Additivity of the entropies of black holes and matter in equilibrium

Erik A. Martinez and James W. York, Jr.
Institute of Field Physics, Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27599-3255
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We consider static spherical spacetimes in which a black hole is in equilibrium with surrounding matter. The grand canonical ensemble is defined by the geometry of the boundary of a region of spacetime and the chemical potential evaluated on this boundary. Additivity of the actions of gravity and matter and the principle of equivalence then jointly imply that the total entropy is the simple sum of the accepted black-hole entropy and the ordinary matter entropy. However, energy and thermodynamic potentials exhibit interaction terms and are not simply additive..

In this work we address two fundamental questions concerning the nature of thermodynamic equilibrium between black holes and matter. (1) Is there gravitational entropy associated with matter in addition to its ordinary thermodynamic entropy? We shall treat only the case in which black hole and matter are in thermodynamic equilibrium. (2) If a black hole and matter are in equilibrium, what is the total entropy? In particular we want to see if their entropies are simply additive.

The two questions are related. The first one has been discussed to some extent by others. Gibbons and Hawking[1] considered a stationary fluid "star" and argued that the only entropy would be that of the fluid. Davies, Ford, and Page[2] considered a static spherical black hole surrounded concentrically by a thin shell of matter and found no inconsistency in taking the gravitational entropy as only that of the black hole. We also shall begin by treating the thin-shell model, but unlike Ref. 2 we shall take explicitly into account the thermodynamics of the matter. Later we generalize to a thick distribution of spherically symmetric matter. We find no gravitational entropy associated with the matter in either case.

The second question is part of a broader one that asks in effect how the entropy concept is to be understood in strongly self-gravitating systems. The possibility of such an understanding is a consequence of the established result that the extreme end point of gravitational collapse, the black hole, has a description as a thermal equilibrium state.[3,4] Furthermore, it has been demonstrated that this state can be stable in the canonical ensemble.[5]

Because gravity is an infinite-range, unscreenable interaction, it always affects to some extent the thermal and chemical or diffusive contact of the system with its environment. This consequence of the principle of equivalence was recognized explicitly long ago by Tolman,[6] who pointed out that the temperature $T=\beta^{-1}$ and chemical potential μ suffer a relative "redshift" as they are measured by static observers in one place or another, even in a state of thermodynamic equilibrium. It is, therefore, essential to recognize explicitly that equilibrium is characterized by spatially varying β and μ. In order to have definite single values to work with, one can choose these quantities to be specified at a given locus that can be thought of as the boundary surface of an effective heat reservoir, or as an interface between what is designated as "the system" and its environment.[5] Thus boundary conditions can be expected to play a very important role when the system contains a black hole.

We adopt as our basic hypotheses that the (grand) canonical ensemble is fundamental and that the actions of gravity and matter are to be added. The latter is in accord with the way actions are combined in computing transition amplitudes and partition functions in the path-integral forms of quantum theory and statistical thermodynamics.[7] As we shall see, this is very different from an assertion of simple additivity for thermodynamic potentials and energies, which in fact does not hold in general when gravity is taken into account. We obtain as a consequence of our hypotheses that the total entropy is the simple sum of the ordinary matter entropy and the accepted value of the black-hole entropy.

In the static spherical metrics we consider, it is natural to use the periodically identified Euclidean or "imaginary time" description well known from the treatment of thermal problems in Minkowski spacetime.[7] In spacetimes with black holes, this leads as in Ref. 1 to manifolds of topology $R^2\times S^2$. We explicitly attach and consider a boundary $S^1\times S^2$ as in Ref. 5. The round S^1 has proper circumference $\beta_0\hbar$, where $\beta_0\equiv\beta(r_0)$ is the inverse proper local temperature measured on the round S^2 with proper area $4\pi r_0^2$ ($k_B=c=1$). Likewise, the proper local chemical potential of the matter $\mu_0\equiv\mu(r_0)$ is specified on the boundary S^2. Thus, β_0, μ_0, and r_0 specify the free boundary data of the grand canonical ensemble we shall employ.

For the Euclidean action of matter, we adopt as Lagrangian the density of the grand potential, as is appropriate whenever the matter admits a thermodynamic description. The grand potential in ordinary thermodynamics is *defined* by $\Omega=\langle E\rangle_{\text{matter}}-TS_{\text{matter}}-\mu N$, where $\langle E\rangle$ is energy, S denotes entropy, and N is a conserved quantity such as the total particle number. In curved spacetime, one adopts the assumption of *minimal* *coupling* and employs the local, proper, "per unit volume" definition of Ω: namely,

$$\omega = \epsilon - T s - \mu n \ . \tag{1}$$

For example, if the matter were a perfect fluid (which we do not assume[8]), the value of the Lagrangian density would be the negative of the pressure.[9] Therefore, the action is

$$I_{\text{matter}} = \int_{r_+}^{r_0} dr \int_{\partial M} d\tau\, d\theta\, d\phi\, g^{1/2}(\epsilon - T s - \mu n) \ , \tag{2}$$

where $r_+ \equiv 2GM$ is the gravitational radius of the black hole and $\partial M = S^1 \times S^2$. The action for gravity, when the metric is taken as the fundamental variable, is[10,1,5]

$$I_{\text{grav}} = -\frac{1}{16\pi G}\int_M g^{1/2} R\, d^4x + \frac{1}{8\pi G}\int_{\partial M} \gamma^{1/2}(K - K^0)\, d^3x \ , \tag{3}$$

where M is the spacetime region explicitly indicated in (2), γ is the determinant of the metric induced on the boundary, K is the trace of extrinsic curvature of the boundary, and K^0 is a constant that normalizes this action to zero for flat spacetime with the given boundary.[1] Note that the constant K^0, which is the only "arbitrary" element in this action, has *no* effect on the entropy.[5]

We assume initially that the matter has support only on a very thin shell of radius α. Then the proper quantities T and μ in (1) and (2) take their values at $r = \alpha$, that is, they are not the boundary data T_0 and μ_0. Nevertheless, we shall see that only the boundary data enter into the final results. We can take, as the metric of M,

$$ds^2 = U_1(r)\left[\frac{U_2(\alpha)}{U_1(\alpha)}\right] d\tau^2 + V_1^{-1}(r)\, dr^2 + r^2 d\Omega^2 \quad (r < \alpha) \ , \tag{4}$$

$$ds^2 = U_2(r)\, d\tau^2 + V_2^{-1}(r)\, dr^2 + r^2 d\Omega^2 \quad (r > \alpha) \ , \tag{5}$$

where subscripts 1 and 2 refer, respectively, to the regions inside and outside the thin shell. Clearly we must have $r_+ < \alpha \le r_0$. We must also require that α is greater than the gravitational radius of the combined system of black hole and shell in order that the metrics (4) and (5) be physically valid. The proper length around the S^1 of the boundary $r = r_0$ is

$$\beta_0 \hbar = P U_2^{1/2}(r_0) \ , \tag{6}$$

where the period P of the Euclidean time τ is arbitrary, in accord with the free choice of β_0 at r_0. In order to avoid a conical singularity at the "center" $r = r_+$ of M, we require

$$[V_1^{1/2}(U_1^{1/2})']_{r=r_+} = \frac{2\pi}{P}\left[\frac{U_1(\alpha)}{U_2(\alpha)}\right]^{1/2} , \tag{7}$$

where a prime denotes differentiation with respect to r. Lastly, we note that the Euler number of the $R^2 \times S^2$ is $\chi = 2$ ("black-hole topology").

The total action of the system is $I = I_{\text{grav}} + I_{\text{matter}}$. The matter action can be simplified further because the supports of ϵ, s, and n are only on the shell $r = \alpha$. Hence, we can express I_{matter} in terms of the energy σ, entropy s, and number n per unit proper area on the shell. Then it is possible to write

$$\begin{aligned} I = &-\frac{1}{16\pi G}\int_M (g_1^{1/2} R_1 + g_2^{1/2} R_2)\, d^4x \\ &-\frac{1}{16\pi G}\int_{r=\alpha} (R_{\text{shell}} - 16\pi G\sigma)\gamma^{1/2} d^3x \\ &-\int_{r=\alpha} [T(\alpha)s + \mu(\alpha)n]\gamma^{1/2} d^3x \\ &+\frac{1}{8\pi G}\int_{r=r_0} (K - K^0)\gamma^{1/2} d^3x \ , \end{aligned} \tag{8}$$

where $\gamma^{1/2} d^3x = (U^{1/2} r^2 \sin\theta)\, d\theta\, d\phi\, d\tau$ evaluated at the appropriate value of r.

In examining the consequences of an action such as (8), by means of a path integral for the partition function, one wishes to enforce the gravitational constraints. The relative simplicity of the metrics we are considering allows us to solve the constraints explicitly and incorporate the results directly into I, producing a "reduced action" I_* as introduced and exemplified in Ref. 11. Calculations using I_* then involve only configurations lying on the constraint hypersurface of the dynamical phase space.

The momentum constraints for the τ=const slices of the metric defined by (4) and (5) are satisfied trivially. The Hamiltonian constraint in both regions 1 and 2 has a simple form which is easily solved by $V_a = 1 - C_a r^{-1}$ ($a = 1,2$), where C_a are constants of integration. To evaluate C_1 we may consider region 1 as a manifold with boundary, the (mathematical) boundary being (say) any r = const surface between $r = r_+$ and $r = \alpha$. From (7) and the fact that the Euler number is $\chi = 2$, one can show that $C_1 = r_+ = 2GM$. Subsequent arguments assume that suitable junction conditions are satisfied at $r = \alpha$ and to this consideration we now turn.

The junction conditions require, as already satisfied, that the metric induced on $r = \alpha$ from $r > \alpha$ and $r < \alpha$ be the same. Furthermore, the discontinuity in the extrinsic curvature K^i_j of the shell must be proportional to the stress energy in this hypersurface. In keeping with the construction of the reduced action, we need only the "Hamiltonian constraint" part of this condition: namely,

$$\Delta(K^\tau_\tau - K) = -8\pi G\sigma \ , \tag{9}$$

where Δ indicates the discontinuity at $r = \alpha$ of the quantity in parentheses.

Note that in both regions 1 and 2, the scalar curvature has the form

$$R = -g^{-1/2}(g^{1/2} V U' U^{-1})' + 2G^\tau_\tau \ , \tag{10}$$

with the last term vanishing because we have solved the constraints. Recalling that $R_{\text{shell}} = 2\Delta K$ and combining (9) and (10) enables us to evaluate the first two integrals in (8) to obtain

$$\frac{1}{4G}P\left[r^2\frac{V_2^{1/2}}{U_2^{1/2}}U_2{}'\right]_{r=r_0} - \frac{1}{4G}P\left[r^2\frac{V_1^{1/2}U_2^{1/2}(\alpha)}{U_1^{1/2}U_1^{1/2}(\alpha)}U_1'\right]_{r=r_+} . \quad (11)$$

The third integral (8) can be evaluated by using (6) and the consequence of the principle of equivalence that μ and T are "redshifted" in the same manner;[6] for example,

$$T(\alpha)U_2^{1/2}(\alpha)=T_0U_2^{1/2}(r_0) , \quad (12)$$

and consequently one obtains $\beta(\alpha)\mu(\alpha)=\beta_0\mu_0$. Therefore, the third integral in (8) becomes

$$-4\pi\alpha^2 s\hbar-\beta_0\mu_0(4\pi\alpha^2 n)\hbar=-S_{\text{matter}}\hbar-\beta_0\mu_0 N\hbar , \quad (13)$$

where $S_{\text{matter}}=(4\pi\alpha^2)s$ is the ordinary thermodynamic entropy of the matter and $N=(4\pi\alpha^2)n$. In (13) the location of the shell is unimportant in the sense that the only values of β and μ that appear are the boundary values. The last integral in (8) takes the same form it has in the absence of a thin shell:[11] namely,

$$\frac{\beta_0 r_0}{G}[1-V_2^{1/2}(r_0)]-\left[\frac{1}{4G}Pr^2\frac{V_2^{1/2}}{U_2^{1/2}}U_2'\right]_{r=r_0} . \quad (14)$$

To obtain the total action, we combine (11), (13), (14), and (7) to obtain

$$I_*\hbar^{-1}=\frac{\beta_0 r_0}{G}[1-V_2^{1/2}(r_0)]-\frac{\pi r_+^2}{G\hbar}-S_{\text{matter}}-\beta_0\mu_0 N . \quad (15)$$

The zero-loop approximation[1,5,12] in the present context identifies $I_*\hbar^{-1}$ with $\beta_0\Theta=\beta_0\langle E\rangle-S-\beta_0\mu_0 N$, where Θ is the total grand potential of black hole plus matter. We then infer our main result:

$$S_{\text{total}}=\frac{\pi r_+^2}{G\hbar}+S_{\text{matter}}=S_{\text{BH}}+S_{\text{matter}} . \quad (16)$$

Thus we obtain both the simple additivity of the constituent black-hole (BH) and matter entropies and the fact that there is no further term indicating an "extra" gravitational entropy. The additivity of entropies suggests that the "number of states" associated with the system is given by $\mathcal{N}_{\text{syst}}\approx\mathcal{N}_{\text{matter}}\mathcal{N}_{\text{BH}}$ despite reservations expressed in Ref. 13, but in keeping with the interpretation of $\mathcal{N}_{\text{BH}}$ indicated in Ref. 14.

It is straightforward to show that the simple additivity of entropy is not dependent on the use of a thin shell as a model for matter around a black hole. If we have a distribution of spherical static (or effectively static) matter filling a region between radii α_1 and α_2 ($r_+\le\alpha_1\le\alpha_2\le r_0$), we can again calculate the reduced action. The only nontrivial constraint is $G_\tau^\tau+8\pi G\epsilon(r)=0$. Solving this for $V(r)$ and following a procedure such as the one above yields

$$\begin{aligned}I_*\hbar^{-1}&=\frac{\beta_0 r_0}{G}[1-V^{1/2}(r_0)]-\frac{\pi r_+^2}{G\hbar}\\&-4\pi\int_{\alpha_1}^{\alpha_2}s(r)V^{-1/2}(r)r^2dr\\&-4\pi\beta_0\mu_0\int_{\alpha_1}^{\alpha_2}n(r)V^{-1/2}(r)r^2dr\\&=\beta_0\langle E\rangle-(S_{\text{BH}}+S_{\text{matter}})-\beta_0\mu_0 N . \end{aligned}\quad (17)$$

Notice that S_{matter} and N are obtained from integrals of proper densities over proper spatial volumes and are, therefore, obtained by simple "counting and adding," which, of course, is quite unlike the way the total energy is calculated. The failure of energy to be simply additive is illustrated below.

Let us return to the model of black hole plus thin shell. Imposing *all* of the Einstein equations (not just the constraints) gives us the metric

$$ds^2=\left[1-\frac{r_+}{r}\right]\left[\frac{1-\tilde{r}_+\alpha^{-1}}{1-r_+\alpha^{-1}}\right]d\tau^2+\left[1-\frac{r_+}{r}\right]^{-1}dr^2+r^2d\Omega^2 \quad (r<\alpha) , \quad (18)$$

$$ds^2=\left[1-\frac{\tilde{r}_+}{r}\right]d\tau^2+\left[1-\frac{\tilde{r}_+}{r}\right]^{-1}dr^2+r^2d\Omega^2 \quad (r>\alpha) , \quad (19)$$

where $r_+=2GM$ is the gravitational radius of the black hole and $\tilde{r}_+\equiv 2G\tilde{M}$ is the gravitational radius of the whole system. The junction conditions at the shell are given by (9) and by a similar equation for the angular components in which $-\sigma$ is replaced by λ_{shell}, the surface pressure for the shell. Defining the proper mass of the shell by $m=4\pi\alpha^2\sigma$ one finds[2]

$$\tilde{M}=M+m(1-2GM\alpha^{-1})^{1/2}-Gm^2(2\alpha)^{-1} . \quad (20)$$

Observe that this relation is not valid unless $\alpha>2G\tilde{M}$. The total thermodynamic energy $\langle E\rangle$ in (15) is given by[5]

$$\tilde{M}=\langle E\rangle-\frac{G\langle E\rangle^2}{2r_0} . \quad (21)$$

Combining the last two equations shows the failure of energy to be simply additive because of the interaction and binding-energy terms.

The period of τ for which the metric (18) is regular at $r=r_+$ is found from (7) to be

$$P=4\pi r_+[(1-r_+\alpha^{-1})(1-\tilde{r}_+\alpha^{-1})^{-1}]^{1/2} , \quad (22)$$

which is $2\pi\kappa^{-1}$, κ=surface gravity of the black hole, in agreement with Ref. 2. The inverse temperature at r_0 has the blueshifted Hawking value $\beta_H(r_0)=P(1-\tilde{r}_+r_0^{-1})^{1/2}$, which can derived by extremizing the reduced action (15) with respect to r_+ (all other quantities fixed), analogously to the procedure of Ref. 11. Then if we examine r_+ as a function of the data β_H, r_0, m, and α similarly to Ref. 5, we see that there do exist locally stable roots for r_+. As a consequence, to proceed from Euclidean action to "zero-loop partition function" as we have done here makes physical sense.[1,5,12]

From the junction conditions one can obtain the surface pressure of the shell:

$$\lambda_{\text{shell}}=\frac{1}{8\pi\alpha^2}[\tilde{M}(1-\tilde{r}_+\alpha^{-1})^{-1/2}-M(1-r_+\alpha^{-1})^{-1/2}-m]\ . \tag{23}$$

Likewise, for the total system one has, as in Ref. 5, a surface pressure λ_{system} conjugate to the boundary area $A_0=4\pi r_0^2$:

$$\lambda_{\text{system}}(r_0)=\frac{1}{8\pi G r_0}\left[\left[1-\frac{\tilde{r}_+}{r_0}\right]^{-1/2}\left[1-\frac{\tilde{r}_+}{2r_0}\right]-1\right], \tag{24}$$

which has the same form found in Ref. 5, but with $\tilde{r}_+=2G\tilde{M}$ given by (20).

We have seen that separating black-hole and matter contributions to the energy in a simple way is impossible and the same clearly holds for the surface pressures and, consequently, for any thermodynamic potentials that one might attempt to associate individually with either black hole or shell. Gravitational coupling is pervasive, affects the temperature and chemical potential, and violates the hypothesis of "weak coupling" that allows one simply to add thermodynamic potentials for separate systems. It is, therefore, remarkable that the simple additivity of the actions leads to the simple additivity of the entropies while the other simplicities of equilibrium drop away.[15] It should be noted that the temperature, chemical potentials, and radius needed to compute the actions refer to no individual constituent of the system, but rather only to a common locus that may be disjoint from the constituents, that is, to the boundary of the spacetime domain in which the system is contained.

There is, however, a special case in which all black hole and shell variables do decouple; this occurs when the shell coincides with the boundary in the limit $\alpha\rightarrow r_0$. In this case we can think of the shell as modeling a simple "heat reservoir" for the hole. Thus, one can show that

$$\lim_{\alpha\rightarrow r_0}\langle E\rangle=\langle E_{\text{BH}}\rangle+4\pi r_0^2\sigma\ , \tag{25}$$

$$\lim_{\alpha\rightarrow r_0}\lambda_{\text{system}}=\lambda_{\text{BH}}+\lambda_{\text{shell}}\ , \tag{26}$$

where BH denotes the black-hole quantities as computed in an idealized massless cavity. One, therefore, recovers the thermodynamic properties of black holes previously computed.[5,11,12] In this same limit ($\alpha\rightarrow r_0$), one can obtain a decoupled Euler relation: namely,

$$\langle E\rangle=2(T_0S_{\text{BH}}-\lambda_{\text{BH}}A_0)+(T_0S_{\text{matter}}-\lambda_{\text{shell}}A_0+\mu_0N)\ , \tag{27}$$

which shows clearly the contrasting scaling properties of black holes[5] and ordinary matter.

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