Vacuum polarization in gravitational and electromagnetic fields around a superconducting string

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We have calculated the polarization current induced in the physical vacuum around a superconducting cosmic string taking into account the gravitational field of the string. The current can be calculated as an expansion in powers of the inverse of the electron mass. In the region far from the string, where it is justified to keep only the lowest term of this expansion, the polarization current turns out to screen the original current in the string, but the effect is very weak. A direct calculation of terms due to the presence of the gravitational field shows that they are dominated, for realistic string parameters, by the purely electromagnetic contribution.

The idea of cosmic-string creation developed when gauge theories with spontaneous symmetry breaking were taken into account in cosmological models. As first suggested by Kibble,[1] phase transitions due to symmetry breaking at early stages of the Universe may lead to production of vacuum domain structures. Cosmic strings seem to be the most interesting example of these structures. A detailed review of this matter, involving results up to 1985, has been presented by Vilenkin.[2] Several authors[3] have suggested that the interaction of strings with surrounding matter may play an important role in galaxy formation. New aspects of this interaction appeared when Witten[4] showed that, under certain assumptions, strings may carry currents up to 10^{20} A. The magnetic field created by such a large current would modify the influence of a string on matter in its neighborhood. It is, however, important that in the presence of such strong magnetic fields the vacuum-polarization effects may be relevant in determining the shape of the field and its actual strength. This problem has been analyzed by the authors of Refs. 5 and 6 with the conclusion that QED vacuum polarization leads to a screening of the current in the string. However, the calculated effect turns out to be very weak, at least in the framework of the applied approximations. In both papers the gravitational field of the string has been neglected, which is not necessarily a good approximation since the linear mass density of a cosmic string in many models is of order 10^{22} g/cm (this corresponds to the width of the string of order 10^{-28} cm). The purpose of this paper is to take into account the gravitational field of the string.

In order to consider the vacuum-polarization effects around a string we can approximate it by a straight wire of radius ρ_0 carrying a current of intensity I. The energy-momentum tensor of such a wire can be written in the form[7]

$$T_{\mu\nu}=\mathrm{diag}(\mu_F+\mu_M,0,0,\mu_F)\Theta_H(\rho_0-\rho) \ . \tag{1}$$

Here ρ denotes the distance from the string oriented along the z axis, and μ_F the string tension, which is of the same order as its linear energy density $\mu_F+\mu_M$. Θ_H denotes the Heaviside function. The particular value of the parameter μ_M depends on the internal structure of the string. In most of the papers on this subject the linear energy density of a string is taken to be equal to its tension, which follows from the simplest model of a string.[8] This implies $\mu_M=0$. In Ref. 9 the parameter μ_M is introduced explicitly, and is interpreted as the mass density of ordinary particles (protons, neutrons, etc.) trapped by a string. Its value is then many orders of magnitude lower than μ_F. However, following Witten's considerations,[10] one can expect an appearance of other, much heavier particles inside a string, which might make μ_M comparable to μ_F. The actual value of μ_M in such a case is difficult to determine and, moreover, is model dependent. In the following we shall assume only a reasonable upper bound for μ_M, i.e., $\mu_M \lesssim \mu_F$, leaving aside the question of whether μ_M can actually reach this bound.

Demiański[11] has generalized the cylindrically symmetric metric around a straight wire with current I (Witten[12]) for a cosmic string in the straight-line approximation. The energy-momentum tensor generating the gravitational field is in this case the sum of the tensor (1) and the energy-momentum tensor of the electromagnetic field generated by the string. In the region $\rho>\rho_0$ the metric takes the form

$$ds^2=(\rho/\rho_0)^{2a^2}f^2(\rho)(dt^2-d\rho^2) -\rho^2f^2(\rho)\gamma^2d\phi^2-f^{-2}(\rho)dz^2 \quad \text{for } \rho>\rho_0 \ , \tag{2}$$

where $f(\rho)=(\rho/\rho_0)^a+A(\rho/\rho_0)^{-a}$, and the electromagnetic field is given by

$$F_{\rho z}=-F_{z\rho}=C\rho^{-1}f^{-2}(\rho) \quad (\text{all the other } F_{\mu\nu} \text{ vanish}) \ . \tag{3}$$

The constants γ, a, A, and C are defined as follows: $\gamma=1-4G\mu_F$, $a=2G\mu_M$, $A=I^2/(16\pi G\mu_M^2\gamma^2)=I^2G/$

$(4\pi a^2\gamma^2)$, $C=I/2\pi\gamma$ (Ref. 13). The influence of virtual electrons on the dynamics of electromagnetic and gravitational fields is described by the effective action

$$S_{\rm eff}=-i\ln\int D\psi D\bar{\psi}\, e^{iS}\ , \tag{4}$$

where S denotes the action for QED in a background gravitational field, and ψ is the electron wave function.

The first terms of the expansion of $S_{\rm eff}$ (4) in powers of m^{-1} (the inverse of electron mass) have been calculated in Ref. 14. Their renormalized form is

$$\begin{aligned} S_{\rm eff}&=\int(-g)^{1/2}dx\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}^{(1)}_{\rm eff}\right)\\ \mathcal{L}^{(1)}_{\rm eff}&=-\frac{e^2}{2880\pi^2m^2}\left(5RF_{\mu\nu}F^{\mu\nu}-26R^{\mu\nu}F_{\mu\rho}F_{\nu}{}^{\rho}+2R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}+24F^{\mu\nu}{}_{;\nu}F_{\mu\rho}{}^{;\rho}\right)\\ &\quad+(\text{terms involving only gravitational field})+O(m^{-4})\ . \end{aligned} \tag{5}$$

Here $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$, and R denote the curvature tensor and its contractions, respectively, and e is the electron charge. The inverse proportionality of the nonkinetic term in (5) to the fermion mass squared justifies neglecting other, heavier fermions in our considerations.

The expansion (5) of $\mathcal{L}^{(1)}_{\rm eff}$ in powers of m^{-1}, called the adiabatic approximation, is legitimate only if both electromagnetic and gravitational fields are sufficiently small and slowly changing. More precisely, it can be used only in regions where $eF_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ and all their covariant derivatives are much smaller than the mass m to a power that can be determined on dimensional grounds. In the case of a cosmic string this restriction does not allow us to use Eq. (5) in regions closer than m^{-1} to the string, for any value of the current I. For currents greater than $I_0=m/2e\approx 600$ A this restriction becomes stronger, and the region in which Eq. (5) is applicable is bounded by the inequality $\rho>2Ie/m^2$.

Having these restrictions in mind we can calculate the vacuum-polarization effects in the region of validity of the adiabatic approximation. Assuming that in this region the correction to the electromagnetic field due to vacuum polarization is small (which will turn out to be the case) we can calculate the vacuum-polarization current from the equation

$$j^{\nu}_{\rm vac}=-D_{\mu}\frac{\partial\mathcal{L}^{(1)}_{\rm eff}}{\partial(\partial_{\mu}A_{\nu})}=2D_{\mu}\frac{\partial\mathcal{L}^{(1)}_{\rm eff}}{\partial F_{[\mu\nu]}}\ , \tag{6}$$

where, after taking the derivative with respect to $F_{\mu\nu}$, and necessary antisymmetrization, the fields $F_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are put equal to the external ones determined by (2) and (3). Having obtained $j^{\nu}_{\rm vac}$ from (6) we insert it into the usual Maxwell equations (in curved space) and calculate the correction to the external electromagnetic field this way. This procedure is obviously an iterative method (in m^{-1}) of solving the Euler-Lagrange equations for the effective Lagrangian.

In the general case, combining (5) and (6), we obtain

$$\begin{aligned} j^{\nu}_{\rm vac}=&-\frac{\alpha}{90\pi m^2}R^{\mu\nu}{}_{\alpha\beta}F_{\rm ext}{}^{\alpha\beta}{}_{;\mu}+\frac{\alpha}{15\pi m^2}\Box j^{\nu}_{\rm ext}\\ &+\frac{2\alpha G}{45m^2}\Big(-2Tj^{\nu}_{\rm ext}+T^{\nu}{}_{\mu}j^{\mu}_{\rm ext}\\ &\qquad+13T_{\alpha}{}^{\mu}F_{\rm ext}{}^{\alpha\nu}{}_{;\mu}-9F^{\alpha\mu}_{\rm ext}T_{\alpha}{}^{\nu}{}_{;\mu}\\ &\qquad-6F^{\mu\nu}_{\rm ext}T_{;\mu}\Big)+\cdots\ , \end{aligned} \tag{7}$$

where $\alpha=e^2/4\pi$, $T_{\mu\nu}$, and $j^{\nu}_{\rm ext}$ are sources for the external gravitational and electromagnetic fields, respectively, and $T=T^{\mu}{}_{\mu}$. For the field around a cosmic string the equalities $j^{\nu}_{\rm ext}=0$ and $T_{\mu\nu}\approx 0$ hold in the whole region where the adiabatic approximation is valid (the energy-momentum tensor of the electromagnetic field can be neglected in this region). Thus, in the orthonormal frame

$$e_{\hat{0}}=\left[\frac{\rho}{\rho_0}\right]^{-a^2}\frac{1}{f(\rho)}\frac{\partial}{\partial t},\quad e_{\hat{1}}=\left[\frac{\rho}{\rho_0}\right]^{-a^2}\frac{1}{f(\rho)}\frac{\partial}{\partial\rho}\ ,$$

$$e_{\hat{2}}=\frac{1}{\rho f(\rho)\gamma}\frac{\partial}{\partial\phi},\quad e_{\hat{3}}=f(\rho)\frac{\partial}{\partial z}$$

we have

$$\begin{aligned} j^{\hat{3}}_{\rm vac}(\rho)&=\frac{-\alpha}{90\pi m^2}R^{\hat{\mu}\hat{3}}{}_{\hat{\alpha}\hat{\beta}}F^{\hat{\alpha}\hat{\beta}}_{\rm ext}\\ &=-\frac{I\alpha a m^{-2}}{45\pi\gamma}\rho_0^{-4}\left[\frac{\rho}{\rho_0}\right]^{-4(a^2+1)}f^{-6}(\rho)h(\rho) \end{aligned} \tag{8}$$

and

$$j^{\hat{0}}_{\rm vac}=j^{\hat{1}}_{\rm vac}=j^{\hat{2}}_{\rm vac}=0\ ,$$

where $h(\rho)=(\rho/\rho_0)^{a}-A(\rho/\rho_0)^{-a}$. (The part of $j^{\mu}_{\rm vac}$ due to purely electromagnetic effects is of order m^{-4}.) Inserting $j^{\mu}_{\rm vac}$ (8) into Maxwell's equations we obtain the following solution for the total electromagnetic field around the string:

$$\begin{aligned} F_{\hat{1}\hat{3}}(\rho)=&\frac{I}{2\pi\rho_0\gamma}f^{-2}(\rho)(\rho/\rho_0)^{-a^2-1}\\ &\times\left[1-\frac{\alpha a}{45\pi(m\Lambda)^2}\left[\frac{\Lambda}{\rho_0}\right]^{-2a^2}\frac{h(\Lambda)}{f^3(\Lambda)}-\frac{\alpha a}{45\pi(m\rho_0)^2}\left[\frac{\rho}{\rho_0}\right]^{-2a^2-2}\frac{h(\rho)}{f^3(\rho)}\right]+O(a^2,m^{-4})\\ &+(\text{corrections due to vacuum polarization at distances to the string smaller than }\Lambda)\ \text{ for }\rho>\Lambda\ . \end{aligned} \tag{9}$$

Expression (9) can be simplified when one realizes that for the considered values of μ_M, μ_F, and I [i.e., $\mu_F \sim 10^{22}$ g/cm, $\mu_M \lesssim \mu_F$, $I \lesssim m/2e$ (Ref. 15)], even for $\rho_0 \sim 10^{-28}$ cm and at distances ρ comparable with the present dimension of the Universe, we have $f(\rho)\approx 1$, $h(\rho)\approx 1$, and $(\rho/\rho_0)^{a^2} \approx 1$. This means that the last three approximate equalities are valid in the whole region where the string can be treated as straight. Thus, taking $\Lambda = m^{-1}$, we obtain

$$F_{\hat{1}\hat{3}}(\rho) = \frac{I}{2\pi\gamma\rho}\left[1 - \frac{\alpha a}{45\pi} - \frac{\alpha a}{45\pi(m\rho)^2}\right] + \cdots \, . \tag{10}$$

The current seen far away from the string is lowered in this approximation by the amount of $\Delta I = (2G\mu_M \alpha / 45\pi) I$. Even if the linear mass density μ_M is of order μ_F, then $G\mu_M \lesssim 10^{-7} - 10^{-5}$, and so the correction in the considered approximation is very small. Taking naively $\Lambda \sim \rho_0$ in (9) would make the correction big, which suggests that probably a nonperturbative calculation (in m^{-1}) of the induced current would give a reliable result. Unfortunately, the adiabatic approximation seems to be the only known method of dealing with vacuum-polarization effects in a general gravitational field. For a given field one can try to use other methods similar to those developed for the electromagnetic field[16] at the expense of general covariance. However, in the case of metric (2) the presence of the linear singularity leads to insurmountable technical difficulties.

We can compare our result with the purely electromagnetic vacuum-polarization effect calculated in Ref. 5. When the gravitational field is neglected the first nonvanishing term in the expansion (in powers of m^{-1}) of the correction ΔI to the current I seen at infinity reads[5]

$$\frac{\Delta I}{I} = \frac{2\alpha^2 I^2}{45\pi^2 m^4 \Lambda^2} \, , \tag{11}$$

where Λ denotes the minimal distance from the string at which the adiabatic approximation is still applicable. The correction (11) cannot be larger than about 5×10^{-5}. On the other hand, our result

$$\frac{\Delta I}{I} = \frac{2G\mu_M \alpha}{45\pi} \approx G\mu_M \times 10^{-4} \tag{12}$$

turns out to be much smaller at the considered values of μ_M. Thus, the second-order (in m^{-2} or in α) effect appears to be greater than the first-order one since it does not depend on the strength of the gravitational field around a string. Of course, we can make the two corrections comparable by varying arbitrarily the parameter μ_M, but it seems physically unreasonable.

In conclusion, the influence of gravity on vacuum-polarization effects around a cosmic string (in the region where the adiabatic approximation is applicable) will lead to a screening of the current in the string. However, this effect will always be negligible. The presented calculations do not allow us to tell anything about phenomena that take place very close to the string, where our approximations are not valid.

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[13]A more careful analysis of the metric (2) leads to the observation that the expressions for the constants a, A, and C given above (on the basis of Ref. 6) are correct only if $a \ll 1$ and $A \ll 1$. Justification of this fact which requires a detailed study of the metric (2) will not be given here. It does not, however, change the final conclusions of this paper.

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