PHYSICAL REVIEW D VOLUME 40, NUMBER 6 15 SEPTEMBER 1989

Shell-focusing singularities in spherically symmetric self-similar spacetimes

B. Waugh and Kayll Lake
Department of Physics, Queen's University at Kingston, Kingston, Ontario, Canada K7L 3N6
(Received 2 May 1989)

We examine spherically symmetric self-similar spacetimes in comoving coordinates, subject to a monotonically increasing comoving time to the future, and find the necessary conditions for the formation of naked strong curvature shell-focusing singularities. The strength is shown to be persistent (not instantaneous). On the basis of eikonal perturbations along the associated Cauchy horizons we find no reason to suspect that the singularities are unstable.

Recently,[1,2] it has been shown that the naked shell-focusing singularity,[3,4] which arises in the collapse of marginally bound self-similar Tolman spacetimes, is strong in the sense of Tipler, Clarke, and Ellis.[5] Prior to this Ori and Piran[6] showed that a self-similar spherical collapse of an adiabatic perfect fluid with a soft enough equation of state can give rise to a naked shell-focusing singularity. This work extended beyond dust known fluid collapse histories which have nakedly singular end states which are not instantaneous. Lake[7] showed that the associated singularity is strong.

Considering the formation of singularities only to the future of regular initial conditions, the examples of strong naked shell-focusing singularities presently known consist of those mentioned plus the linear-mass Vaidya solution.[8] The fact that *all* the known examples are in spherical self-similar spacetimes suggests that a general study of the strength, persistence, and stability of these singularities would be of use. In the present paper we undertake such a study within the limitations of comoving coordinates, subject to a monotonically increasing comoving time to the future, and find the criteria necessary for the formation of a persistent strong shell-focusing singularity. We also examine the stability of the singularities by way of massless field perturbations, within the eikonal approximation, along the associated Cauchy horizons. This analysis suggests that the singularities are stable.

The self-similar spherically symmetric metric (in standard geometrical units) in comoving coordinates[9] is given by

$$ds^2=-e^{\Psi}dT^2+e^{\Lambda}dR^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \qquad (1)$$

where Ψ, Λ, and $\tilde{r}\equiv r/T$ are functions of the self-similarity variable $y\equiv R/T$. We take the future orientation $dT>0$ along timelike trajectories.

A shell-focusing singularity in these coordinates is the node $R=T=0$ associated with the radial null geodesic equations. (The critical direction of the trajectories originating at the node is the future Cauchy horizon. This is the first outgoing null geodesic extending from $R=T=0$.) Such a node is a scalar polynomial singularity[5] when the Kretschmann scalar diverges. For metric (1) this scalar is given by

$$\begin{aligned}K&=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\\&=(4m/r^3)^2+2(2m/r^3+G^R_R)^2+2(2m/r^3+G^T_T)^2\\&\quad+(4m/r^3-2G^\theta_\theta+G^T_T+G^R_R)^2-4e^{\Psi-\Lambda}(G^T_R)^2\ ,\end{aligned} \qquad (2)$$

where G^α_β is the Einstein tensor and m is the "mass" $[\equiv\frac{1}{2}(g_{\theta\theta})^{3/2}R_{\theta\phi}{}^{\theta\phi}$ which is of the form $m=g(y)R$ here]. Given $r=\tilde{r}T$, the first term in K is $16g^2y^2/\tilde{r}^6T^4$. Thus, for vanishing heat flux ($G^T_R=0$), K diverges along homothetic Killing trajectories like $1/T^4$.

Consider a radial homothetic Killing trajectory (not necessarily null or geodesic) defined by $y=$const. The Lagrangian is given by

$$2L=e^{\Psi}\left[\frac{dT}{d\lambda}\right]^2\chi\ , \qquad (3)$$

where $\chi\equiv y^2e^{\Lambda-\Psi}-1$. Thus for $\chi=0$ the trajectory is null, and for χ positive (negative) the trajectory is spacelike (timelike). In the null case the trajectory can be shown to be geodesic. If $\chi=0$ has no roots then clearly $R=T=0$ is not naked.

Consider $y>0$. Assuming that a root to $\chi=0$ does exist (call the largest positive root $y_{\rm FC}$ and take $0<y_{\rm FC}<\infty$) then for $y=\text{const}>y_{\rm FC}$ the trajectories are spacelike. This implies that no null geodesic originating at the singularity will extend into the region where $y>y_{\rm FC}$ and $y=y_{\rm FC}$ is necessarily the first null geodesic to escape the singularity. We call $y_{\rm FC}$ the future Cauchy horizon of the singularity.

It is instructive to consider part of the spacetimes in double-null coordinates. The metric is given by

$$ds^2=-2f\,du\,dv+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \qquad (4)$$

where f and r are functions of u and v. Above the event horizon the functions and the coordinates are set by the requirement that $\xi^\alpha=(u,0,0,v)$ be a homothetic Killing vector ($\nabla_{(\alpha}\xi_{\beta)}=g_{\alpha\beta}$). Then f and $\alpha\equiv r/v$ are functions of the self-similarity variable $x\equiv u/v$. We take the future orientation $du,dv>0$ where $v=$const defines ingoing and $u=$const defines outgoing radial null geodesics. The shell-focusing singularity arises at ($v=0,u\geq 0$). The future Cauchy horizon lies along $u=0$ and the past along $v=0$ and both are homothetic. In comoving coordinates

 ©1989 The American Physical Society

they are given by $y=y_{\rm FC}>0$ for the future Cauchy horizon and $y=y_{\rm PC}<0$ for the past Cauchy horizon ($y_{\rm FC}$ and $y_{\rm PC}$ are constants).

Following the work of Clarke and Królak[10] we consider the null geodesics along the Cauchy horizons. The singularity is a strong curvature singularity (as defined by Tipler, Clarke, and Ellis[5]) if

$$\lim_{\lambda\to 0}\lambda^2 G_{\alpha\beta}k^\alpha k^\beta \neq 0 \ . \tag{5}$$

For metric (1) it follows for the Einstein tensor that

$$\lambda^2 G_{\alpha\beta}k^\alpha k^\beta = 2C\left[\frac{\lambda}{T}\frac{dT}{d\lambda}\right]^2 , \tag{6}$$

where C is a constant defined by

$$C=\left.\frac{y\left[\dfrac{d\Psi}{dy}-\dfrac{d\Lambda}{dy}\right]}{2}\right|_{y_c}=1-\left.\frac{y}{2}\frac{d\chi}{dy}\right|_{y_c} , \tag{7}$$

and k^α is tangent to the null geodesic $y_c=y_{\rm FC}$ or $y_{\rm PC}$. The null geodesic equations integrate explicitly along the future Cauchy horizon to give $T=\lambda^\delta$ where $\delta=1/(1+C)$. As a result

$$\lambda^2 G_{\alpha\beta}k^\alpha k^\beta = \frac{2C}{(1+C)^2} \ . \tag{8}$$

That is, the Cauchy horizons terminate at $R=T=0$ in a *strong* curvature singularity for $C\neq 0$. [For an interpretation of C we can turn to a perfect-fluid description of the background. From the symmetry the equation of state must be of the form $p=(\Gamma-1)\epsilon$ and $\epsilon=D(y)/T^2$. It follows that[7] $C=\Gamma(4\pi D)^{(2-\Gamma)/\Gamma}=4\pi D\Gamma e^{\Psi}$.]

The radial null geodesics to the immediate future of the future Cauchy horizon must terminate at $R=T=0$ in a strong curvature singularity to ensure that the singularity is not simply instantaneously strong. By the nature of the future Cauchy horizon it is either the only radial null geodesic originating at the node with $y=y_{\rm FC}$ (where more than one positive root exists for $\chi=0$), or it is the first of a family of geodesics with $y=y_{\rm FC}$ at the node (where $y_{\rm FC}$ is the only positive root for $\chi=0$). In the first instance there must be an infinite number of radial null geodesics originating at the node with $y=y_b$, y_b being the next largest root of $y^2e^{\Lambda-\Psi}=1$. Trajectories satisfying $y_b<y=\text{const}<y_{\rm FC}$ must be timelike. Thus (defining C along $y=y_b$ now) $d\chi/dy|_{y_b}\leq 0$ so $C\geq 1$ and the null geodesics necessarily terminate at $R=T=0$ in a *strong* curvature singularity.[11] In the second instance, trajectories satisfying $y=\text{const}<y_{\rm FC}$ must be spacelike, thus $d\chi/dy|_{y_{\rm FC}}=0$ and $C=1$ along $y=y_{\rm FC}$. (We consider this the degenerate case.) The future Cauchy horizon is necessarily strong in this case.[11] In either case it is clear that the trajectories approaching the node immediately to the future of the future Cauchy horizon necessarily terminate at $R=T=0$ in a *strong* curvature singularity.

We now wish to test the singularity for stability. This is not a straightforward task. Here we limit ourselves to the injection of massless fields along the Cauchy horizons in the high-frequency (eikonal) limit. The frequency shift of this radiation is given by

$$\frac{\nu_e}{\nu_o}=\frac{(u_\alpha k^\alpha)_e}{(u_\alpha k^\alpha)_o} , \tag{9}$$

where ν_e (ν_o) is the emitted (observed) frequency and u_α the four-velocity tangent to the observer or emitter. By considering radiation along the Cauchy horizons and by taking the observer and emitter to be comoving, we find

$$\frac{\nu_e}{\nu_o}=\left[\frac{T_o}{T_e}\right]^c=\left[\frac{R_o}{R_e}\right]^c . \tag{10}$$

Thus for the future Cauchy horizon stability is ensured as long as $C\geq 0$ which, by continuity [see Eq. (8)], we take to be the weak energy condition.[12] It is important to note that $d\chi/dy\geq 0$ at $y=y_{\rm FC}>0$ and that $d\chi/dy\leq 0$ at $y=y_{\rm PC}<0$ so that $C\leq 1$ in both cases. (For dust $C<\frac{2}{3}$.)

Notice that the test field injected along the past Cauchy horizon develops a blueshift. However, the energy density of the test field relative to that of the background matter evolves like $T^{2(1-C)}$ and so the perturbation actually dies away (or, if $C=1$, does not grow) as the singularity is approached.[13] On the basis of our analysis, therefore, there is no reason to suspect that the singularity is unstable.[14]

In summary, we have found that *all* spherically symmetric self-similar spacetimes, subject to a monotonically increasing comoving time to the future, and with real finite positive roots to $y^2e^{\Lambda-\Psi}=1$, admit globally naked strong shell-focusing singularities with Cauchy horizon $y=y_{\rm FC}$, where $y_{\rm FC}$ is the largest root. Further we have shown that radial null trajectories immediately "inside" the Cauchy horizon always terminate in a strong singularity, thus ensuring that the node is not simply instantaneously strong. Finally, on the basis of an eikonal perturbation, we find no reason to suspect that the singularities are unstable.

Note added in proof. Shell-focusing singularities in homothetically self-similar spacetimes can be viewed in terms of the fixed points of the homotheties [see G. S. Hall, Gen. Relativ. Gravit. **20**, 671 (1988)]. Given a proper homothetic Killing vector ξ^α and the associated integral curve L along which $\xi_\alpha\xi^\alpha=\nabla_\beta\xi_\alpha\xi^\alpha=0$, it follows that L is a null geodesic which terminates in a strong curvature singularity. All presently known examples of strong naked shell-focusing singularities are therefore simply a kinematical result of homotheticity [T. Zannias and K. Lake (in preparation)].

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

[1]B. Waugh and K. Lake, Phys. Rev. D **38**, 1315 (1988).

[2]A. Ori and T. Piran (private communication).

[3]D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979).

[4]D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984).

[5]F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. 2.

[6]A. Ori and T. Piran, Phys. Rev. Lett. **59**, 2137 (1987).

[7]K. Lake, Phys. Rev. Lett. **60**, 241 (1988).

[8]K. Rajagopal and K. Lake, Phys. Rev. D **35**, 1531 (1987).

[9]For example, M. E. Cahill and A. H. Taub, Commun. Math. Phys. **21**, 1 (1971). For a systematic analysis of self-similarity in general relativity, see D. M. Eardley, *ibid.* **37**, 287 (1974). For an application see, for example, G. V. Bicknell and R. N. Henriksen, Astrophys. J. **225**, 237 (1978).

[10]C. J. S. Clarke and A. Królak, J. Geom. Phys. **2**, 127 (1986).

[11]Equation (8) holds at the node along all radial null geodesics which reach it. To see this, in the first case, along the geodesics write $R = y_b T + F(T)$, with $F = dF/dT = 0$ at $T = 0$, where $(y_b + dF/dT)^2 e^{\Lambda - \Psi} = 1$. A straightforward calculation then reproduces Eq. (6) as the node is approached. To evaluate $[(\lambda/T)(dT/d\lambda)]^2$, we write out the radial null geodesic equation which, as the node is approached, gives $d^2T/d\lambda^2 + (C/T)(dT/d\lambda)^2 = 0$. From l'Hôpital's rule, at the node, $\lambda(d^2T/d\lambda^2)/(dT/d\lambda) + 1 = (\lambda/T)(dT/d\lambda)$, so that Eq. (8) follows. In the second case replace y_b by $y_{\rm FC}$ in the foregoing.

[12]For example, R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[13]We thank A. Ori and T. Piran for pointing this out to us.

[14]In the time-reversed "white-hole" case ($T \to -T$, outgoing null geodesics $\to$ ingoing) the opposite conclusion holds. That is, the energy density of the test field relative to the background still evolves like $T^{2(1-C)}$, but along $y_b < 0$. Unless $C = 1$, the ratio diverges as the node is approached ($T \to 0^-$).