qquad (2.29)$$

Notice that $I_{2,\mathrm{NS}}$ has no collinear singularity when $\bar{\boldsymbol{\Omega}} \to 0$ because of the theta functions. Furthermore it has no soft singularity when $p_3 \to 0$ because of the subtraction.

D. Decomposition of the singular contribution

Now let us consider the singular piece. We write it in the form

$$I_{2,\mathrm{S}} = I_{2,\mathrm{coll}} + I_{2,\mathrm{soft}} + I_{2,\mathrm{double}} + I_{2,\mathrm{finite}}. \qquad (2.30)$$

Here the first term contains the collinear singularity and is obtained by setting $\bar{\boldsymbol{\Omega}} = 0$ everywhere in (2.28) except in the denominator and subtracting the same term with p_3 also set equal to zero except in the denominator:

$$I_{2,\mathrm{coll}} = \mathcal{I}_2(\epsilon) \int dy_1 \int \frac{dp_3}{p_3} \left(\frac{p_3}{\mu}\right)^{-2\epsilon} [F_2(y_1; p_J - p_3, \boldsymbol{\Omega}_J; p_3, \boldsymbol{\Omega}_J) - F_2(y_1; p_J, \boldsymbol{\Omega}_J; 0, \boldsymbol{\Omega}_J)\ \theta(p_3 < Q_2)], \qquad (2.31)$$

where

$$\mathcal{I}_2(\epsilon) = \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}}\ \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})}$$
$$= 2^{1-2\epsilon}\pi^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^\pi d\bar{\phi}\ (\sin\bar{\phi})^{-2\epsilon} \int_{-\infty}^{\infty} d\bar{y}\ \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})} = -\frac{2\pi}{\epsilon}\ 2^{-4\epsilon}\pi^{-\epsilon}\frac{\Gamma(1-\epsilon)^3}{\Gamma(1-2\epsilon)^2}. \qquad (2.32)$$

Here

$$V_{-2\epsilon} = 2^{1-2\epsilon}\pi^{-\epsilon}\,\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{2.33}$$

is the area of a (-2ϵ)-dimensional sphere.

In the second term in (2.31), we have inserted a $\theta(p_3 < Q_2)$ since the theta function contained in F_2 that limits the p_3 integration was lost when we set $p_3 = 0$ in F_2. An upper limit on p_3 is necessary to make the integral finite, although the actual value, Q_2, of the limit will cancel from the calculation. One might choose $Q_2 = p_J/2$ as the upper limit because that is the limit in the first term in (2.31): $p_3 < p_2 = p_J - p_3$.

The next piece $I_{2,\rm soft}$ reflects the infrared singularity and is obtained by setting $p_3 = 0$ in Eq. (2.28) everywhere except in the denominator and subtracting the same expression with $\bar{\mathbf{\Omega}}$ also set equal to zero except in the denominator:

$$I_{2,\rm soft} = \mathcal{J}_2(\epsilon)\int dy_1 \int d^{2-2\epsilon}\bar{\mathbf{\Omega}}\,\frac{F_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J+\bar{\mathbf{\Omega}}) - F_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J)}{\cosh(\bar{y}) - \cos(\bar{\phi})}, \tag{2.34}$$

where

$$\mathcal{J}_2(\epsilon) = \int_0^{Q_2}\frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} = -\frac{1}{2\epsilon}\left(\frac{Q_2}{\mu}\right)^{-2\epsilon}. \tag{2.35}$$

The next piece is obtained by setting both $\bar{\mathbf{\Omega}}$ and p_3 to zero except in the denominator:

$$I_{2,\rm double} = \mathcal{I}_2(\epsilon)\;\mathcal{J}_2(\epsilon)\;\int dy_1\; F_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J). \tag{2.36}$$

Finally, $I_{2,\rm finite}$ is simply the remainder defined by Eq. (2.30):

$$\begin{aligned} I_{2,\rm finite} = \int dy_1 \int \frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} \int d^{2-2\epsilon}\bar{\mathbf{\Omega}}\;\frac{1}{\cosh(\bar{y})-\cos(\bar{\phi})} \\ \times\Bigg[F_2\left(y_1;p_J-p_3,\mathbf{\Omega}_J-\frac{p_3}{p_J}\bar{\mathbf{\Omega}};p_3,\mathbf{\Omega}_J+\frac{p_J-p_3}{p_J}\bar{\mathbf{\Omega}}\right) \\ -F_2(y_1;p_J-p_3,\mathbf{\Omega}_J;p_3,\mathbf{\Omega}_J) - F_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J+\bar{\mathbf{\Omega}})\;\theta(p_3<Q_2) \\ +F_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J)\;\theta(p_3<Q_2)\Bigg]. \end{aligned} \tag{2.37}$$

We see that because of the subtractions, $I_{2,\rm finite}$ has neither collinear nor infrared divergences.

We now introduce a notation for $I_{2,\rm S}$ that is analogous to that for the $2 \to 2$ cross section. We write it as an integral of a function $G_2^{(2\to3)}(y_1,p_J,y_J,\phi_J)$ over y_1:

$$I_{2,\rm S} = \frac{d\sigma}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = \int dy_1\; G_2^{(2\to3)}(y_1,p_J,y_J,\phi_J). \tag{2.38}$$

Here $G_2^{(2\to3)}$ is given by a three-dimensional integral over $(p_3,\bar{y},\bar{\phi})$. It is divided into pieces:

$$G_2^{(2\to3)} = G_{2,\rm coll}^{(2\to3)} + G_{2,\rm soft}^{(2\to3)} + G_{2,\rm double}^{(2\to3)} + G_{2,\rm finite}^{(2\to3)}\,, \tag{2.39}$$

as specified in this section. The finite term, as given in Eq. (2.37), is computed by numerical integration with $\epsilon = 0$. In the following sections, we extract the $1/\epsilon^2$ and $1/\epsilon$ pieces from the collinear, soft, and double integrals, so that they can be canceled against $1/\epsilon^2$ and $1/\epsilon$ terms in $G^{(2\to2)}(y_1,p_J,y_J,\phi_J)$. We will be left with contributions to $G_2^{(2\to3)}$ that are finite as $\epsilon \to 0$ and are expressed either analytically, or as one-dimensional integrals that can be computed numerically.

E. The double singular contribution

From Eq. (2.36), we have for $G_{2,\rm double}^{(2\to3)}$ the simple expression

$$G_{2,\rm double}^{(2\to3)} = \mathcal{I}_2(\epsilon)\;\mathcal{J}_2(\epsilon)\;H_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J)\;f_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J). \tag{2.40}$$

When evaluated at the double singular point, the function H_2 [Eq. (2.14)] is simply

$$H_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J) = (4\pi^2)^\epsilon \frac{C(\epsilon)}{2\pi} L(X_A,X_B) \frac{\alpha_s}{2\pi} . \tag{2.41}$$

The function f_2, when evaluated at the double singular point, is also very simple:

$$f_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J;\epsilon) = 2N d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon). \tag{2.42}$$

[This relation is a consequence of the more general relation given in Eq. (1.66)].

We now assemble these ingredients, extracting an ϵ-dependent factor that appears in the $2 \to 2$ cross section, Eq. (1.25). (This extracted factor includes a factor $Q_{\rm ES}^{2\epsilon}$, where $Q_{\rm ES}$ is an arbitrary scale factor, as in the $2 \to 2$ cross section.) We obtain

$$G_{2,\rm double}^{(2\to3)} = C(\epsilon)\, L(X_A,X_B)\, \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q_{\rm ES}^2}\right)^\epsilon \Gamma_K(\epsilon)\, d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon) \left[\frac{N}{\epsilon^2}\left(\frac{Q_{\rm ES}^2}{16Q_2^2}\right)^\epsilon \Gamma_2(\epsilon)\right], \tag{2.43}$$

where

$$\Gamma_2(\epsilon) = \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-2\epsilon)} = 1 - \frac{\pi^2}{3}\epsilon^2 + O(\epsilon^3). \tag{2.44}$$

Expanding the factor in square brackets gives

$$G_{2,\rm double}^{(2\to3)} = C(\epsilon)\, L(X_A,X_B)\, \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q_{\rm ES}^2}\right)^\epsilon \Gamma_K(\epsilon) d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon) \times \left\{\frac{N}{\epsilon^2} + \frac{N}{\epsilon}\ln\left(\frac{Q_{\rm ES}^2}{16Q_2^2}\right) + N\left[\frac{1}{2}\ln^2\left(\frac{Q_{\rm ES}^2}{16Q_2^2}\right) - \frac{\pi^2}{3}\right] + O(\epsilon)\right\}. \tag{2.45}$$

F. The collinear contribution

From Eqs. (2.31) and (2.13), we have the following expression for $G_{2,\rm coll}^{(2\to3)}$:

$$G_{2,\rm coll}^{(2\to3)} = \mathcal{I}_2(\epsilon) \int \frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} \{\, H_2(y_1;p_J-p_3,\mathbf{\Omega}_J;p_3,\mathbf{\Omega}_J) f_2(y_1;p_J-p_3,\mathbf{\Omega}_J;p_3,\mathbf{\Omega}_J) - H_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J) f_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J)\theta(p_3<Q_2)\}. \tag{2.46}$$

In the singular configuration at which the momenta are evaluated in the first term of Eq. (2.46), gluons 2 and 3 are parallel and make up a jet with lightlike momentum

$$p_J^\mu = \left(\frac{1}{\sqrt{2}}\, p_J e^{y_J}, \frac{1}{\sqrt{2}}\, p_J e^{-y_J}, \mathbf{p}_J\right). \tag{2.47}$$

Let us change integration variables in Eq. (2.46) from p_3 to z defined by

$$p_3 = (1-z)p_J. \tag{2.48}$$

Then $p_3^\mu = (1-z)p_J^\mu$ and $p_2^\mu = z\, p_J^\mu$.

Then H_2 is given simply by

$$H_2(y_1;zp_J,\mathbf{\Omega}_J;(1-z)p_J,\mathbf{\Omega}_J) = z^{1-2\epsilon}\,(4\pi^2)^\epsilon\, \frac{C(\epsilon)}{2\pi}\, L(X_A,X_B)\, \Theta(z>\tfrac{1}{2})\, \frac{\alpha_s}{2\pi}\,, \tag{2.49}$$

where X_A and X_B are determined from y_1, y_J and p_J according to Eq. (1.16), as in the $2 \to 2$ process. The restriction that $z > \frac{1}{2}$ comes from the factor $\theta(p_3 < p_2)$. From the explicit form (2.32) of $\mathcal{I}_2(\epsilon)$ we have

$$\mathcal{I}_2(\epsilon) = -\frac{2\pi}{\epsilon}(16\pi)^{-\epsilon}\, \Gamma_K(\epsilon)\, [1+O(\epsilon^2)]. \tag{2.50}$$

Thus

$$G_{2,\rm coll}^{(2\to3)} = -\, C(\epsilon)\, L(X_A,X_B)\, \frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q_{\rm ES}^2}\right)^\epsilon \Gamma_K(\epsilon)\, \frac{1}{\epsilon}\left(\frac{Q_{\rm ES}^2}{16p_J^2}\right)^\epsilon \times \int_0^1 \frac{dz}{1-z}\,(1-z)^{-2\epsilon}\{z^{1-2\epsilon}\,\theta(z>\tfrac{1}{2})\, f_2(y_1;zp_J,\mathbf{\Omega}_J;(1-z)p_J,\mathbf{\Omega}_J) - f_2(y_1;p_J,\mathbf{\Omega}_J;0,\mathbf{\Omega}_J)\,\theta(z>1-Q_2/p_J)\}[1+O(\epsilon^2)], \tag{2.51}$$

At the collinear point, the function f_2 has a very simple form given by Eq. (1.66):

$$f_2(y_1; zp_J, \Omega_J; (1-z)p_J, \Omega_J; \epsilon) = d^{(4)}(\hat{S}, \hat{T}, \hat{U}; \epsilon)\, \frac{1-z}{z}\, \tilde{P}_{gg}(z), \tag{2.52}$$

where $\tilde{P}_{gg}(z)$ is the Altarelli-Parisi kernel for $g \to g$, but without the usual regulation for $z \to 1$, as given in Eq. (1.62). Thus

$$G_{2,\rm coll}^{(2\to3)} = -C(\epsilon)\, L(X_A, X_B)\, \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q_{\rm ES}^2}\right)^\epsilon \Gamma_K(\epsilon)\, d^{(4)}(\hat{S}, \hat{T}, \hat{U}; \epsilon) \left[\frac{1}{\epsilon} \left(\frac{Q_{\rm ES}^2}{16p_J^2}\right)^\epsilon Z_2(\epsilon)[1 + O(\epsilon^2)]\right], \tag{2.53}$$

where Z_2 is the integral

$$Z_2 = \int_0^1 dz\, \{\theta(z > \tfrac{1}{2})\, [z(1-z)]^{-2\epsilon}\, \tilde{P}_{gg}(z) - \theta(z > 1 - Q_2/p_J)\, (1-z)^{-1-2\epsilon}\, P_{gg}^{\rm soft}\,\}, \tag{2.54}$$

and where we have denoted

$$P_{gg}^{\rm soft} \equiv \lim_{z\to1}\, (1-z)\tilde{P}_{gg}(z) = 2N. \tag{2.55}$$

The integration can easily be performed, yielding

$$Z_2(\epsilon) = 2N\left\{-\frac{11}{12} - \frac{1}{2}\ln\left(\frac{Q_2^2}{p_J^2}\right) + \epsilon\left[-\frac{67}{18} + \frac{\pi^2}{3} + \frac{1}{4}\ln^2\left(\frac{Q_2^2}{p_J^2}\right)\right] + O(\epsilon^2)\right\}. \tag{2.56}$$

Expanding in powers of ϵ in Eq. (2.53), we obtain

$$\begin{aligned} G_{2,\rm coll}^{(2\to3)} &= C(\epsilon)\, L(X_A, X_B)\, \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q_{\rm ES}^2}\right)^\epsilon \Gamma_K(\epsilon)\, d^{(4)}(\hat{S}, \hat{T}, \hat{U}; \epsilon) \\ &\times 2N\left\{\frac{1}{\epsilon}\left[\frac{11}{12} + \frac{1}{2}\ln\left(\frac{Q_2^2}{p_J^2}\right)\right] + \left[\frac{11}{12} + \frac{1}{2}\ln\left(\frac{Q_2^2}{p_J^2}\right)\right]\ln\left(\frac{Q_{\rm ES}^2}{16p_J^2}\right) + \frac{67}{18} - \frac{\pi^2}{3} - \frac{1}{4}\ln^2\left(\frac{Q_2^2}{p_J^2}\right) + O(\epsilon)\right\}. \end{aligned} \tag{2.57}$$

G. The soft contribution

In this section we wish to calculate $G_{2,\rm soft}^{(2\to3)}$, which is given by Eqs. (2.34) and (2.13). Making use of the fact that the function H_2 as given in Eq. (2.14) is simple in the soft limit, we have

$$\begin{aligned} G_{2,\rm soft}^{(2\to3)} &= (4\pi^2)^\epsilon\, \frac{C(\epsilon)}{2\pi}\, L(X_A, X_B)\, \frac{\alpha_s}{2\pi}\, \mathcal{J}_2(\epsilon) \\ &\times \int d\bar{y} \int d^{1-2\epsilon}\bar{\phi}\, \frac{1}{\cosh(\bar{y}) - \cos(\bar{\phi})}[f_2(y_1; p_J, \Omega_J; 0, \Omega_J + \bar{\Omega}) - f_2(y_1; p_J, \Omega_J; 0, \Omega_J)]. \end{aligned} \tag{2.58}$$

According to Eqs. (1.53) and (1.54), the function f_2 has the structure

$$f_2 = \frac{f_{2A}}{d_{2A}} + \frac{f_{2B}}{d_{2B}} + \frac{f_{21}}{d_{21}}, \tag{2.59}$$

where the f_{nm} are defined in Eq. (1.50) and

$$d_{2m} = \frac{p_J}{p_3}\, (p_2 \cdot p_3 + p_m \cdot p_3). \tag{2.60}$$

In (2.60), we evaluate the dot products and divide by p_3, then take the limit $p_3 \to 0$ with constant $(y_3, \phi_3) = (y_J + \bar{y}, \phi_J + \bar{\phi})$. This gives

$$d_{2A} = p_J^2 D_1(\bar{y}, \bar{\phi}, y_1 - y_J), \quad d_{2B} = p_J^2 D_1(-\bar{y}, \bar{\phi}, y_J - y_1), \quad d_{21} = p_J^2 D_2(\bar{y}, y_1 - y_J), \tag{2.61}$$

where

$$\begin{aligned} D_1(\bar{y}, \bar{\phi}, y_1 - y_J) &= \tfrac{1}{2}(2 + e^{y_1 - y_J})e^{-\bar{y}} + \tfrac{1}{2}e^{\bar{y}} - \cos\bar{\phi}, \\ D_2(\bar{y}, y_1 - y_J) &= \tfrac{1}{2}(1 + e^{y_J - y_1})(e^{\bar{y}} + e^{y_1 - y_J - \bar{y}}). \end{aligned} \tag{2.62}$$

The functions f_{nm} with $p_3 = 0$ are given by Eq. (1.67):

$$f_{2A} = 4VN^3(1-\epsilon)^2\,(-\hat{U})\,f_S(\hat{U},\hat{S},\hat{T}),\quad f_{2B} = 4VN^3(1-\epsilon)^2\,(-\hat{T})\,f_S(\hat{T},\hat{U},\hat{S}),$$
$$f_{21} = 4VN^3(1-\epsilon)^2\,\hat{S}\,f_S(\hat{S},\hat{T},\hat{U}). \tag{2.63}$$

Finally, we use Eq. (2.35) for $\mathcal{J}_2(\epsilon)$ and show explicitly the factor $V_{-2\epsilon}$, given in Eq. (2.33), that appears in the relation

$$d^{1-2\epsilon}\bar{\phi} = V_{-2\epsilon}\,d\bar{\phi}[\sin(\bar{\phi})]^{-2\epsilon}. \tag{2.64}$$

Assembling this, we obtain

$$G^{(2\to3)}_{2,\mathrm{soft}} = -\frac{1}{\epsilon}\,C(\epsilon)\,L(X_A,X_B)\,\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2_{\mathrm{ES}}}\right)^{\epsilon}\Gamma_K(\epsilon)\left(\frac{Q^2_{\mathrm{ES}}}{4Q_2^2}\right)^{\epsilon}\frac{(1-\epsilon)^2}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}\frac{4VN^3}{2\pi p_J^2}$$
$$\times[-\hat{U}\,f_S(\hat{U},\hat{S},\hat{T})\,V_1(y_1-y_J)-\hat{T}\,f_S(\hat{T},\hat{U},\hat{S})\,V_1(y_J-y_1)+\hat{S}\,f_S(\hat{S},\hat{T},\hat{U})\,V_2(y_1-y_J)]. \tag{2.65}$$

Here $\Gamma_K(\epsilon)$ is given in Eq. (1.29), and we note that the factor $1/[\Gamma(1+\epsilon)\Gamma(1-\epsilon)]$ will not enter the final result because it equals $1+O(\epsilon^2)$. The functions $V_1(y_1-y_J)$ and $V_2(y_1-y_J)$ are the integrals of $1/D_1$ and $1/D_2$. These integrals are defined more precisely and are evaluated in Appendix B. Using the results of Appendix B together with Eq. (1.18) relating (p_J, y_J, y_1) to $(\hat{S},\hat{T},\hat{U})$ we have

$$-\frac{\hat{U}}{2\pi p_J^2}\,V_1(y_1-y_J) = \ln\left(\frac{-\hat{U}}{16p_J^2}\right)+\epsilon P_{1N}(y_1-y_J)+O(\epsilon),$$
$$-\frac{\hat{T}}{2\pi p_J^2}\,V_1(y_J-y_1) = \ln\left(\frac{-\hat{T}}{16p_J^2}\right)+\epsilon P_{1N}(y_J-y_1)+O(\epsilon), \tag{2.66}$$
$$\frac{\hat{S}}{2\pi p_J^2}\,V_2(y_1-y_J) = \ln\left(\frac{\hat{S}}{16p_J^2}\right)+\epsilon P_{2N}(y_1-y_J)+O(\epsilon).$$

The functions $P_{1N}(y_1-y_J)$ and $P_{2N}(y_1-y_J)$ are rather complicated, and are given in Eq. (B7).

Using these results, $G^{(2\to3)}_{2,\mathrm{soft}}$ becomes

$$G^{(2\to3)}_{2,\mathrm{soft}} = -\,C(\epsilon)\,L(X_A,X_B)\,\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2_{\mathrm{ES}}}\right)^{\epsilon}\Gamma_K(\epsilon)\,4VN^3$$
$$\times\Bigg(\frac{1}{\epsilon}\left[f_S(\hat{U},\hat{S},\hat{T})\,\ln\left(\frac{-\hat{U}}{16p_J^2}\right)+f_S(\hat{T},\hat{U},\hat{S})\,\ln\left(\frac{-\hat{T}}{16p_J^2}\right)+f_S(\hat{S},\hat{T},\hat{U})\,\ln\left(\frac{\hat{S}}{16p_J^2}\right)\right]$$
$$+f_S(\hat{U},\hat{S},\hat{T})\,\left\{\left[\ln\left(\frac{Q^2_{\mathrm{ES}}}{4Q_2^2}\right)-2\right]\ln\left(\frac{-\hat{U}}{16p_J^2}\right)+P_{1N}(y_1-y_J)\right\}$$
$$+f_S(\hat{T},\hat{U},\hat{S})\,\left\{\left[\ln\left(\frac{Q^2_{\mathrm{ES}}}{4Q_2^2}\right)-2\right]\ln\left(\frac{-\hat{T}}{16p_J^2}\right)+P_{1N}(y_J-y_1)\right\}$$
$$+f_S(\hat{S},\hat{T},\hat{U})\,\left\{\left[\ln\left(\frac{Q^2_{\mathrm{ES}}}{4Q_2^2}\right)-2\right]\ln\left(\frac{\hat{S}}{16p_J^2}\right)+P_{2N}(y_1-y_J)\right\}+O(\epsilon)\Bigg). \tag{2.67}$$

In order to make the cancellation of divergences easier to see, we add and subtract logarithms of the scale factor Q^2_{ES} in the singular terms. Then we use the identity (1.33) relating $f_S(\hat{s},\hat{t},\hat{u})$ to $d^{(4)}(\hat{s},\hat{t},\hat{u};\epsilon)$ to obtain our final result:

$$
\begin{aligned}
G_{2,\text{soft}}^{(2\to3)} = &-\; C(\epsilon)\; L(X_A,X_B)\;\frac{\alpha_s}{2\pi}\;\left(\frac{4\pi\mu^2}{Q_{\text{ES}}^2}\right)^{\epsilon}\Gamma_K(\epsilon)\\
&\times\Bigg(\frac{N}{\epsilon}\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\ln\!\left(\frac{Q_{\text{ES}}^2}{16p_J^2}\right)+\frac{4VN^3}{\epsilon}\left[f_{\text{S}}(\hat U,\hat S,\hat T)\;\ln\!\left(\frac{-\hat U}{Q_{\text{ES}}^2}\right)+f_{\text{S}}(\hat T,\hat U,\hat S)\;\ln\!\left(\frac{-\hat T}{Q_{\text{ES}}^2}\right)\right.\\
&\qquad\left.+f_{\text{S}}(\hat S,\hat T,\hat U)\;\ln\!\left(\frac{\hat S}{Q_{\text{ES}}^2}\right)\right]\\
&+4VN^3\; f_{\text{S}}(\hat U,\hat S,\hat T)\;\left\{\left[\ln\!\left(\frac{Q_{\text{ES}}^2}{4Q_2^2}\right)-2\right]\ln\!\left(\frac{-\hat U}{16p_J^2}\right)+P_{1N}(y_1-y_J)\right\}\\
&+4VN^3\; f_{\text{S}}(\hat T,\hat U,\hat S)\;\left\{\left[\ln\!\left(\frac{Q_{\text{ES}}^2}{4Q_2^2}\right)-2\right]\ln\!\left(\frac{-\hat T}{16p_J^2}\right)+P_{1N}(y_J-y_1)\right\}\\
&+4VN^3\; f_{\text{S}}(\hat S,\hat T,\hat U)\;\left\{\left[\ln\!\left(\frac{Q_{\text{ES}}^2}{4Q_2^2}\right)-2\right]\ln\!\left(\frac{\hat S}{16p_J^2}\right)+P_{2N}(y_1-y_J)\right\}\\
&+2N\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\;\ln\!\left(\frac{Q_{\text{ES}}^2}{16p_J^2}\right)+O(\epsilon)\Bigg).
\end{aligned}
\tag{2.68}
$$

III. TERM 1

A. Kinematics

In this section we are concerned with the term $\langle|\mathcal{M}|^2\rangle_1$ in the invariant matrix element with a $1/p_3\cdot p_1$ singularity. We are thus interested in the integration region in which 1 and 3 may become collinear, or in which gluon 3 may become soft. Our treatment will be similar to the treatment above of term 2. However, there are some differences, which arise ultimately from the definition (1.38) that gluon 2 always lies nearer the jet axis than gluon 1.

As in the case of term 2, we take the seven integration variables to be

$$y_1;p_2,y_2,\phi_2;p_3,y_3,\phi_3. \tag{3.1}$$

The crucial denominator factor $p_1\cdot p_3$ has a form analogous to that given in Eq. (2.9):

$$p_1\cdot p_3 = p_1p_3[\cosh(y_3-y_1)-\cos(\phi_3-\phi_1)], \tag{3.2}$$

where p_1 and ϕ_1 are to be expressed in terms of the seven integration variables, as discussed below. In order to simplify the notation for the manipulations to follow, we shall define the integrand to be a function F_1 times $(p_3/\mu)^{-2\epsilon}/p_3[\cosh(y_3-y_1)-\cos(\phi_3-\phi_1)]$:

$$
\begin{aligned}
(p_3/\mu)^{-2\epsilon}\frac{F_1(y_1;p_2,\Omega_2;p_3;\Omega_3)}{p_3[\cosh(y_3-y_1)-\cos(\phi_1-\phi_3)]} &= x_Af_A(x_A)\;x_Bf_B(x_B)\frac{p_2p_3}{8(2\pi)^5\hat s^2}\left(\frac{p_2p_3}{(2\pi)^2\mu^2}\right)^{-2\epsilon}\langle|\mathcal{M}|^2\rangle_1\\
&\times\theta(p_3<p_1)\;\theta(p_3<p_2)\;\theta(|\phi_2-\phi_J|<|\phi_1-\phi_J|).
\end{aligned}
\tag{3.3}
$$

Using Eq. (1.54), we can express F_1 in terms of f_1:

$$F_1(y_1;p_2,\Omega_2;p_3;\Omega_3) = H_1f_1, \tag{3.4}$$

where

$$H_1 = \left(\frac{p_2}{p_J}\right)^{1-2\epsilon}(4\pi^2)^{\epsilon}\;\frac{C(\epsilon)}{2\pi}\;L(x_A,x_B)\;\tilde\Theta\frac{\alpha_s}{2\pi}. \tag{3.5}$$

The variables p_1, x_A, x_B, and $p_i\cdot p_j$ can be expressed in terms of the seven integration variables of Eq. (3.1) as in Eqs. (2.2) and (2.6)–(2.8). In particular, we shall want to be able to express the denominator $p_1\cdot p_3/p_1p_3$ in terms of the seven integration variables. We name this denominator function D_{13} and find

$$
\begin{aligned}
D_{13}(y_1;p_2,\Omega_2;p_3,\Omega_3) &\equiv \cosh(y_3-y_1)-\cos(\phi_3-\phi_1)\\
&=\frac{p_1\cdot p_3}{p_1p_3}=\cosh(y_3-y_1)+\frac{\cos(\phi_3-\phi_2)+(p_3/p_2)}{[1+(p_3/p_2)^2+2(p_3/p_2)\cos(\phi_3-\phi_2)]^{1/2}}.
\end{aligned}
\tag{3.6}
$$

B. Jet definition

With the notation established so far, we can write the cross section in the form

$$d\sigma_1 = dy_1\, dp_2\, \frac{dp_3}{p_3}\, d^{2-2\epsilon}\boldsymbol{\Omega}_2\, d^{2-2\epsilon}\boldsymbol{\Omega}_3 \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{F_1(y_1;p_2,\boldsymbol{\Omega}_2;p_3,\boldsymbol{\Omega}_3)}{D_{13}(y_1;p_2,\boldsymbol{\Omega}_2;p_3,\boldsymbol{\Omega}_3)}. \tag{3.7}$$

Now we have to insert the jet definition. This is just the same as for term 2. We find

$$\frac{d\sigma_1}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = \int dy_1 \int dp_2 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\boldsymbol{\Omega}_2 \int d^{2-2\epsilon}\boldsymbol{\Omega}_3 \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{F_1(y_1;p_2,\boldsymbol{\Omega}_2;p_3,\boldsymbol{\Omega}_3)}{D_{13}(y_1;p_2,\boldsymbol{\Omega}_2;p_3,\boldsymbol{\Omega}_3)}$$
$$\times\Bigg[\delta(p_2+p_3-p_J)\delta^{2-2\epsilon}\left(\frac{1}{p_J}(p_2\boldsymbol{\Omega}_2+p_3\boldsymbol{\Omega}_3)-\boldsymbol{\Omega}_J\right)$$
$$\times\theta\left(|\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_3| < \frac{p_2+p_3}{p_2}R\right)$$
$$+\delta(p_2-p_J)\delta^{2-2\epsilon}(\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_J)\theta\left(|\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_3| > \frac{p_2+p_3}{p_2}R\right)$$
$$+\delta(p_3-p_J)\delta^{2-2\epsilon}(\boldsymbol{\Omega}_3-\boldsymbol{\Omega}_J)\theta\left(|\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_3| > \frac{p_2+p_3}{p_2}R\right)\Bigg]. \tag{3.8}$$

C. Extraction of the singular contribution

We now manipulate the expression in (3.8) in order to separate it into a nonsingular piece with theta functions plus a singular piece in which the theta functions have been eliminated.

We make use of the variables defined in Eqs. (2.24) and (2.25). We use the jet defining delta functions to perform $3-2\epsilon$ of the integrations in (3.8), as in Sec. II B. This gives

$$\frac{d\sigma_1}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon}$$
$$\times\Bigg[\frac{F_1(y_1;p_J-p_3,\boldsymbol{\Omega}_J-(p_3/p_J)\bar{\boldsymbol{\Omega}};p_3,\boldsymbol{\Omega}_J+(1-p_3/p_J)\bar{\boldsymbol{\Omega}})}{D_{13}(y_1;p_J-p_3,\boldsymbol{\Omega}_J-(p_3/p_J)\bar{\boldsymbol{\Omega}};p_3,\boldsymbol{\Omega}_J+(1-p_3/p_J)\bar{\boldsymbol{\Omega}})}$$
$$\times\theta\left(|\bar{\boldsymbol{\Omega}}| < \frac{p_J}{p_J-p_3}R\right) + \frac{F_1(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_J+\bar{\boldsymbol{\Omega}})}{D_{13}(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_J+\bar{\boldsymbol{\Omega}})}\theta\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J+p_3}{p_J}R\right)\Bigg]$$
$$+\int dy_1 \int dp_2 \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}}\, \frac{1}{p_J}\left(\frac{p_J}{\mu}\right)^{-2\epsilon} \frac{F_1(y_1;p_2,\boldsymbol{\Omega}_J-\bar{\boldsymbol{\Omega}};p_J,\boldsymbol{\Omega}_J)}{D_{13}(y_1;p_2,\boldsymbol{\Omega}_J-\bar{\boldsymbol{\Omega}};p_J,\boldsymbol{\Omega}_J)}\theta\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J+p_2}{p_2}R\right). \tag{3.9}$$

We now separate this into a singular piece with no jet defining theta functions plus a nonsingular piece:

$$\frac{d\sigma_1}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = I_{1,\mathrm{S}} + I_{1,\mathrm{NS}}, \tag{3.10}$$

where

$$I_{1,\mathrm{S}} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon} \frac{F_1(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_J+\bar{\boldsymbol{\Omega}})}{D_{13}(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_J+\bar{\boldsymbol{\Omega}})} \tag{3.11}$$

and

$$I_{1,\mathrm{NS}} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon}$$
$$\times\Bigg[\frac{F_1(y_1;p_J-p_3,\boldsymbol{\Omega}_J-(p_3/p_J)\bar{\boldsymbol{\Omega}};p_3,\boldsymbol{\Omega}_J+(1-p_3/p_J)\bar{\boldsymbol{\Omega}})}{D_{13}(y_1;p_J-p_3,\boldsymbol{\Omega}_J-(p_3/p_J)\bar{\boldsymbol{\Omega}};p_3,\boldsymbol{\Omega}_J+(1-p_3/p_J)\bar{\boldsymbol{\Omega}})}$$
$$\times\theta\left(|\bar{\boldsymbol{\Omega}}| < \frac{p_J}{p_J-p_3}R\right) - \frac{F_1(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_J+\bar{\boldsymbol{\Omega}})}{D_{13}(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_J+\bar{\boldsymbol{\Omega}})}\theta\left(|\bar{\boldsymbol{\Omega}}| < \frac{p_J+p_3}{p_J}R\right)\Bigg]$$
$$+\int dy_1 \int dp_2 \int d^{2-2\epsilon}\bar{\boldsymbol{\Omega}}\, \frac{1}{p_J}\left(\frac{p_J}{\mu}\right)^{-2\epsilon} \frac{F_1(y_1;p_2,\boldsymbol{\Omega}_J-\bar{\boldsymbol{\Omega}};p_J,\boldsymbol{\Omega}_J)}{D_{13}(y_1;p_2,\boldsymbol{\Omega}_J-\bar{\boldsymbol{\Omega}};p_J,\boldsymbol{\Omega}_J)}\theta\left(|\bar{\boldsymbol{\Omega}}| > \frac{p_J+p_2}{p_2}R\right). \tag{3.12}$$

Notice that $I_{1,\mathrm{NS}}$ has no collinear singularity when gluon 3 becomes collinear with gluon 1 because of the theta functions, including (for the third term) the theta function in the definition of F_1 that guarantees that gluon 2 is nearer to the jet direction than gluon 1. Furthermore $I_{1,\mathrm{NS}}$ has no soft singularity when $p_3 \to 0$ because of the subtraction.

D. Simplification of the denominator

We must now extract the singular terms from $I_{1,\mathrm{S}}$ in the form of integrations that can be performed analytically. Our first task will be to simplify the denominator.

Let us examine the crucial denominator factor D_{13}. We recall from Eq. (3.6) (with $p_2 = p_J, \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_J$) that

$$\begin{aligned} D_{13}(y_1; p_J, \boldsymbol{\Omega}_J; p_3, \boldsymbol{\Omega}_3) &= \cosh(y_3 - y_1) - \cos(\phi_3 - \phi_1) \\ &= \cosh(y_3 - y_1) + \frac{\cos(\phi_3 - \phi_J) + (p_3/p_J)}{[1 + (p_3/p_J)^2 + 2(p_3/p_J)\cos(\phi_3 - \phi_J)]^{1/2}}. \end{aligned} \tag{3.13}$$

This expression is rather complicated. We are, however, primarily interested in the behavior of this denominator near the collinear singularity at $y_3 - y_1 = 0, \phi_3 - \phi_J = \pi$. Let us therefore rewrite the right-hand side of (3.13) in a more convenient form that has the same expansion about the collinear point to second order in $(y_3 - y_1)^2, (\phi_3 - \phi_J - \pi)^2$:

$$\begin{aligned} D_{13}(y_1; p_J, \boldsymbol{\Omega}_J; p_3, \boldsymbol{\Omega}_3) &= \cosh(y_3 - y_1) - \frac{[1 - (p_3/p_J)] - [1 + \cos(\phi_3 - \phi_J)]}{\{[1 - (p_3/p_J)]^2 + 2(p_3/p_J)[1 + \cos(\phi_3 - \phi_J)]\}^{1/2}} \\ &\sim \cosh(y_3 - y_1) - 1 + \frac{1 + \cos(\phi_3 - \phi_J)}{[1 - (p_3/p_J)]^2} \\ &\sim \frac{1}{[1 - (p_3/p_J)]^2} \left(\cosh\{[1 - (p_3/p_J)](y_3 - y_1)\} + \cos(\phi_3 - \phi) \right). \end{aligned} \tag{3.14}$$

Let us define

$$\begin{aligned} &\rho(p_3/p_J,\ y_3 - y_1,\ \phi_3 - \phi_J) \\ &\quad = \frac{1}{[1 - (p_3/p_J)]^2} \frac{\cosh\{[1 - (p_3/p_J)](y_3 - y_1)\} + \cos(\phi_3 - \phi_J)}{\cosh(y_3 - y_1) - \cos(\phi_3 - \phi_1)} \\ &\quad = \frac{\left(\cosh\{[1 - (p_3/p_J)](y_3 - y_1)\} + \cos(\phi_3 - \phi_J) \right)/[1 - (p_3/p_J)]^2}{\cosh(y_3 - y_1) + [\cos(\phi_3 - \phi_J) + (p_3/p_J)]/[1 + (p_3/p_J)^2 + 2(p_3/p_J)\cos(\phi_3 - \phi_J)]^{1/2}}, \end{aligned} \tag{3.15}$$

where we have used Eq. (3.13) to express ρ explicitly in terms of the variables $p_3/p_J,\ y_3 - y_1,\ \phi_3 - \phi_J$. Then the analysis given above shows that ρ has a smooth limit at the collinear point, with

$$\rho(p_3/p_J,\ 0,\ \pi) = 1. \tag{3.16}$$

Inspection of Eq. (3.15) shows that ρ is also simple in the soft limit

$$\rho(0,\ y_3 - y_1,\ \phi_3 - \phi_J) = 1. \tag{3.17}$$

E. Decomposition of the singular contribution

We can use the results of the previous section to rewrite our singular integral as

$$\begin{aligned} I_{1,\mathrm{S}} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\boldsymbol{\Omega}_3 \left(\frac{p_3}{\mu}\right)^{-2\epsilon} [1 - (p_3/p_J)]^2 \\ \times \frac{F_1(y_1; p_J, \boldsymbol{\Omega}_J; p_3, \boldsymbol{\Omega}_3)}{\cosh\{[1 - (p_3/p_J)](y_3 - y_1)\} + \cos(\phi_3 - \phi_J)}\ \rho(p_3/p_J,\ y_3 - y_1,\ \phi_3 - \phi_J), \end{aligned} \tag{3.18}$$

where we have used $\boldsymbol{\Omega}_3$ as the angular integration variable. An alternative form with a simpler denominator is obtained by changing variables from $\boldsymbol{\Omega}_3$ to $\tilde{\boldsymbol{\Omega}} = (\tilde{y}, \tilde{\phi})$ defined by $\tilde{y} = [1 - (p_3/p_J)](y_3 - y_1)$, $\tilde{\phi} = \phi_3 - \phi_J$. This gives a Jacobian factor $[1 - (p_3/p_J)]^{-1}$, with the result

$$\begin{aligned} I_{1,\mathrm{S}} = \int dy_1 \int \frac{dp_3}{p_3} \int d^{2-2\epsilon}\tilde{\boldsymbol{\Omega}} \left(\frac{p_3}{\mu}\right)^{-2\epsilon} [1 - (p_3/p_J)] \\ \times \frac{F_1(y_1; p_J, \boldsymbol{\Omega}_J; p_3, y_1 + \tilde{y}/[1 - (p_3/p_J)], \phi_J + \tilde{\phi})}{\cosh(\tilde{y}) + \cos(\tilde{\phi})}\ \rho(p_3/p_J,\ \tilde{y}/[1 - (p_3/p_J)],\ \tilde{\phi}). \end{aligned} \tag{3.19}$$

We now extract the singularities from $I_{1,\mathrm{S}}$ by writing it in the form

$$I_{1,\mathrm{S}} = I_{1,\mathrm{coll}} + I_{1,\mathrm{soft}} + I_{1,\mathrm{double}} + I_{1,\mathrm{finite}}. \tag{3.20}$$

The first term in (3.20) contains the collinear singularity and is obtained by setting $\tilde{\Omega} = (0,\pi)$ in F_1 and ρ in (3.19), so that Ω_3 lies in the singular direction $\Omega_3 = \Omega_s$, where

$$y_s = y_1, \quad \phi_s = \phi_J + \pi. \tag{3.21}$$

We subtract from this the same term with p_3 also set equal to zero except in the $1/p_3$ denominator in order to eliminate the soft singularity. This gives

$$I_{1,\mathrm{coll}} = \mathcal{I}_1(\epsilon)\int dy_1 \int \frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} \{[1-(p_3/p_J)]F_1(y_1;p_J,\Omega_J;p_3,\Omega_s) - F_1(y_1;p_J,\Omega_J;0,\Omega_s)\ \theta(p_3<Q_1)\}, \tag{3.22}$$

where $\mathcal{I}_1(\epsilon)$ is the integral already encountered in Eq. (2.32) except for a change of sign from $-\cos\phi$ to $+\cos\phi$, which does not change the result:

$$\begin{aligned}\mathcal{I}_1(\epsilon) &= \int d^{2-2\epsilon}\tilde{\Omega}\ \frac{1}{\cosh(\tilde{y})+\cos(\tilde{\phi})} \\ &= 2^{1-2\epsilon}\pi^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^{\pi} d\tilde{\phi}\ (\sin\tilde{\phi})^{-2\epsilon}\int_{-\infty}^{\infty} d\tilde{y}\ \frac{1}{\cosh\tilde{y}+\cos\tilde{\phi}} \\ &= -\frac{2\pi}{\epsilon}\ 2^{-4\epsilon}\pi^{-\epsilon}\frac{\Gamma(1-\epsilon)^3}{\Gamma(1-2\epsilon)^2}.\end{aligned} \tag{3.23}$$

In the second term in (3.22), we have inserted a $\theta(p_3<Q_1)$ since the theta function contained in F_1 that limits the p_3 integration was lost when we set $p_3=0$ in F_1. An upper limit on p_3 is necessary to make the integral finite, although the actual value of the limit will cancel from the calculation. One might choose $Q_1 = p_J/2$ as the upper limit because that is the limit in the first term in (3.22): $p_3 < p_1 = p_J - p_3$.

The second term in (3.20), $I_{1,\mathrm{soft}}$, reflects the infrared singularity and is obtained by setting $p_3 = 0$ in Eq. (3.19) everywhere except in the $1/p_3$ denominator and subtracting the same expression with Ω_3 also set equal to Ω_s except in the denominator:

$$I_{1,\mathrm{soft}} = \mathcal{J}_1(\epsilon)\int dy_1 \int d^{2-2\epsilon}\tilde{\Omega}\ \frac{F_1(y_1;p_J,\Omega_J;0,y_1+\tilde{y},\phi_J+\tilde{\phi}) - F_1(y_1;p_J,\Omega_J;0,\Omega_s)}{\cosh(\tilde{y})+\cos(\tilde{\phi})}, \tag{3.24}$$

where $\mathcal{J}_1(\epsilon)$ is the same integral encountered in Eq. (2.35):

$$\mathcal{J}_1(\epsilon) = \int_0^{Q_1}\frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} = -\frac{1}{2\epsilon}\left(\frac{Q_1}{\mu}\right)^{-2\epsilon}. \tag{3.25}$$

The final term in (3.20), $I_{1,\mathrm{double}}$, is obtained by setting both $\Omega_3 = \Omega_s$ and $p_3 = 0$ except in the denominator factors:

$$I_{1,\mathrm{double}} = \mathcal{I}_1(\epsilon)\ \mathcal{J}_1(\epsilon)\ \int dy_1\ F_1(y_1;p_J,\Omega_J;0,\Omega_s). \tag{3.26}$$

Finally, $I_{1,\mathrm{finite}}$ is simply the remainder defined by Eq. (3.20):

$$\begin{aligned}I_{1,\mathrm{finite}} = \int dy_1 \int \frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} \int d^{2-2\epsilon}\Omega_3 \\ \times\Bigg(&\frac{[1-(p_3/p_J)]^2}{\cosh\{[1-(p_3/p_J)](y_3-y_1)\}+\cos(\phi_3-\phi_J)} \\ &\times[F_1(y_1;p_J,\Omega_J;p_3,\Omega_3)\rho(p_3/p_J,\ y_3-y_1,\ \phi_3-\phi_J) - F_1(y_1;p_J,\Omega_J;p_3,\Omega_s)] \\ &+\frac{\theta(p_3<Q_1)}{\cosh(y_3-y_1)+\cos(\phi_3-\phi_J)}[-F_1(y_1;p_J,\Omega_J;0,\Omega_3) \\ &\qquad +F_1(y_1;p_J,\Omega_J;0,\Omega_s)]\Bigg).\end{aligned} \tag{3.27}$$

We see that because of the subtractions, $I_{1,\mathrm{finite}}$ has neither collinear nor infrared divergences.

We now introduce a notation in which $I_{1,\mathrm{S}}$ is written as an integral over y_1:

$$I_{1,\mathrm{S}} = \int dy_1\ G_1^{(2\to 3)}(y_1,p_J,y_J,\phi_J). \tag{3.28}$$

Here $G_1^{(2\to 3)}$ is given by a three-dimensional integral over $(p_3, \tilde{y}, \tilde{\phi})$. It is divided into pieces:

$$G_1^{(2\to3)} = G_{1,\text{coll}}^{(2\to3)} + G_{1,\text{soft}}^{(2\to3)} + G_{1,\text{double}}^{(2\to3)} + G_{1,\text{finite}}^{(2\to3)}\,, \tag{3.29}$$

as specified in this section. We analyze the pieces, other than the finite term, in the following sections.

F. The double singular contribution

From Eq. (3.26), we have for $G_{1,\text{double}}^{(2\to3)}$ the simple expression

$$G_{1,\text{double}}^{(2\to3)} = \mathcal{I}_1(\epsilon)\ \mathcal{J}_1(\epsilon)\ H_1(y_1;p_J,\boldsymbol{\Omega}_J;0,\boldsymbol{\Omega}_s)f_1(y_1;p_J,\boldsymbol{\Omega}_J;0,\boldsymbol{\Omega}_s). \tag{3.30}$$

When evaluated at the double singular point, the functions H_1 and f_1 [cf. Eqs. (1.65) and (3.5)] are simply

$$H_1(y_1;p_J,\boldsymbol{\Omega}_J;0,\boldsymbol{\Omega}_s) = (4\pi^2)^\epsilon \frac{C(\epsilon)}{2\pi} L(X_A,X_B)\frac{\alpha_s}{2\pi}, \tag{3.31}$$

$$f_1(y_1;p_J,\boldsymbol{\Omega}_J;0,\boldsymbol{\Omega}_J;\epsilon) = 2N\ d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon). \tag{3.32}$$

We now assemble these ingredients and obtain

$$G_{1,\text{double}}^{(2\to3)} = C(\epsilon)L(X_A,X_B)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q_{\text{ES}}^2}\right)^\epsilon \Gamma_K(\epsilon)d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\left[\frac{N}{\epsilon^2}\left(\frac{Q_{\text{ES}}^2}{16Q_1^2}\right)^\epsilon \Gamma_1(\epsilon)\right], \tag{3.33}$$

where

$$\Gamma_1(\epsilon) = \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-2\epsilon)} = 1 - \frac{\pi^2}{3}\epsilon^2 + O(\epsilon^3). \tag{3.34}$$

Expanding the factor in square brackets gives

$$\begin{aligned} G_{1,\text{double}}^{(2\to3)} = {} & C(\epsilon)\ L(X_A,X_B)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q_{\text{ES}}^2}\right)^\epsilon \Gamma_K(\epsilon)d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon) \\ & \times\left\{\frac{N}{\epsilon^2} + \frac{N}{\epsilon}\ln\left(\frac{Q_{\text{ES}}^2}{16Q_1^2}\right) + N\left[\frac{1}{2}\ln^2\left(\frac{Q_{\text{ES}}^2}{16Q_1^2}\right) - \frac{\pi^2}{3}\right] + O(\epsilon)\right\}. \end{aligned} \tag{3.35}$$

G. The collinear contribution

From Eqs. (3.22) and (3.4), we have the following expression for $G_{1,\text{coll}}^{(2\to3)}$:

$$\begin{aligned} G_{1,\text{coll}}^{(2\to3)} = \mathcal{I}_1(\epsilon)\int\frac{dp_3}{p_3}\left(\frac{p_3}{\mu}\right)^{-2\epsilon} [\ & (1-p_3/p_J)H_1(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_s)f_1(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_s) \\ & -H_1(y_1;p_J,\boldsymbol{\Omega}_J;0,\boldsymbol{\Omega}_s)f_1(y_1;p_J,\boldsymbol{\Omega}_J;0,\boldsymbol{\Omega}_s)\theta(p_3<Q_1)]. \end{aligned} \tag{3.36}$$

In the singular configuration at which the momenta are evaluated in the first term of Eq. (3.36), gluons 1 and 3 are parallel and recoil against the observed jet. Let us change integration variables in Eq. (3.36) from p_3 to z defined by

$$p_3 = (1-z)p_J. \tag{3.37}$$

Then $p_3^\mu = [(1-z)/z]p_1^\mu$.

The function H_1 at the collinear point is given simply by

$$H_1(y_1;p_J,\boldsymbol{\Omega}_J;p_3,\boldsymbol{\Omega}_s) = (4\pi^2)^\epsilon\frac{C(\epsilon)}{2\pi}L(X_A,X_B)\Theta(z>\tfrac{1}{2})\frac{\alpha_s}{2\pi}, \tag{3.38}$$

where X_A and X_B are determined from y_1, y_J, and p_J according to Eq. (1.16), as in the $2\to2$ process. The restriction that $z>\frac{1}{2}$ comes from the factor $\theta(p_3<p_1)$. From the explicit form (3.23) of $\mathcal{I}_1(\epsilon)$ we have

$$\mathcal{I}_1(\epsilon) = -\frac{2\pi}{\epsilon}(16\pi)^{-\epsilon}\ \Gamma_K(\epsilon)\ [1+O(\epsilon^2)]. \tag{3.39}$$

At the collinear point, the function f_1 has the simple form given by Eq. (I.65):

$$f_1(y_1;p_J,\mathbf{\Omega}_J;(1-z)p_J,\mathbf{\Omega}_s;\epsilon) = d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\frac{1-z}{z}\,\tilde{P}_{gg}(z), \tag{3.40}$$

where $\tilde{P}_{gg}(z)$ is given by Eq. (1.62).

Combining these results, we obtain

$$G_{1,\text{coll}}^{(2\to3)} = -C(\epsilon)L(X_A,X_B)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q_{\text{ES}}^2}\right)^\epsilon \Gamma_K(\epsilon)d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon)\left[\frac{1}{\epsilon}\left(\frac{Q_{\text{ES}}^2}{16p_J^2}\right)^\epsilon Z_1(\epsilon)[1+O(\epsilon^2)]\right], \tag{3.41}$$

where Z_1 is the integral

$$Z_1 = \int_0^1 dz[\theta(z>\tfrac{1}{2})(1-z)^{-2\epsilon}\tilde{P}_{gg}(z) - \theta(z>1-Q_1/p_J)\,(1-z)^{-1-2\epsilon}P_{gg}^{\text{soft}}] \tag{3.42}$$

and where we have denoted $P_{gg}^{\text{soft}} = 2N$ as in Eq. (2.55). The integration can easily be performed, yielding

$$Z_1(\epsilon) = 2N\left\{-\frac{11}{12} - \frac{1}{2}\ln\left(\frac{Q_1^2}{p_J^2}\right) + \epsilon\left[-\tfrac{137}{72} - \tfrac{11}{6}\ln(2) + \frac{\pi^2}{6} + \frac{1}{4}\ln^2\left(\frac{Q_1^2}{p_J^2}\right)\right] + O(\epsilon^2)\right\}. \tag{3.43}$$

Expanding in powers of ϵ in Eq. (3.41), we obtain

$$\begin{aligned} G_{1,\text{coll}}^{(2\to3)} = {} & C(\epsilon)\;L(X_A,X_B)\;\frac{\alpha_s}{2\pi}\;\left(\frac{4\pi\mu^2}{Q_{\text{ES}}^2}\right)^\epsilon \Gamma_K(\epsilon)\;d^{(4)}(\hat{S},\hat{T},\hat{U};\epsilon) \\ & \times 2N\left\{\frac{1}{\epsilon}\left[\frac{11}{12}+\frac{1}{2}\ln\left(\frac{Q_1^2}{p_J^2}\right)\right] + \left[\frac{11}{12}+\frac{1}{2}\ln\left(\frac{Q_1^2}{p_J^2}\right)\right]\ln\left(\frac{Q_{\text{ES}}^2}{16p_J^2}\right)\right. \\ & \left. +\tfrac{137}{72} + \tfrac{11}{6}\ln(2) - \frac{\pi^2}{6} - \frac{1}{4}\ln^2\left(\frac{Q_1^2}{p_J^2}\right) + O(\epsilon)\right\}. \end{aligned} \tag{3.44}$$

H. The soft contribution

In this section we wish to calculate $G_{1,\text{soft}}^{(2\to3)}$, which is given by Eqs. (3.24) and (3.4). This task is very easy. We invite the reader to check that $G_{1,\text{soft}}^{(2\to3)}$ is the same as $G_{2,\text{soft}}^{(2\to3)}$, Eq. (2.68), with the substitutions $y_1 \leftrightarrow y_J$, $\hat{T} \leftrightarrow \hat{U}$, and $Q_2 \to Q_1$. Furthermore, $G_{2,\text{soft}}^{(2\to3)}$ is symmetric under the interchange $y_1 \leftrightarrow y_J$, $\hat{T} \leftrightarrow \hat{U}$. Thus

$$G_{1,\text{soft}}^{(2\to3)} = (G_{2,\text{soft}}^{(2\to3)})_{Q_2\to Q_1}. \tag{3.45}$$

IV. TERM A

A. Kinematics

In this section we are concerned with the term in the invariant matrix element with a $1/p_3\cdot p_A$ singularity. We are thus interested in the integration region in which gluon 3 may become collinear with hadron A, or in which gluon 3 may become soft. For this purpose, we define integration variables $\mathbf{W},\xi$ for gluon 3 as follows:

$$p_3^\mu = (\xi\sqrt{s/2}, \xi W^2/\sqrt{2s}, \xi\mathbf{W})\;. \tag{4.1}$$

Note that $\mathbf{W}$ is a vector in $2-2\epsilon$ dimensions. Its magnitude gives the rapidity of gluon 3:

$$e^{2y_3} = \frac{\xi\sqrt{s/2}}{\xi W^2/\sqrt{2s}} = \frac{s}{W^2}. \tag{4.2}$$

Let us investigate the appearance of the singularities corresponding to p_3^μ being parallel to p_A^μ and to p_3^μ being soft. The collinear singularity corresponds to $W\to 0$ at fixed ξ. Using $p_A^\mu = (x_A\sqrt{s/2},0,\mathbf{0})$ we find that

$$p_3\cdot p_A = \tfrac{1}{2}x_A\xi\mathbf{W}^2\;. \tag{4.3}$$

The soft singularity corresponds to $\xi\to 0$ at fixed $\mathbf{W}$, that is, just scaling p_3^μ. It arises from factors such as $p_3\cdot(p_A+p_B)$ in the matrix element (after rewriting by partial fractions). Using p_A^μ as above and $p_B^\mu = (0, x_B\sqrt{s/2},\mathbf{0})$, one finds, for instance,

$$p_3 \cdot (p_A + p_B) = \frac{\xi}{2}(x_B s + x_A W^2) . \tag{4.4}$$

We shall therefore extract a factor

$$\frac{1}{\xi^2 W^2} \tag{4.5}$$

from the matrix element to account for the collinear and soft singularities.

Let us now look at the needed integrations. We can adapt Eq. (2.11), using

$$dy_3\ p_3^{1-2\epsilon}\ dp_3\ d^{1-2\epsilon}\phi_3 = dy_3\ d^{2-2\epsilon}\mathbf{p}_3 = \frac{d\xi}{\xi} d^{2-2\epsilon}\mathbf{p}_3 = \xi^{1-2\epsilon} d\xi\ d^{2-2\epsilon}\mathbf{W} . \tag{4.6}$$

This gives

$$\begin{aligned} d\sigma_A = {} & dy_1\ dy_2\ dp_2\ d^{1-2\epsilon}\phi_2\ \xi^{1-2\epsilon} d\xi\ d^{2-2\epsilon}\mathbf{W}\ x_A f_A(x_A)\ x_B f_B(x_B) \\ & \times \frac{p_2}{8(2\pi)^5 \hat{s}^2} \left(\frac{p_2}{(2\pi)^2\mu^2}\right)^{-2\epsilon} \langle |\mathcal{M}|^2\rangle_A \theta(p_3 < p_1)\ \theta(p_3 < p_2)\ \theta(|\phi_2 - \phi_J| < |\phi_1 - \phi_J|) . \end{aligned} \tag{4.7}$$

Let us define a function F_A to be the integrand with the singularities factored out:

$$\begin{aligned} \mu^{2\epsilon}\frac{F_A(y_1; p_2, y_2, \phi_2; \xi, \mathbf{W})}{\xi^2 W^2} = {} & x_A f_A(x_A)\ x_B f_B(x_B) \frac{p_2}{8(2\pi)^5 \hat{s}^2} \left(\frac{p_2}{(2\pi)^2\mu^2}\right)^{-2\epsilon} \langle |\mathcal{M}|^2\rangle_A \\ & \times \theta(p_3 < p_1)\ \theta(p_3 < p_2)\ \theta(|\phi_2 - \phi_J| < |\phi_1 - \phi_J|). \end{aligned} \tag{4.8}$$

Using (1.54), we can express F_A in terms of f_A. One finds [cf. Eq. (2.13)]

$$F_A(y_1; p_2, y_2, \phi_2; \xi, \mathbf{W}) = H_A f_A , \tag{4.9}$$

where

$$H_A(y_1; p_2, y_2, \phi_2; \xi, \mathbf{W}) = \left(\frac{p_2}{p_J}\right)^{1-2\epsilon} (4\pi^2)^\epsilon\ \frac{C(\epsilon)}{2\pi}\ L(x_A, x_B)\ \tilde{\Theta}\ \frac{\alpha_s}{2\pi}\ 2 . \tag{4.10}$$

We shall want to express the variables of the problem in terms of the chosen integration variables

$$y_1, p_2, y_2, \phi_2, \xi, W, \phi_3 . \tag{4.11}$$

The required relations are similar to those in Sec. II A:

$$\begin{aligned} & p_3 = \xi W\ ,\ \ e^{y_3} = \frac{\sqrt{s}}{W}\ ,\ \ p_1 = [p_2^2 + p_3^2 + 2p_2p_3\cos(\phi_2 - \phi_3)]^{1/2}\ ,\ \ x_A = \frac{1}{\sqrt{s}}(p_1 e^{y_1} + p_2 e^{y_2} + p_3 e^{y_3}), \\ & x_B = \frac{1}{\sqrt{s}}(p_1 e^{-y_1} + p_2 e^{-y_2} + p_3 e^{-y_3}),\ \ p_{AB} = x_A\ x_B\ s/2,\ \ p_{A1} = x_A\sqrt{s}\ p_1\ e^{-y_1}/2,\ \ p_{A2} = x_A\sqrt{s}\ p_2\ e^{-y_2}/2, \\ & p_{A3} = x_A\sqrt{s}\ p_3\ e^{-y_3}/2\ ,\ \ p_{B1} = x_B\sqrt{s}\ p_1\ e^{+y_1}/2\ ,\ \ p_{B2} = x_B\sqrt{s}\ p_2\ e^{+y_2}/2\ ,\ \ p_{B3} = x_B\sqrt{s}\ p_3\ e^{+y_3}/2\ , \\ & p_{12} = p_1p_2\cosh(y_1 - y_2) + p_2^2 + p_2p_3\cos(\phi_2 - \phi_3)\ ,\ \ p_{13} = p_1p_3\cosh(y_1 - y_3) + p_3^2 + p_2p_3\cos(\phi_2 - \phi_3)\ , \\ & p_{23} = p_2p_3\cosh(y_2 - y_3) - p_2p_3\cos(\phi_2 - \phi_3)\ . \end{aligned} \tag{4.12}$$

It is useful for computational purposes to rewrite the dot products involving p_3^μ in a form that makes it explicit that they contain a factor ξ as $\xi \to 0$. One finds

$$\begin{aligned} & p_{A3} = \xi\left(\frac{x_A W^2}{2}\right)\ ,\ \ p_{B3} = \xi\left(\frac{x_B s}{2}\right), \\ & p_{13} = \xi\left(e^{-y_1}\frac{p_1\sqrt{s}}{2} + e^{y_1}\frac{p_1 W^2}{2\sqrt{s}} + \xi W^2 + p_2 W\cos(\phi_2 - \phi_3)\right)\ , \\ & p_{23} = \xi\left(e^{-y_2}\frac{p_2\sqrt{s}}{2} + e^{y_2}\frac{p_2 W^2}{2\sqrt{s}} - p_2 W\cos(\phi_2 - \phi_3)\right)\ . \end{aligned} \tag{4.13}$$

B. Jet definition and extraction of the singular contribution

With the notation established in (4.8), we can write the cross section in the form

$$d\sigma_A = dy_1\ dp_2\ dy_2\ d^{1-2\epsilon}\phi_2\ \frac{d\xi}{\xi}\ \mu^{2\epsilon}\frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\ \xi^{-2\epsilon} F_A(y_1; p_2, y_2, \phi_2; \xi, \mathbf{W}) . \tag{4.14}$$

As we argued in Sec. I C, there are three possibilities for which gluons constitute the jet: gluon 2, gluon 3, or gluons 2 and 3. Following Eq. (2.23), we find the following contribution to the jet cross section corresponding to these three possibilities:

$$\frac{d\sigma_A}{dy_J\ dp_J\ d^{1-2\epsilon}\phi_J} = \int dy_1 \int dp_2 \int dy_2 \int d^{1-2\epsilon}\phi_2 \int \frac{d\xi}{\xi}\ \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\ \xi^{-2\epsilon} F_A(y_1;p_2,y_2,\phi_2;\xi,\mathbf{W})$$
$$\times\left[\delta(p_2+p_3-p_J)\ \delta^{2-2\epsilon}\left(\frac{1}{p_J}(p_2\mathbf{\Omega}_2+p_3\mathbf{\Omega}_3)-\mathbf{\Omega}_J\right)\ \theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_3|<\frac{p_2+p_3}{p_2}R\right)\right.$$
$$+\delta(p_2-p_J)\ \delta^{2-2\epsilon}(\mathbf{\Omega}_2-\mathbf{\Omega}_J)\ \theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_3|>\frac{p_2+p_3}{p_2}R\right)$$
$$\left.+\delta(p_3-p_J)\ \delta^{2-2\epsilon}(\mathbf{\Omega}_3-\mathbf{\Omega}_J)\ \theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_3|>\frac{p_2+p_3}{p_2}R\right)\right]. \quad (4.15)$$

In the first and second terms here, we use the delta functions to perform the integration over the momentum of gluon 2. In the case of the first term, this gives

$$p_2=p_J-\xi W\ ,\ \ \phi_2=\frac{p_J}{p_J-\xi W}\ \phi_J-\frac{\xi W}{p_J-\xi W}\ \phi_3\ ,\ \ y_2=\frac{p_J}{p_J-\xi W}\ y_J-\frac{\xi W}{p_J-\xi W}\ \ln\sqrt{s/W^2}\ , \quad (4.16)$$

and a Jacobian equal to $[p_J/(p_J-\xi W)]^{2-2\epsilon}$. In the third term, we use the delta functions to perform the integrations over ξ and $\mathbf{W}$ so that

$$\xi=e^{y_J}p_J/\sqrt{s}\ ,\ \ W=e^{-y_J}\sqrt{s}\ ,\ \ \phi_3=\phi_J \quad (4.17)$$

with a Jacobian equal to $(p_J/\xi)^{1-2\epsilon}$. Thus Eq. (4.15) becomes

$$\frac{d\sigma_A}{dy_J\ dp_J\ d^{1-2\epsilon}\phi_J} = \int dy_1 \int \frac{d\xi}{\xi}\ \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\ \xi^{-2\epsilon}\left(\frac{p_J}{p_J-\xi W}\right)^{2-2\epsilon}$$
$$\times F_A\left(y_1;p_J-\xi W,\frac{p_Jy_J-\xi W\ln\sqrt{s/W^2}}{p_J-\xi W},\frac{p_J\phi_J-\xi W\phi_3}{p_J-\xi W};\xi,\mathbf{W}\right)$$
$$\times\theta[(y_J-\ln\sqrt{s/W^2})^2+(\phi_J-\phi_3)^2<R^2]$$
$$+\int dy_1 \int \frac{d\xi}{\xi}\ \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\ \xi^{-2\epsilon} F_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{W})$$
$$\times\theta\left[(y_J-\ln\sqrt{s/W^2})^2+(\phi_J-\phi_3)^2>\left(\frac{p_J+\xi W}{p_J}\right)^2R^2\right]$$
$$+\int dy_1 \int dp_2 \int dy_2 \int d^{1-2\epsilon}\phi_2\ \frac{1}{p_J}\left(\frac{\mu}{p_J}\right)^{2\epsilon}$$
$$\times F_A(y_1;p_2,y_2,\phi_2;e^{y_J}p_J/\sqrt{s},e^{-y_J}\sqrt{s}\ \cos\phi_J,e^{-y_J}\sqrt{s}\ \sin\phi_J)$$
$$\times\theta\left(|\mathbf{\Omega}_2-\mathbf{\Omega}_J|>\frac{p_2+p_J}{p_2}R\right). \quad (4.18)$$

The only term with a collinear singularity here is the second. In this term, let us first eliminate the theta function by adding and subtracting a piece with the opposite theta function. In this way, we divide the cross section into a singular piece with no theta function plus a nonsingular piece:

$$\frac{d\sigma_A}{dy_J\ dp_J\ d^{1-2\epsilon}\phi_J} = I_{A,\mathrm{NS}}+I_{A,\mathrm{S}}, \quad (4.19)$$

where

$$I_{A,\mathrm{NS}} = \int dy_1 \int \frac{d\xi}{\xi}\ \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\ \xi^{-2\epsilon}\left(\frac{p_J}{p_J-\xi W}\right)^{2-2\epsilon}$$
$$\times F_A\left(y_1;p_J-\xi W,\frac{p_Jy_J-\xi W\ln\sqrt{s/W^2}}{p_J-\xi W},\frac{p_J\phi_J-\xi W\phi_3}{p_J-\xi W};\xi,\mathbf{W}\right)$$
$$\times\theta[(y_J-\ln\sqrt{s/W^2})^2+(\phi_J-\phi_3)^2<R^2]$$
$$-\int dy_1 \int \frac{d\xi}{\xi}\ \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\ \xi^{-2\epsilon} F_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{W})$$

$$\times\theta\left[(y_J - \ln\sqrt{s/W^2})^2 + (\phi_J - \phi_3)^2 < \left(\frac{p_J + \xi W}{p_J}\right)^2 R^2\right]$$

$$+\int dy_1 \int dp_2 \int dy_2 \int d^{1-2\epsilon}\phi_2 \,\frac{1}{p_J}\left(\frac{\mu}{p_J}\right)^{2\epsilon}$$
$$\times F_A(y_1;p_2,y_2,\phi_2;e^{y_J}p_J/\sqrt{s},e^{-y_J}\sqrt{s}\,\cos\phi_J,e^{-y_J}\sqrt{s}\,\sin\phi_J)$$
$$\times\theta\left(|\Omega_2 - \Omega_J| > \frac{p_2 + p_J}{p_2}R\right) . \tag{4.20}$$

Notice that there is no collinear ($W \to 0$) singularity in $I_{A,\rm NS}$ because the theta functions now do not allow it. Also, there is no soft ($\xi \to 0$) singularity because the first and second terms cancel when $\xi \to 0$.

The remaining term is

$$I_{A,\rm S} = \int dy_1 \int \frac{d\xi}{\xi}\,\mu^{2\epsilon}\int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\,\xi^{-2\epsilon}F_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{W}) . \tag{4.21}$$

C. Decomposition of the singular contribution

Now we separate our integral $I_{A,\rm S}$ into pieces. The piece containing the collinear singularity is defined to be

$$I_{A,\rm coll} = \mathcal{I}_A(\epsilon)\int dy_1 \int \frac{d\xi}{\xi}\,\xi^{-2\epsilon}[F_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{0}) - \theta(\xi < \Xi_A)\,F_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})] , \tag{4.22}$$

where

$$\mathcal{I}_A(\epsilon) = \mu^{2\epsilon}\int d^{2-2\epsilon}\mathbf{W}\,\theta(W^2 < Q_A^2)\,\frac{1}{W^2} = -\frac{\pi^{1-\epsilon}}{\epsilon\,\Gamma(1-\epsilon)}\left(\frac{Q_A^2}{\mu^2}\right)^{-\epsilon} . \tag{4.23}$$

We have inserted a theta function $\theta(W^2 < Q_A^2)$ to provide an upper cutoff on the W integration. The final result will not depend on the arbitrary parameter Q_A^2. A sensible choice might be $Q_A^2 = s$, which corresponds to $y_3 = 0$. Note also that there is a subtraction at $\xi = 0$ to remove the soft singularity from $I_{\rm coll}$. In this term, we have also inserted a factor $\theta(\xi < \Xi_A)$ to provide an upper cutoff on the ξ integration. For instance, one might use $\Xi_A = 1 - 4p_J^2/s$. The final result will not depend on Ξ_A.

We define the soft-gluon subtraction $I_{A,\rm soft}$ as

$$I_{A,\rm soft} = \mathcal{J}_A(\epsilon)\int dy_1\,\mu^{2\epsilon}\int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\,[F_A(y_1;p_J,y_J,\phi_J;0,\mathbf{W}) - \theta(W^2 < Q_A^2)\,F_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})] , \tag{4.24}$$

where

$$\mathcal{J}_A(\epsilon) = \int_0^{\Xi_A}\frac{d\xi}{\xi}\xi^{-2\epsilon} = -\frac{1}{2\epsilon}\,\Xi_A^{-2\epsilon} . \tag{4.25}$$

The final subtraction is for gluon 3 being both collinear to the beam and soft. It yields

$$I_{A,\rm double} = \mathcal{I}_A(\epsilon)\mathcal{J}_A(\epsilon)\int dy_1\;F_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0}) . \tag{4.26}$$

We have now treated the singular terms and can turn our attention to the finite remainder. We define $I_{A,\rm finite}$ by

$$I_{A,\rm S} = I_{A,\rm coll} + I_{A,\rm soft} + I_{A,\rm double} + I_{A,\rm finite}. \tag{4.27}$$

Then

$$I_{A,\rm finite} = \int dy_1 \int \frac{d\xi}{\xi}\,\mu^{2\epsilon}\int \frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\,\xi^{-2\epsilon}$$
$$\times[F_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{W}) - \theta(W^2 < Q_A^2)\,F_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{0})$$
$$-\theta(\xi < \Xi_A)F_A(y_1;p_J,y_J,\phi_J;0,\mathbf{W}) + \theta(W^2 < Q_A^2)\,\theta(\xi < \Xi_A)\,F_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})]. \tag{4.28}$$

One easily checks that the subtractions remove the singularities, so that $I_{A,\rm finite}$ is indeed finite.

We now introduce a notation in which $I_{A,\rm S}$ is written as an integral over y_1:

$$I_{A,\mathrm{S}} = \int dy_1\; G_A^{(2\to3)}(y_1,p_J,y_J,\phi_J)\,. \tag{4.29}$$

Here $G_A^{(2\to3)}$ is given by a three-dimensional integral over (p_3,ξ,W,ϕ_3). It is divided into pieces

$$G_A^{(2\to3)} = G_{A,\mathrm{coll}}^{(2\to3)} + G_{A,\mathrm{soft}}^{(2\to3)} + G_{A,\mathrm{double}}^{(2\to3)} + G_{A,\mathrm{finite}}^{(2\to3)}\,, \tag{4.30}$$

as specified in this section. We analyze the pieces, other than the finite term, in the following sections.

D. The double singular contribution

From Eqs. (4.26) and (4.9), we have for $G_{A,\mathrm{double}}^{(2\to3)}$ the simple expression

$$G_{A,\mathrm{double}}^{(2\to3)} = \mathcal{I}_A(\epsilon)\; \mathcal{J}_A(\epsilon)\; H_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})\; f_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})\,. \tag{4.31}$$

When evaluated at the double singular point, functions H_A and f_A [cf. Eq. (1.61)] are simply

$$H_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0}) = (4\pi^2)^\epsilon \frac{C(\epsilon)}{2\pi} L(X_A,X_B)\; \frac{\alpha_s}{2\pi}\; 2, \tag{4.32}$$

$$f_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0};\epsilon) = 2N\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon). \tag{4.33}$$

We now assemble these ingredients and obtain

$$G_{A,\mathrm{double}}^{(2\to3)} = C(\epsilon)\; L(X_A,X_B)\frac{\alpha_s}{2\pi}\; \left(\frac{4\pi\mu^2}{Q_{\mathrm{ES}}^2}\right)^\epsilon \Gamma_K(\epsilon)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon) \left[\frac{N}{\epsilon^2}\left(\frac{Q_{\mathrm{ES}}^2}{\Xi_A^2 Q_A^2}\right)^\epsilon \Gamma_A(\epsilon)\right]\,, \tag{4.34}$$

where

$$\Gamma_A(\epsilon) = \frac{\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^3} = 1 + O(\epsilon^3)\,. \tag{4.35}$$

Expanding the factor in square brackets gives

$$G_{A,\mathrm{double}}^{(2\to3)} = C(\epsilon)\; L(X_A,X_B)\frac{\alpha_s}{2\pi}\; \left(\frac{4\pi\mu^2}{Q_{\mathrm{ES}}^2}\right)^\epsilon \Gamma_K(\epsilon)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon) \left[\frac{N}{\epsilon^2} + \frac{N}{\epsilon}\ln\!\left(\frac{Q_{\mathrm{ES}}^2}{\Xi_A^2 Q_A^2}\right) + N\frac{1}{2}\ln^2\!\left(\frac{Q_{\mathrm{ES}}^2}{\Xi_A^2 Q_A^2}\right) + O(\epsilon)\right]\,. \tag{4.36}$$

E. The collinear contribution

In this section, we examine the term $G_{A,\mathrm{coll}}^{(2\to3)}(y_1,p_J,y_J,\phi_J)$ given by Eqs. (4.22), (4.9), and (4.30):

$$\begin{aligned} G_{A,\mathrm{coll}}^{(2\to3)} = \mathcal{I}_A(\epsilon) \int \frac{d\xi}{\xi}\xi^{-2\epsilon}[&H_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{0}) f_A(y_1;p_J,y_J,\phi_J;\xi,\mathbf{0}) \\ &-H_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0}) f_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})\theta(\xi<\Xi_A)]\,. \end{aligned} \tag{4.37}$$

The functions appearing here are evaluated in the collinear configuration in which gluon 2 constitutes the jet and gluon 3 is exactly collinear to incoming gluon A, carrying a fraction ξ of the incoming proton momentum.

It will be helpful to recall some notation from Sec. I B. We defined momentum fractions X_A, X_B by

$$X_A = \frac{p_J}{\sqrt{s}}\,(e^{y_1}+e^{y_J}),\quad X_B = \frac{p_J}{\sqrt{s}}\,(e^{-y_1}+e^{-y_J}). \tag{4.38}$$

If only gluons 1 and 2 were present, X_A and X_B would be the momentum fractions of the incoming gluons. In the collinear configuration at issue here, gluon 3 also takes up some + component of momentum, so that the momentum fractions of the incoming gluons are

$$x_A = X_A + \xi,\quad x_B = X_B\,. \tag{4.39}$$

We also introduced variables $\hat S,\hat T,\hat U$ formed from the momenta of gluons 1 and 2 in such a way that, in the absence of gluon 3, these would be the Mandelstam variables of the elementary gluon scattering. Thus [cf. Eq. (1.18)],

$$\hat{S} = 2p_J^2[1+\cosh(y_J - y_1)]\ , \quad \hat{T} = -p_J^2(1+e^{y_J - y_1})\ , \quad \hat{U} = -p_J^2(1+e^{y_1 - y_J})\ . \tag{4.40}$$

Finally, let us supplement this notation by letting z denote the fraction of the momentum of gluon A that is left for the hard interaction after the collinear gluon 3 is emitted:

$$z = \frac{x_A - \xi}{x_A}, \tag{4.41}$$

so

$$x_A = \frac{X_A}{z}\ . \tag{4.42}$$

We can now return to an examination of Eq. (4.37). In the collinear configuration at hand, the function H_A, Eq. (4.10), becomes

$$H_A(y_1; p_2, y_2, \phi_2; \xi, \mathbf{W}) = (4\pi^2)^\epsilon \frac{C(\epsilon)}{2\pi} L\left(\frac{X_A}{z}, X_B\right) \frac{\alpha_s}{2\pi}\, 2\ . \tag{4.43}$$

The function f_A has the simple form given by Eq. (1.61):

$$f_A(y_1; p_J, y_J, \phi_J; \xi, 0; \epsilon) = d^{(4)}(\hat{S}, \hat{T}, \hat{U}; \epsilon)\ \frac{1-z}{z}\ \tilde{P}_{gg}(z), \tag{4.44}$$

where $\tilde{P}_{gg}(z)$ is given in Eq. (1.62).

We shall want to rewrite $G^{(2\to3)}_{A,\text{coll}}$ using z instead of ξ as the integration variable. The relation between the two is

$$z = \frac{X_A}{x_A} = \frac{X_A}{X_A + \xi}\ , \tag{4.45}$$

so

$$\xi = \frac{1-z}{z} X_A\ . \tag{4.46}$$

The Jacobian is

$$\frac{d\xi}{\xi} = \frac{dz}{z(1-z)}\ . \tag{4.47}$$

Using all of this information, we can write $G^{(2\to3)}_{A,\text{coll}}$ as

$$G^{(2\to3)}_{A,\text{coll}} = \mathcal{I}_A(\epsilon)(4\pi^2)^\epsilon \frac{C(\epsilon)}{\pi} \frac{\alpha_s}{2\pi}\ d^{(4)}(\hat{S}, \hat{T}, \hat{U}; \epsilon)$$
$$\times \int_0^1 \frac{dz}{z(1-z)} \left(\frac{1-z}{z} X_A\right)^{-2\epsilon} \left[\frac{1-z}{z} \tilde{P}_{gg}(z) L\left(\frac{X_A}{z}, X_B\right) - \theta(z > z_{\min}) P^{\text{soft}}_{gg} L(X_A, X_B)\right]\ . \tag{4.48}$$

Here we have defined

$$z_{\min} = \frac{X_A}{X_A + \Xi_A} \tag{4.49}$$

and we have used Eq. (2.55),

$$(1-z)\ \tilde{P}_{gg}(z) \longrightarrow P^{\text{soft}}_{gg} = 2N \quad \text{as} \quad z \to 1\ . \tag{4.50}$$

Inserting the value of $\mathcal{I}_A(\epsilon)$ from (4.23) gives

$$G^{(2\to3)}_{A,\text{coll}} = -\frac{1}{\epsilon} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} C(\epsilon) \frac{\alpha_s}{2\pi}\ d^{(4)}(\hat{S}, \hat{T}, \hat{U}; \epsilon)$$
$$\times \int dz\ \left(\frac{Q_A X_A (1-z)}{\mu z}\right)^{-2\epsilon} \left[\tilde{P}_{gg}(z)\ \frac{1}{z^2}\ L\left(\frac{X_A}{z}, X_B\right) - \theta(z > z_{\min})\ \frac{P^{\text{soft}}_{gg}}{z(1-z)}\ L(X_A, X_B)\right]\ . \tag{4.51}$$

Then expanding in powers of ϵ gives

$$G_{A,\mathrm{coll}}^{(2\to3)} = -\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\, C(\epsilon)\frac{\alpha_s}{2\pi}\, d^{(4)}(\hat S,\hat T,\hat U;\epsilon)$$
$$\times\int dz\left[\tilde P_{gg}(z)\,\frac{1}{z^2}\,L\!\left(\frac{X_A}{z},X_B\right) - \theta(z>z_{\min})\,\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\,L(X_A,X_B)\right]$$
$$+C(0)\frac{\alpha_s}{\pi}\, d^{(4)}(\hat S,\hat T,\hat U;0)\int dz\,\ln\!\left(\frac{Q_A X_A(1-z)}{\mu z}\right)$$
$$\times\left[\tilde P_{gg}(z)\,\frac{1}{z^2}L\!\left(\frac{X_A}{z},X_B\right) - \theta(z>z_{\min})\,\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\,L(X_A,X_B)\right] + O(\epsilon)\,. \tag{4.52}$$

Now we have to add the counterterm that removes the collinear divergence, using the $\overline{\mathrm{MS}}$ definition of parton distribution functions.[15] A general discussion of the calculational prescription can be found in Ref. 16. To use this prescription, we note from Eqs. (1.24) and (1.25) that the Born cross section is

$$\frac{d\sigma_{\mathrm{Born}}}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = C(\epsilon)\int dy_1\; L(X_A,X_B)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,. \tag{4.53}$$

We write this as

$$\frac{d\sigma_{\mathrm{Born}}}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = \int dx_A\; dx_B\; f_A(x_A)\; f_B(x_B) C(\epsilon)\int dy_1\frac{1}{X_A X_B}\delta(x_A - X_A)\;\delta(x_B - X_B)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,. \tag{4.54}$$

Thus we can identify the hard-scattering cross section as

$$\frac{d\hat\sigma_{\mathrm{Born}}}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J} = C(\epsilon)\int dy_1\frac{1}{X_A X_B}\delta(x_A - X_A)\;\delta(x_B - X_B)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,. \tag{4.55}$$

The $\overline{\mathrm{MS}}$ counterterm to be added to $I_{A,\mathrm{coll}}$ is thus

$$\begin{aligned} I_{A,\mathrm{CT}} &= \frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\alpha_s}{2\pi}\int dx_A\; dx_B\; f_A(x_A)\; f_B(x_B)\int_0^1 dz\; P_{gg}(z)\left(\frac{d\hat\sigma_{\mathrm{Born}}}{dp_J\,dy_J\,d^{1-2\epsilon}\phi_J}\right)_{x_A\to z x_A} \\ &= \frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\alpha_s}{2\pi}\int dx_A\; dx_B\; f_A(x_A)\; f_B(x_B) \\ &\qquad\times\int_0^1 dz\; P_{gg}(z)\; C(\epsilon)\int dy_1\frac{1}{X_A X_B}\delta(z x_A - X_A)\;\delta(x_B - X_B)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon) \\ &= \frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\alpha_s}{2\pi}\; C(\epsilon)\int_0^1 dz\; P_{gg}(z)\;\int dy_1\frac{1}{z^2}\; L\left(\frac{X_A}{z},X_B\right)\; d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,. \end{aligned} \tag{4.56}$$

Removing the integral over y_1 gives the counterterm for $G_{A,\mathrm{coll}}^{(2\to3)}$.

Now we want to break this up, explicitly displaying the $z\to1$ regulation of $P_{gg}(z)$:

$$\begin{aligned} G_{A,\mathrm{CT}}^{(2\to3)} &= \frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\alpha_s}{2\pi}\; C(\epsilon) d^{(4)}(\hat S,\hat T,\hat U;\epsilon) \\ &\times\left\{\int_0^1 dz\; P_{gg}(z)\;\left[\frac{1}{z^2}\; L\left(\frac{X_A}{z},X_B\right) - \theta(z>z_{\min})L(X_A,X_B)\right] + L(X_A,X_B)\int_{z_{\min}}^1 dz\; P_{gg}(z)\right\}\,. \end{aligned} \tag{4.57}$$

We need the integral

$$\begin{aligned} \int_{z_{\min}}^1 dz\; P_{gg}(z) &= \int_{z_{\min}}^1 dz\;\left[2N\left(\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right) + \tfrac{1}{2}\beta_0\;\delta(1-z)\right] \\ &= \int_{z_{\min}}^1 dz\;\left(\tilde P_{gg}(z) - \frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\right) + \int_{z_{\min}}^1 dz\;\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)_+} + \tfrac{1}{2}\beta_0 \\ &= \int_{z_{\min}}^1 dz\;\left(\tilde P_{gg}(z) - \frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\right) + P_{gg}^{\mathrm{soft}}\,\ln\!\left(\frac{1-z_{\min}}{z_{\min}}\right) + \tfrac{1}{2}\beta_0 \\ &= \int_{z_{\min}}^1 dz\;\left(\tilde P_{gg}(z) - \frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\right) + P_{gg}^{\mathrm{soft}}\,\ln\!\left(\frac{\Xi_A}{X_A}\right) + \tfrac{1}{2}\beta_0\,. \end{aligned} \tag{4.58}$$

Thus we find that the counterterm is

$$G_{A,\mathrm{CT}}^{(2\to3)} = +\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,C(\epsilon)\frac{\alpha_s}{2\pi}\,d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\int dz\left\{\tilde P_{gg}(z)\left[\frac{1}{z^2}\,L\!\left(\frac{X_A}{z},X_B\right)-\theta(z>z_{\min})\,L(X_A,X_B)\right]\right.$$
$$\left.+\theta(z>z_{\min})\,L(X_A,X_B)\left(\tilde P_{gg}(z)-\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\right)\right\}$$
$$+\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}C(\epsilon)\frac{\alpha_s}{2\pi}\,d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,L(X_A,X_B)\left[P_{gg}^{\mathrm{soft}}\,\ln\left(\frac{\Xi_A}{X_A}\right)+\tfrac{1}{2}\beta_0\right]. \quad (4.59)$$

Noting that two of the terms cancel, we write this as

$$G_{A,\mathrm{CT}}^{(2\to3)} = +\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,C(\epsilon)\frac{\alpha_s}{2\pi}\,d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\int dz\left[\tilde P_{gg}(z)\,\frac{1}{z^2}L\!\left(\frac{X_A}{z},X_B\right)-\theta(z>z_{\min})\,\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\,L(X_A,X_B)\right]$$
$$+\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}C(\epsilon)\frac{\alpha_s}{2\pi}\,d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,L(X_A,X_B)\left[P_{gg}^{\mathrm{soft}}\,\ln\left(\frac{\Xi_A}{X_A}\right)+\tfrac{1}{2}\beta_0\right]. \quad (4.60)$$

We thus obtain for $G_{A,\mathrm{coll}}^{(2\to3)}+G_{A,\mathrm{CT}}^{(2\to3)}$ the result

$$G_{A,\mathrm{coll}}^{(2\to3)}+G_{A,\mathrm{CT}}^{(2\to3)} = \frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,C(\epsilon)\frac{\alpha_s}{2\pi}\,d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,L(X_A,X_B)\left[P_{gg}^{\mathrm{soft}}\,\ln\left(\frac{\Xi_A}{X_A}\right)+\tfrac{1}{2}\beta_0\right]$$
$$+C(0)\frac{\alpha_s}{\pi}\,d^{(4)}(\hat S,\hat T,\hat U;0)\int dz\ln\left(\frac{Q_AX_A(1-z)}{\mu z}\right)$$
$$\times\left[\tilde P_{gg}(z)\,\frac{1}{z^2}L\!\left(\frac{X_A}{z},X_B\right)-\theta(z>z_{\min})\,\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\,L(X_A,X_B)\right]+O(\epsilon). \quad (4.61)$$

We can modify the $1/\epsilon$ term to make it match the form of our other $1/\epsilon$ terms by multiplying it by

$$\Gamma(1-\epsilon)\,\Gamma_K(\epsilon)\left(\frac{\mu^2}{Q_{\mathrm{ES}}^2}\right)^\epsilon = \frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{Q_{\mathrm{ES}}^2}\right)^\epsilon = 1+\epsilon\,\ln\left(\frac{\mu^2}{Q_{\mathrm{ES}}^2}\right)+O(\epsilon^2). \quad (4.62)$$

This produces an extra finite term to compensate for the change, giving

$$G_{A,\mathrm{coll}}^{(2\to3)}+G_{A,\mathrm{CT}}^{(2\to3)} = C(\epsilon)L(X_A,X_B)\,\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q_{\mathrm{ES}}^2}\right)^\epsilon\Gamma_K(\epsilon)\frac{1}{\epsilon}\,d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left[P_{gg}^{\mathrm{soft}}\ln\left(\frac{\Xi_A}{X_A}\right)+\tfrac{1}{2}\beta_0\right]$$
$$-C(0)\frac{\alpha_s}{\pi}\,\frac{1}{2}\ln\left(\frac{\mu^2}{Q_{\mathrm{ES}}^2}\right)d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\,L(X_A,X_B)\left[P_{gg}^{\mathrm{soft}}\,\ln\left(\frac{\Xi_A}{X_A}\right)+\tfrac{1}{2}\beta_0\right]$$
$$+C(0)\frac{\alpha_s}{\pi}\,d^{(4)}(\hat S,\hat T,\hat U;0)\int dz\,\ln\left(\frac{Q_AX_A(1-z)}{\mu z}\right)$$
$$\times\left[\tilde P_{gg}(z)\,\frac{1}{z^2}L\!\left(\frac{X_A}{z},X_B\right)-\theta(z>z_{\min})\,\frac{P_{gg}^{\mathrm{soft}}}{z(1-z)}\,L(X_A,X_B)\right]+O(\epsilon). \quad (4.63)$$

The first term is saved for cancellation against other divergent pieces, while the remaining two terms give finite results that can be computed numerically.

If one wishes to distinguish between the renormalization scale μ_{UV} and the factorization scale μ_{coll}, then the μ appearing in Eq. (4.63) is μ_{coll}.

F. The soft contribution

In this section we wish to calculate $G_{A,\mathrm{soft}}^{(2\to3)}$, which is given by Eqs. (4.24) and (4.9). Making use of the fact that the function H_A as given in Eq. (4.10) is simple in the soft limit, we have

$$G_{A,\mathrm{soft}}^{(2\to3)} = (4\pi^2)^\epsilon\,\frac{C(\epsilon)}{2\pi}L(X_A,X_B)\,\frac{\alpha_s}{2\pi}\,2\mathcal{J}_A(\epsilon)\,\mu^{2\epsilon}$$
$$\times\int\frac{d^{2-2\epsilon}\mathbf{W}}{W^2}\,[f_A(y_1;p_J,y_J,\phi_J;0,\mathbf{W})-\theta(W^2<Q_A^2)\,f_A(y_1;p_J,y_J,\phi_J;0,\mathbf{0})]. \quad (4.64)$$

According to Eqs. (1.53) and (1.54), the function f_A has the structure

$$f_A = \frac{f_{AB}}{d_{AB}}+\frac{f_{A1}}{d_{A1}}+\frac{f_{A2}}{d_{A2}}, \quad (4.65)$$

where the f_{nm} are defined in Eq. (1.50) and

$$d_{Am} = \frac{x_A}{\xi}(p_A \cdot p_3 + p_m \cdot p_3) \,. \tag{4.66}$$

In (4.66), we use (4.13) to evaluate the dot products and divide by ξ, then take the limit $\xi \to 0$ with constant (W, ϕ_3). This gives

$$d_{AB} = p_J^2 D_4(W, y_1, y_J) \,, \quad d_{A1} = p_J^2 D_3(W, \phi_3 - \phi_J + \pi, y_1, y_J) \,, \quad d_{A2} = p_J^2 D_3(W, \phi_3 - \phi_J, y_J, y_1) \,, \tag{4.67}$$

where

$$\begin{aligned} D_3(W, \phi, y_1, y_J) &= (e^{y_1} + e^{y_J}) \left(\frac{W^2}{s}(e^{y_1} + \tfrac{1}{2}e^{y_J}) + \tfrac{1}{2}e^{-y_1} - \frac{W}{\sqrt{s}} \cos\phi \right) \,, \\ D_4(W, y_1, y_J) &= \tfrac{1}{2}(2 + e^{y_1 - y_J} + e^{y_J - y_1}) \left(\frac{W^2}{s} e^{y_1 + y_J} + 1 \right) \,. \end{aligned} \tag{4.68}$$

The functions f_{nm} with $p_3^\mu = 0$ are given by Eq. (1.67):

$$\begin{aligned} f_{AB} &= 4VN^3(1-\epsilon)^2 \hat{S}\; f_S(\hat{S}, \hat{T}, \hat{U}), \\ f_{A1} &= 4VN^3(1-\epsilon)^2 (-\hat{T})\; f_S(\hat{T}, \hat{U}, \hat{S}), \\ f_{A2} &= 4VN^3(1-\epsilon)^2 (-\hat{U})\; f_S(\hat{U}, \hat{S}, \hat{T}) \,. \end{aligned} \tag{4.69}$$

Finally, we make use of Eq. (4.25) for $\mathcal{J}_A(\epsilon)$ and show explicitly the factor $V_{-2\epsilon}$, given in Eq. (2.33), that appears in the relation

$$\mu^{2\epsilon} d^{2-2\epsilon}\mathbf{W} = V_{-2\epsilon} \left(\frac{W}{\mu} \right)^{-2\epsilon} W\, dW d\phi [\sin(\phi)]^{-2\epsilon} \,. \tag{4.70}$$

Assembling this, we obtain

$$\begin{aligned} G_{A,\text{soft}}^{(2\to3)} = &-\frac{1}{\epsilon}\, C(\epsilon) L(X_A, X_B) \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q_{\text{ES}}^2} \right)^{\epsilon} \Gamma_K(\epsilon) \left(\frac{Q_{\text{ES}}^2}{4\mu^2 \Xi_A^2} \right)^{\epsilon} \frac{(1-\epsilon)^2}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)} \frac{4VN^3}{\pi p_J^2} \\ &\times [\hat{S}\; f_S(\hat{S}, \hat{T}, \hat{U})\; V_4(y_1, y_J, Q_A) - \hat{T}\; f_S(\hat{T}, \hat{U}, \hat{S})\; V_3(y_1, y_J, Q_A) - \hat{U}\; f_S(\hat{U}, \hat{S}, \hat{T})\; V_3(y_J, y_1, Q_A)] \,. \end{aligned} \tag{4.71}$$

Here $\Gamma_K(\epsilon)$ is given in Eq. (1.29), and we note that the factor $1/[\Gamma(1+\epsilon)\Gamma(1-\epsilon)]$ equals $1 + O(\epsilon^2)$. The functions $V_3(y_1, y_J, Q_A)$ and $V_4(y_1, y_J, Q_A)$ are the integrals of $1/D_3$ and $1/D_4$. These integrals are defined more precisely and are evaluated in Appendix B. We use the results of Appendix B together with Eq. (1.18) relating (p_J, y_J, y_1) to $(\hat{S}, \hat{T}, \hat{U})$ and the relations $p_J^2 = \hat{T}\hat{U}/\hat{S}$ and $s = \hat{S}/(X_A X_B)$, which follow from Eqs. (1.16) and (1.18), to express the integrals V in the form

$$\begin{aligned} \frac{\hat{S}}{\pi p_J^2}\, V_4(y_1, y_J, Q_A) &= \ln\left(\frac{\hat{S}}{Q_{\text{ES}}^2} \right) - \ln\left(\frac{X_A X_B Q_A^2}{Q_{\text{ES}}^2} \right) - y_1 - y_J + \epsilon P_{4N}(y_1, y_J, Q_A, \mu, s) + O(\epsilon^2) \,, \\ -\frac{\hat{T}}{\pi p_J^2}\, V_3(y_1, y_J, Q_A) &= \ln\left(\frac{-\hat{T}}{Q_{\text{ES}}^2} \right) - \ln\left(\frac{X_A X_B Q_A^2}{Q_{\text{ES}}^2} \right) - y_1 - y_J + \epsilon P_{3N}(y_1, y_J, Q_A, \mu, s) + O(\epsilon^2) \,, \\ -\frac{\hat{U}}{\pi p_J^2}\, V_3(y_J, y_1, Q_A) &= \ln\left(\frac{-\hat{U}}{Q_{\text{ES}}^2} \right) - \ln\left(\frac{X_A X_B Q_A^2}{Q_{\text{ES}}^2} \right) - y_1 - y_J + \epsilon P_{3N}(y_J, y_1, Q_A, \mu, s) + O(\epsilon^2) \,. \end{aligned} \tag{4.72}$$

The functions $P_{3N}(y_J, y_1, Q, s)$ and $P_{4N}(y_1, y_J, Q, \mu, s)$ are rather complicated, and are given in Eq. (B8) of Appendix B.

Using these results we obtain, for $G_{A,\text{soft}}^{(2\to3)}$,

$$G^{(2\to3)}_{A,\rm soft} = -\frac{1}{\epsilon}\, C(\epsilon)\, L(X_A,X_B)\, \frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2_{\rm ES}}\right)^{\epsilon} \Gamma_K(\epsilon)\left(\frac{Q^2_{\rm ES}}{4\mu^2\Xi^2_A}\right)^{\epsilon} 4VN^3$$
$$\times\Bigg\{-(1-\epsilon)^2\,[f_S(\hat S,\hat T,\hat U)+f_S(\hat T,\hat U,\hat S)+f_S(\hat U,\hat S,\hat T)]\left[\ln\left(\frac{X_AX_BQ_A^2}{Q^2_{\rm ES}}\right)+y_1+y_J\right]$$
$$+f_S(\hat S,\hat T,\hat U)\left[(1-2\epsilon)\ln\left(\frac{\hat S}{Q^2_{\rm ES}}\right)+\epsilon P_{4N}(y_1,y_J,Q_A,\mu,s)\right]$$
$$+f_S(\hat T,\hat U,\hat S)\left[(1-2\epsilon)\ln\left(\frac{-\hat T}{Q^2_{\rm ES}}\right)+\epsilon P_{3N}(y_1,y_J,Q_A,\mu,s)\right]$$
$$+f_S(\hat U,\hat S,\hat T)\left[(1-2\epsilon)\ln\left(\frac{-\hat U}{Q^2_{\rm ES}}\right)+\epsilon P_{3N}(y_J,y_1,Q_A,\mu,s)\right]\Bigg\}+O(\epsilon). \quad (4.73)$$

We use the identity (1.33),

$$4VN^2(1-\epsilon)^2[f_S(\hat s,\hat t,\hat u)+f_S(\hat t,\hat u,\hat s)+f_S(\hat u,\hat s,\hat t)] = d^{(4)}(\hat s,\hat t,\hat u;\epsilon), \quad (4.74)$$

and expand in powers of ϵ to obtain the final result:

$$G^{(2\to3)}_{A,\rm soft} = -\, C(\epsilon)\, L(X_A,X_B)\, \frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2_{\rm ES}}\right)^{\epsilon} \Gamma_K(\epsilon)$$
$$\times\Bigg(-\frac{N}{\epsilon}d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left[\ln\left(\frac{X_AX_BQ_A^2}{Q^2_{\rm ES}}\right)+y_1+y_J\right]$$
$$+\frac{4VN^3}{\epsilon}\left[f_S(\hat S,\hat T,\hat U)\ln\left(\frac{\hat S}{Q^2_{\rm ES}}\right)+f_S(\hat T,\hat U,\hat S)\ln\left(\frac{-\hat T}{Q^2_{\rm ES}}\right)+f_S(\hat U,\hat S,\hat T)\ln\left(\frac{-\hat U}{Q^2_{\rm ES}}\right)\right]$$
$$+4VN^3\, f_S(\hat S,\hat T,\hat U)\left\{\left[\ln\left(\frac{Q^2_{\rm ES}}{4\mu^2\Xi^2_A}\right)-2\right]\ln\left(\frac{\hat S}{Q^2_{\rm ES}}\right)+P_{4N}(y_1,y_J,Q_A,\mu,s)\right\}$$
$$+4VN^3\, f_S(\hat T,\hat U,\hat S)\left\{\left[\ln\left(\frac{Q^2_{\rm ES}}{4\mu^2\Xi^2_A}\right)-2\right]\ln\left(\frac{-\hat T}{Q^2_{\rm ES}}\right)+P_{3N}(y_1,y_J,Q_A,\mu,s)\right\}$$
$$+4VN^3\, f_S(\hat U,\hat S,\hat T)\left\{\left[\ln\left(\frac{Q^2_{\rm ES}}{4\mu^2\Xi^2_A}\right)-2\right]\ln\left(\frac{-\hat U}{Q^2_{\rm ES}}\right)+P_{3N}(y_J,y_1,Q_A,\mu,s)\right\}$$
$$-Nd^{(4)}(\hat S,\hat T,\hat U;0)\ln\left(\frac{Q^2_{\rm ES}}{4\mu^2\Xi^2_A}\right)\left[\ln\left(\frac{X_AX_BQ_A^2}{Q^2_{\rm ES}}\right)+y_1+y_J\right]+O(\epsilon)\Bigg). \quad (4.75)$$

The reader may have noticed by comparing this result with the starting expression (4.64) that the quantity in the large bold parentheses in (4.75) should be independent of the scale parameter μ. Using the results of Appendix B for P_{3N} and P_{4N}, one finds after a bit of algebra that this is indeed so.

V. CANCELLATION OF DIVERGENCES

Let us check the cancellation of $1/\epsilon$ and $1/\epsilon^2$ terms. We denote

$$Y(\epsilon) = C(\epsilon)L(X_A,X_B)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2_{\rm ES}}\right)^{\epsilon}\Gamma_K(\epsilon). \quad (5.1)$$

We collect the terms that are divergent as $\epsilon\to 0$. The first is from Eq. (1.25) and (1.31):

$$[G^{(2\to2)}(y_1,p_J,y_J,\phi_J)]_{\rm divergent} = Y(\epsilon)\Bigg\{ d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left(-\frac{4N}{\epsilon^2}-\frac{22N}{3\epsilon}\right)$$
$$+\frac{16VN^3}{\epsilon}\Bigg[\ln\left(\frac{\hat S}{Q^2_{\rm ES}}\right) f_S(\hat S,\hat T,\hat U)+\ln\left(\frac{-\hat T}{Q^2_{\rm ES}}\right) f_S(\hat T,\hat U,\hat S)$$
$$+\ln\left(\frac{-\hat U}{Q^2_{\rm ES}}\right) f_S(\hat U,\hat S,\hat T)\Bigg]\Bigg\}. \quad (5.2)$$

The next contribution is from Eqs. (2.45), (2.57) and (2.68):

$$[G^{(2\to3)}_{2,\mathrm{double}} + G^{(2\to3)}_{2,\mathrm{coll}} + G^{(2\to3)}_{2,\mathrm{soft}}]_{\mathrm{divergent}}$$
$$= Y(\epsilon)\Bigg(d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left\{\frac{N}{\epsilon^2}+\frac{N}{\epsilon}\left[\ln\left(\frac{Q_{\mathrm{ES}}^2}{16Q_2^2}\right)+\frac{11}{6}+\ln\left(\frac{Q_2^2}{p_J^2}\right)-\ln\left(\frac{Q_{\mathrm{ES}}^2}{16p_J^2}\right)\right]\right\}$$
$$-\frac{4VN^3}{\epsilon}\left[f_S(\hat U,\hat S,\hat T)\ \ln\left(\frac{-\hat U}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat T,\hat U,\hat S)\ \ln\left(\frac{-\hat T}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat S,\hat T,\hat U)\ \ln\left(\frac{\hat S}{Q_{\mathrm{ES}}^2}\right)\right]\Bigg). \tag{5.3}$$

The next contribution is from Eqs. (3.35), (3.44), and (3.45):

$$[G^{(2\to3)}_{1,\mathrm{double}} + G^{(2\to3)}_{1,\mathrm{coll}} + G^{(2\to3)}_{1,\mathrm{soft}}]_{\mathrm{divergent}}$$
$$= Y(\epsilon)\Bigg(d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left\{\frac{N}{\epsilon^2}+\frac{N}{\epsilon}\left[\ln\left(\frac{Q_{\mathrm{ES}}^2}{16Q_1^2}\right)+\frac{11}{6}+\ln\left(\frac{Q_1^2}{p_J^2}\right)-\ln\left(\frac{Q_{\mathrm{ES}}^2}{16p_J^2}\right)\right]\right\}$$
$$-\frac{4VN^3}{\epsilon}\left[f_S(\hat U,\hat S,\hat T)\ \ln\left(\frac{-\hat U}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat T,\hat U,\hat S)\ \ln\left(\frac{-\hat T}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat S,\hat T,\hat U)\ \ln\left(\frac{\hat S}{Q_{\mathrm{ES}}^2}\right)\right]\Bigg). \tag{5.4}$$

The next contribution is from Eqs. (4.36), (4.63), and (4.75):

$$[G^{(2\to3)}_{A,\mathrm{double}} + G^{(2\to3)}_{A,\mathrm{coll}} + G^{(2\to3)}_{A,\mathrm{CT}} + G^{(2\to3)}_{A,\mathrm{soft}}]_{\mathrm{divergent}}$$
$$= Y(\epsilon)\Bigg(d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left\{\frac{N}{\epsilon^2}+\frac{N}{\epsilon}\left[\ln\left(\frac{Q_{\mathrm{ES}}^2}{\Xi_A^2Q_A^2}\right)+2\ln\left(\frac{\Xi_A}{X_A}\right)+\frac{11}{6}+\ln\left(\frac{X_AX_BQ_A^2}{Q_{\mathrm{ES}}^2}\right)+y_1+y_J\right]\right\}$$
$$-\frac{4VN^3}{\epsilon}\left[f_S(\hat S,\hat T,\hat U)\ \ln\left(\frac{\hat S}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat T,\hat U,\hat S)\ \ln\left(\frac{-\hat T}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat U,\hat S,\hat T)\ \ln\left(\frac{-\hat U}{Q_{\mathrm{ES}}^2}\right)\right]\Bigg). \tag{5.5}$$

The final contribution comes from term B and is the same as the term A contribution except for the substitutions $A\leftrightarrow B$, $\hat T\leftrightarrow\hat U$, $y_1\to-y_1$, and $y_J\to-y_J$:

$$[G^{(2\to3)}_{B,\mathrm{double}} + G^{(2\to3)}_{B,\mathrm{coll}} + G^{(2\to3)}_{B,\mathrm{CT}} + G^{(2\to3)}_{B,\mathrm{soft}}]_{\mathrm{divergent}}$$
$$= Y(\epsilon)\Bigg(d^{(4)}(\hat S,\hat T,\hat U;\epsilon)\left\{\frac{N}{\epsilon^2}+\frac{N}{\epsilon}\left[\ln\left(\frac{Q_{\mathrm{ES}}^2}{\Xi_B^2Q_B^2}\right)+2\ln\left(\frac{\Xi_B}{X_B}\right)+\frac{11}{6}+\ln\left(\frac{X_AX_BQ_B^2}{Q_{\mathrm{ES}}^2}\right)-y_1-y_J\right]\right\}$$
$$-\frac{4VN^3}{\epsilon}\left[f_S(\hat S,\hat T,\hat U)\ \ln\left(\frac{\hat S}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat U,\hat S,\hat T)\ln\left(\frac{-\hat U}{Q_{\mathrm{ES}}^2}\right)+f_S(\hat T,\hat U,\hat S)\ \ln\left(\frac{-\hat T}{Q_{\mathrm{ES}}^2}\right)\right]\Bigg). \tag{5.6}$$

The reader can now easily check that the sum of the divergent contributions is exactly zero.

VI. CONCLUSIONS

We have seen that the divergent contributions cancel. It remains to add up the finite contributions, now taking $\epsilon = 0$. We have

$$\frac{d\sigma}{dp_J\,dy_J\,d\phi_J} = \sum_i \int dy_1\ G^{(i)}(y_1,p_J,y_J,\phi_J). \tag{6.1}$$

Since we know that the divergent terms in the $G^{(i)}$ cancel, they can simply be dropped. Formulas for the various contributions $G^{(i)}$ are given in the equations

$G^{(2\to2)}$	(1.25)				
$G^{(2\to3)}_{2,\mathrm{NS}}$	(2.29)	$G^{(2\to3)}_{2,\mathrm{finite}}$	(2.37)		
$G^{(2\to3)}_{2,\mathrm{coll}}$	(2.57)	$G^{(2\to3)}_{2,\mathrm{soft}}$	(2.68)	$G^{(2\to3)}_{2,\mathrm{double}}$	(2.45)
$G^{(2\to3)}_{1,\mathrm{NS}}$	(3.12)	$G^{(2\to3)}_{1,\mathrm{finite}}$	(3.27)		
$G^{(2\to3)}_{1,\mathrm{coll}}$	(3.44)	$G^{(2\to3)}_{1,\mathrm{soft}}$	(3.45)	$G^{(2\to3)}_{1,\mathrm{double}}$	(3.35)
$G^{(2\to3)}_{A,\mathrm{NS}}$	(4.20)	$G^{(2\to3)}_{A,\mathrm{finite}}$	(4.28)		
$G^{(2\to3)}_{A,\mathrm{coll}} + G^{(2\to3)}_{A,\mathrm{CT}}$	(4.63)	$G^{(2\to3)}_{A,\mathrm{soft}}$	(4.75)	$G^{(2\to3)}_{A,\mathrm{double}}$	(4.36)

$G^{(2\to3)}_B$ as for $G^{(2\to3)}_B$ with $A\leftrightarrow B, \hat T\leftrightarrow\hat U, y_1\to-y_1, y_J\to-y_J$

These formulas give the $\int dy_1 G^{(i)}$, less the divergent pieces, as integrals over one, two, or four variables. These integrations are all finite, so they can be done numerically. Thus the numerical results are produced by a FORTRAN computer program.

We have checked the calculation and the computer coding for the numerical integrations by checking that the computed result is independent of the variables $Q_{\rm ES}$, Q_1, Q_2, Q_A, Q_B, Ξ_A, and Ξ_B. We have also checked the coding by writing two independent programs and verifying that they produce the same results.

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APPENDIX A: DEFINITION OF THE JET AXIS IN N DIMENSIONS

In Sec. II C we defined the azimuthal components ϕ_J of the jet axis and transformed integration variables to ϕ_J and $\bar{\phi}$ using a vector notation that is really only suitable if the number of transverse dimensions, $n = 2 - 2\epsilon$, is 2. In the case of arbitrary n, the angular variables lie on $(n-1)$-dimensional spheres, which have an intrinsic curvature and thus require further definition. Here we give the required generalization to arbitrary n.

We begin with the azimuthal angles of two gluons, which we denote here by a and b. These variables ϕ_a, ϕ_b represent points on a sphere $\mathcal{S}(n-1)$ having $n-1$ dimensions. Let Θ denote the (one-dimensional) angle between ϕ_a and ϕ_b, and let $x = p_b/(p_a + p_b)$, so $1 - x = p_a/(p_a + p_b)$, as illustrated in Fig. 3.

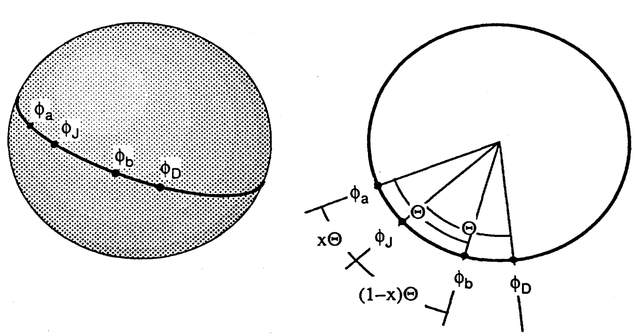

FIG. 3. Definition of the jet axis and related angles.

We then define the azimuthal angles of the jet axis, $\phi_J \in \mathcal{S}(n-1)$, to lie in the plane defined by ϕ_a and ϕ_b a fraction x of the way (as measured in arc length) from ϕ_a to ϕ_b. We define another point $\phi_D \in \mathcal{S}(n-1)$ to also lie in this plane, an angle Θ from the jet axis. (Thus ϕ_D is essentially $\bar{\phi}$, but measured from the jet axis instead of the north pole of the original sphere.)

Evidently, there is a one-to-one transformation between the variables ϕ_a, ϕ_b and the variables ϕ_J, ϕ_D. Thus we can write

$$\int d^{n-1}\phi_a \, d^{n-1}\phi_b \, f(\phi_a, \phi_b) = \int d^{n-1}\phi_J \, d^{n-1}\phi_D \, \rho(\phi_a, \phi_b) \, f(\phi_a, \phi_b), \quad \text{(A1)}$$

where $d^{n-1}\phi$ stands for the usual rotationally invariant integration measure on a sphere. We would like to know the Jacobian $\rho(\phi_a, \phi_b)$. This is easy because we can write

$$\int d^{n-1}\phi_a \, d^{n-1}\phi_b \, f(\phi_a, \phi_b) = \int_0^{2\pi} d\theta_a \int_0^{2\pi} d\theta_b \, \sin(\Theta)^{n-2} \int d^{2n-4}P \, f(\phi_a, \phi_b). \quad \text{(A2)}$$

Here θ_a, θ_b are the one-dimensional angles of ϕ_a and ϕ_b in the plane determined by ϕ_a and ϕ_b. The angles θ_a, θ_b are measured from an arbitrary fixed meridian. The remaining integration, $\int d^{2n-4}P$, denotes integration over this plane. [The factor $\sin(\Theta)^{n-2}$ will drop out of our final result.] We can also write

$$\int d^{n-1}\phi_J \, d^{n-1}\phi_D \, f(\phi_a, \phi_b) = \int_0^{2\pi} d\theta_J \int_0^{2\pi} d\theta_D \sin(\Theta)^{n-2} \int d^{2n-4}dP \, f(\phi_a, \phi_b). \quad \text{(A3)}$$

Finally, it is easy to check from the explicit transformation that

$$\int_0^{2\pi} d\theta_a \int_0^{2\pi} d\theta_b \cdots = \int_0^{2\pi} d\theta_J \int_0^{2\pi} d\theta_D \cdots . \quad \text{(A4)}$$

Comparing Eqs. (A.2)–(A.4), we see that the Jacobian in Eq. (A.1) is

$$\rho(\phi_a, \phi_b) = 1. \quad \text{(A5)}$$

APPENDIX B: THE SOFT INTEGRALS

Only four soft integrals have to be calculated:

$$\begin{aligned}
V_1(y_1-y_J) &= \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\left[\int_{-\infty}^{+\infty}\frac{dy}{\cosh(y)-\cos(\phi)}\left(\frac{1}{D_1(y,\phi,y_1-y_J)}-\frac{1}{D_1(0,0,y_1-y_J)}\right)\right],\\
V_2(y_1-y_J) &= \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\left[\int_{-\infty}^{+\infty}\frac{dy}{\cosh(y)-\cos(\phi)}\left(\frac{1}{D_2(y,y_1-y_J)}-\frac{1}{D_2(0,y_1-y_J)}\right)\right],\\
V_3(y_1,y_J,Q) &= \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\left[\int_0^\infty\frac{dW}{W}\left(\frac{W}{\mu}\right)^{-2\epsilon}\left(\frac{1}{D_3(W,\phi,y_1,y_J)}-\frac{\theta(W<Q)}{D_3(0,0,y_1,y_J)}\right)\right],\\
V_4(y_1,y_J,Q) &= \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\left[\int_0^\infty\frac{dW}{W}\left(\frac{W}{\mu}\right)^{-2\epsilon}\left(\frac{1}{D_4(W,y_1,y_J)}-\frac{\theta(W<Q)}{D_4(0,y_1,y_J)}\right)\right],
\end{aligned}\tag{B1}$$

where

$$\begin{aligned}
&D_1(y,\phi,y_1-y_J)=\tfrac{1}{2}(2+e^{y_1-y_J})e^{-y}+\tfrac{1}{2}e^{y}-\cos\phi,\quad D_2(y,y_1-y_J)=\tfrac{1}{2}(1+e^{y_J-y_1})(e^{y}+e^{y_1-y_J-y}),\\
&D_3(W,\phi,y_1,y_J)=(e^{y_1}+e^{y_J})\left(\frac{W^2}{s}(e^{y_1}+\tfrac{1}{2}e^{y_J})+\tfrac{1}{2}e^{-y_1}-\frac{W}{\sqrt{s}}\cos\phi\right)\\
&D_4(W,y_1,y_J)=\tfrac{1}{2}(2+e^{y_1-y_J}+e^{y_J-y_1})\left(\frac{W^2}{s}e^{y_1+y_J}+1\right).
\end{aligned}\tag{B2}$$

These integrals are finite as $\epsilon\to 0$, but since they multiply a factor $1/\epsilon$ we must evalute them to first order in ϵ.

First we carry out the integrals over y and W. These are the integrals which are enclosed in the square brackets in Eq. (B1). We denote them as $\tilde{V}_i$. We obtain

$$\begin{aligned}
\tilde{V}_1(\phi,y_1-y_J) &= \frac{4}{1+a^2}\left(\tfrac{1}{2}\ln(2+a^2)-\frac{1-\cos\phi}{\sin\phi}(\pi-\phi)-\frac{\cos\alpha}{\sin\alpha}(\pi-\alpha)\right),\\
\tilde{V}_2(\phi,y_1-y_J) &= \frac{4a^2}{(1+a^2)^2}\frac{1}{1+a^4+2a^2\cos(2\phi)}\\
&\quad\times\left[-(\pi-\phi)\tan\left(\frac{\phi}{2}\right)[(1-a^2)^2-4a^2\cos(\phi)]+(a^4-1)\ln(a)-a(1+a^2)\pi\cos(\phi)\right],\\
\tilde{V}_3(\phi,y_1,y_J,Q) &= \frac{2a^2}{1+a^2}(\kappa_1\mu)^{2\epsilon}\left[\frac{\cos\beta}{\sin\beta}(\pi-\beta)-\ln(\kappa_1 Q)+\epsilon\left((\pi-\beta)^2-\frac{\pi^2}{3}+\ln^2(\kappa_1 Q)\right)\right],\\
\tilde{V}_4(y_1,y_J,Q) &= \frac{2a^2}{(1+a^2)^2}(\kappa_2\mu)^{2\epsilon}\left[-\ln(\kappa_2 Q)+\epsilon\left(-\frac{\pi^2}{12}+\ln^2(\kappa_2 Q)\right)\right],
\end{aligned}\tag{B3}$$

where $a^2,\alpha,\beta,\kappa_1,\kappa_2$ are defined as

$$\begin{aligned}
&a^2=e^{y_1-y_J},\cos\,\alpha=(2+e^{y_1-y_J})^{-1/2}\cos\,\phi,\quad \cos\,\beta=(2+e^{y_J-y_1})^{-1/2}\cos\,\phi,\\
&\kappa_1^2=(2+e^{y_J-y_1})\,e^{2y_1}/s,\quad \kappa_2^2=e^{y_J+y_1}/s.
\end{aligned}\tag{B4}$$

In the $\epsilon=0$ limit, the integration over ϕ can also be performed analytically. Consequently, analytic expressions can be obtained for the leading contributions as $\epsilon\to 0$. In the case of the next to leading part, two one-dimensional integrations over ϕ must be performed numerically.

It is convenient to give the results in the form

$$\begin{aligned}
V_1(y_1-y_J) &= \frac{2\pi}{1+e^{y_1-y_J}}[P_{1S}(y_1-y_J)+\epsilon P_{1N}(y_1-y_J)]+O(\epsilon^2),\\
V_2(y_1-y_J) &= \frac{\pi}{1+\cosh(y_1-y_J)}[P_{2S}(y_1-y_J)+\epsilon P_{2N}(y_1-y_J)]+O(\epsilon^2),\\
V_3(y_1,y_J,Q) &= \frac{\pi}{1+e^{y_J-y_1}}[P_{3S}(y_1,y_J,Q,s)+\epsilon P_{3N}(y_1,y_J,Q,\mu,s)]+O(\epsilon^2),\\
V_4(y_1,y_J,Q) &= \frac{\pi}{2[1+\cosh(y_1-y_J)]}[P_{4S}(y_1,y_J,Q,s)+\epsilon P_{4N}(y_1,y_J,Q,\mu,s)]+O(\epsilon^2).
\end{aligned}\tag{B5}$$

We find that the leading contributions have the simple expressions

$$P_{1S}(y_1-y_J)=\ln\left(\frac{1+e^{y_1-y_J}}{16}\right),\quad P_{2S}(y_1-y_J)=\ln\left(\frac{1+\cosh(y_1-y_J)}{8}\right),$$
$$P_{3S}(y_1,y_J,Q,s)=-\ln(1+e^{y_1-y_J})-y_1-y_J-\ln\left(\frac{Q^2}{s}\right),\quad P_{4S}(y_1,y_J,Q,s)=-y_1-y_J-\ln\left(\frac{Q^2}{s}\right). \tag{B6}$$

We also find that the nonleading contributions $P_{1N}(y_1-y_J)$ and $P_{2N}(y_1-y_J)$ are given by

$$P_{1N}(y_1-y_J)=\quad -\frac{\pi^2}{3}+2\ \ln 2\ \ln(2+e^{y_1-y_J})+\frac{4}{\pi}\int_0^\pi d\phi\ln(\sin\phi)\ \frac{\cos\alpha}{\sin\alpha}\ (\pi-\alpha),$$
$$P_{2N}(y_1-y_J)=-\frac{\pi^2}{3}+2\ \ln 2\ln\{2[1+\cosh(y_1-y_J)]\}-2\ln(1+e^{y_1-y_J})\ \ln(1+e^{y_J-y_1}), \tag{B7}$$

where cos α is defined as in Eq. (B4). The integral that remains can be evaluated numerically.

For the nonleading integrals $P_{3N}(y_1,y_J,Q,\mu,s)$ and $P_{4N}(y_1,y_J,Q,\mu,s)$ we obtain

$$\begin{aligned}P_{3N}(y_1,y_J,Q,\mu,s)=&-\frac{2\pi^2}{3}+\frac{1}{2}\ln\left(\frac{Q^2}{s}e^{y_1}(2e^{y_1}+e^{y_J})\right)\ln\left(\frac{Q^2}{16s}e^{y_1}(2e^{y_1}+e^{y_J})\right)\\&-\ln\left(\frac{\mu^2}{s}e^{y_1}(2e^{y_1}+e^{y_J})\right)\ln\left(\frac{Q^2}{s}e^{y_1}(e^{y_1}+e^{y_J})\right)\\&+\frac{2}{\pi}\int_0^\pi d\phi\left(-2\ \ln(\sin\phi)\ \frac{\cos\beta}{\sin\beta}(\pi-\beta)+(\pi-\beta)^2\right),\\P_{4N}(y_1,y_J,Q,\mu,s)=&-\frac{\pi^2}{6}-\frac{1}{2}\ln\left(\frac{Q^2}{s}e^{y_1+y_J}\right)\ln\left(\frac{16\mu^4}{sQ^2}e^{y_1+y_J}\right),\end{aligned} \tag{B8}$$

where cos β is defined as in Eq. (B4).

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