Exact solutions for fermionic Green's functions in the Bloch-Nordsieck approximation of QED

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A set of new closed-form solutions for fermionic Green's functions in the Bloch-Nordsieck approximation of QED is presented. A manifestly covariant phase-space path-integral method is applied for calculating the *n*-fermion Green's function in a classical external field. In the case of one and two fermions, explicit expressions for the full Green's functions are analytically obtained, with renormalization carried out in the modified minimal subtraction scheme. The renormalization constants and the corresponding anomalous dimensions are determined. The mass-shell behavior of the two-fermion Green's function is investigated in detail. No assumptions are made concerning the structure of asymptotic states and no IR cutoff is used in the calculations.

I. INTRODUCTION

Conventional perturbation theory for quantum field theories including massless fields (e.g., photons) produces infrared (IR) divergences, in addition to the ordinary ultraviolet (UV) divergences. While the UV divergences can be removed by renormalization within some appropriate regularization scheme, this is not possible for the IR divergences.[1]

In a classic paper, Bloch and Nordsieck[2] (BN) have shown that in any scattering process involving charged particles, an infinite number of soft (IR) photons may be emitted with a finite total energy. Thus, the concept of a single particle described asymptotically by a free field is no longer applicable. A charged particle is always accompanied by a "cloud" of soft photons[3-6] which is inextricably tied to the source. As a result, the residue of the two-point Green's function of a charged particle near the mass shell is modified, although the position of the physical pole is not shifted. (This reflects the fact that the mass renormalization constant is IR finite.) One finds[4,6-9]

$$G_{\text{ren}}(p) \sim (p^2 - m^2 + i\epsilon)^{-(1+a)} \tag{1.1}$$

with $a = (\alpha/2\pi)(3-\lambda)$, where $\alpha = e^2/4\pi$ ($\hbar = c = 1$) is the fine-structure constant and λ is the gauge parameter.[10] The effect of soft photons is contained in the factor $(p^2-m^2+i\epsilon)^{-a} = e^{-a\ln(p^2-m^2+i\epsilon)}$, which is a logarithmic branch line along the real axis beginning at the mass shell of the charged particle.[11]

The aim of this paper is the analytic calculation of closed-form solutions for the fermionic Green's functions in the BN approximation of QED. One of the crucial features of our approach is that the *n*-fermion Green's function in a given classical external field is not derived by solving a differential equation,[7] but in terms of a phase-space path integral.[12] This treatment[13] has several methodological advantages: (i) it permits clear insights into the physical content of the performed approximations; (ii) it is manifestly covariant and does not depend on assumptions concerning the structure of asymptotic states;[4-6] (iii) in addition to the above, it makes it possible to take into account quantum fluctuations around the classical trajectory. Such contributions cannot be incorporated by other means in a simple manner. Another virtue of our approach is that the use of an IR cutoff is entirely avoided, while the UV divergences are controlled by dimensional regularization.

The work is organized as follows. In Sec. II the path-integral method is illustrated in a discussion of the two-point Green's function. The generalization to *n* fermions is treated subsequently in Sec. III. This problem is solved in *closed form* to all orders in the coupling constant α and for arbitrary gauges. In the case of two fermions, the mass-shell behavior of the momentum-space form of the renormalized four-point Green's function can be investigated in detail because an *explicit* expression is derived. To our knowledge, such an expression which has been calculated by a *nonperturbative* method (the BN model) has not been obtained previously. The correspondence between our solutions and results derived elsewhere by other methods is discussed. Finally, Sec. IV contains a summary of our results. Some details of the calculations are presented in the appendixes.

II. TWO-POINT GREEN'S FUNCTION

We start out with the path-integral treatment of the two-point Green's function. Since the path-integral technique has been discussed extensively elsewhere,[14] our discussion will be brief, concentrating on the points relevant to the present investigation. We work in terms of the generating functional, $Z_E[J,\bar{\eta},\eta]$, of Euclidean QED (indicated by the subscript E in the following) which is written

$$\begin{aligned} Z_E[J,\bar{\eta},\eta] = N \int [dA_\mu] \det[iG_E^{-1}(x,x'|A)] \\ \times \exp[-\langle \bar{\eta}(x) G_E(x,x'|A)\eta(x')\rangle] \\ \times \exp[-\tfrac{1}{2}\langle A_\mu(x) D^{-1}_{\mu\nu,E}(x,x') A_\nu(x')\rangle \\ - \langle J_\mu(x) A_\mu(x)\rangle] , \end{aligned} \tag{2.1}$$

where the functional integration over the fermion fields has already been carried out, $N^{-1}=Z_E[J=\bar{\eta}=\eta=0]$, and the definitions

$$G_E^{-1}(x,x'|A)=[i\gamma_\mu\partial_\mu+e\gamma_\mu A_\mu(x)+m]\delta^4(x-x') , \quad (2.2)$$

$$D_{\mu\nu,E}^{-1}(x,x')=\left[\partial_\mu\partial_\nu\delta_{\mu\nu}+\frac{1-\lambda}{\lambda}\partial_\mu\partial_\nu\right]\delta^4(x-x') \quad (2.3)$$

have been used. Here and below $\langle\ \rangle$ denotes four-dimensional integration over position or momentum variables. $\det[iG_E^{-1}(x,y|A)]$ contains all possible closed loops of fermion lines.

The electron Green's function of the second-quantized theory is obtained from $Z_E[J,\bar{\eta},\eta]$ by functional differentiation with respect to the sources $\bar{\eta},\eta$:

$$\begin{aligned}G_E(x,y)&=(-1)\left[\frac{\delta^2 Z_E[J,\bar{\eta},\eta]}{\delta\bar{\eta}(y)\delta\eta(x)}\right]_{J=\bar{\eta}=\eta=0}\\&=N\int[dA_\mu]\det[iG_E^{-1}(x,y|A)]G_E(x,y|A)\\&\quad\times\exp[-\tfrac{1}{2}\langle A_\mu(x')D_{\mu\nu,E}^{-1}(x',x'')A_\nu(x'')\rangle] .\end{aligned} \quad (2.4)$$

The Green's function $G_E(x,y|A)$ describes the first-quantized motion of an electron in a classical external field and satisfies the equation

$$\int d^4z\, G_E^{-1}(x,z|A)G_E(z,y|A)=\delta^4(x-y) \quad (2.5)$$

from which follows

$$[i\gamma_\mu\partial_\mu+e\gamma_\mu A_\mu(x)+m]G_E(x,y|A)=\delta^4(x-y) \quad (2.6)$$

using (2.2). Upon analytic continuation to Minkowski space, the last equation reads

$$[i\gamma^\mu\partial_\mu+e\gamma^\mu A_\mu(x)-m+i\epsilon]G(x,y|A)=-\delta^4(x-y) . \quad (2.7)$$

Introducing now an auxiliary set of states $|x\rangle$ with $\langle x|y\rangle=\delta^4(x-y)$, $G(x,y|A)$ can be expressed as the matrix element of an operator $G[A]$: $G(x,y|A)=\langle x|G[A]|y\rangle$, where the operators x^μ and $p^\mu=ig^{\mu\nu}\partial_\nu$ satisfy the commutation relations $[p^\mu,x^\nu]=ig^{\mu\nu}$, $[x^\mu,x^\nu]=[p^\mu,p^\nu]=0$. Defining the conjugate momentum operator $\Pi^\mu=iD^\mu+i\epsilon\gamma^\mu=i\partial^\mu+eA^\mu+i\epsilon\gamma^\mu$, the first-order differential equation (2.7) can be converted into a second-order differential equation which allows for a path-integral representation via Fock's method of the fifth parameter[15] ($H^{-1}=-i\int_0^\infty e^{iH\tau-\epsilon\tau}d\tau$):

$$G[A]=\frac{\gamma\cdot\Pi+m}{2m}i\int_0^\infty d\tau\exp\left[i\frac{\tau}{2m}\left[\Pi^2-m^2+\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]-\epsilon\tau\right] . \quad (2.8)$$

In the BN approximation of QED, the matrices $\gamma^\mu=(\beta,\beta\alpha)$ are replaced by four scalars $u^\mu=(\gamma^{-1},\gamma^{-1}\mathbf{v})$ with $u^2=1$, where $\gamma^{-1}=(1-v^2/c^2)^{1/2}$. The physical content of this approximation is that the recoil on the electron can be neglected so that the emitted photons are uncorrelated and their spectrum is given by a Poisson distribution.[2] Thus,

$$\sigma_{\mu\nu}=\frac{i}{2}[\gamma_\mu,\gamma_\nu]\rightarrow\frac{i}{2}[u_\mu,u_\nu]=0$$

and Eq. (2.8) becomes

$$G_{\rm BN}[A]=\frac{u\cdot\Pi+m}{2m}i\int_0^\infty d\tau\exp\left[i\tau\left[\frac{\Pi^2-m^2}{2m}+i\epsilon'\right]\right] . \quad (2.9)$$

To obtain $G_{\rm BN}(x,y|A)$, one has to compute first the spinless Green's function

$$\begin{aligned}\tilde{G}(x,y|A)&=\langle x|\tilde{G}[A]|y\rangle\\&=\left\langle x\left|i\int_0^\infty d\tau e^{-i\tilde{H}\tau}\right|y\right\rangle\equiv i\int_0^\infty d\tau\langle x,\tau|y,0\rangle\end{aligned} \quad (2.10)$$

with $\tilde{H}=-(\Pi^2+m^2)/2m-i\epsilon'$. Following Ref. 13, we apply rather standard techniques[14] to convert the amplitude $\langle x,\tau|y,0\rangle$ into a path integral defined by means of a nine-dimensional phase-space lattice. The result is[13]

$$\langle x,\tau|y,0\rangle=\lim_{n\to\infty}\tilde{N}\int\prod_{\beta=1}^{n}dx_\beta\exp\left\{-i\delta\sum_{\alpha=1}^{n+1}\left[\frac{m}{2}\left[1+\frac{(x_\alpha-x_{\alpha-1})^2}{\delta^2}\right]-eA\left[\frac{x_\alpha+x_{\alpha-1}}{2}\right]\frac{x_\alpha-x_{\alpha-1}}{\delta}\right]\right\} , \quad (2.11)$$

where $\delta=\tau_{\alpha+1}-\tau_\alpha=\tau/(n+1)$ and $\tilde{N}$ is a normalization constant. Taking the continuum limit, we find ($x_{n+1}=x$, $x_0=y$)

$$\langle x,\tau|y,0\rangle=\tilde{N}\int[dx(\tau)]\exp\left[-i\int_0^\tau d\tau'\left[\frac{m}{2}[1+\dot{x}(\tau')^2]-eA(x(\tau'))\cdot\dot{x}(\tau')-i\epsilon\right]\right]. \tag{2.12}$$

The exponent in Eq. (2.12) can now be identified with the classical action in which Fock's fifth parameter τ plays the role of the proper time of the electron. (Note that although this action is not reparametrization invariant[16] it leads to the correct equations of motion.) Thus the amplitude $\langle x,\tau|y,0\rangle$ satisfies a "Schrödinger-type" equation [recall Eq. (2.10) with respect to the proper time: $i(\partial/\partial\tau)\langle x,\tau|y,0\rangle=\tilde{H}\langle x,\tau|y,0\rangle$ with the initial condition $\langle x,0|y,0\rangle=\delta^4(x-y)$].

To proceed, we expand the path integral in (2.12) around the classical path $x^\mu(\tau)=x^\mu_{\rm cl}+\xi^\mu(\tau)$ keeping the end points fixed:

$$\langle x,\tau|y,0\rangle=\tilde{N}\int[d\xi(\tau')]\exp\left[iS[x_{\rm cl}]+\frac{i}{2!}\delta^2S[x_{\rm cl}]+\cdots\right]. \tag{2.13}$$

Retaining only the leading term $\langle x,\tau|y,0\rangle\simeq\tilde{N}e^{iS[x_{\rm cl}]}$, and setting $\tilde{N}=1$, the spinless Green's function is

$$\tilde{G}(x,y|A)=i\int_0^\infty d\tau\exp\left[-i\int_0^\tau d\tau'\left[\frac{m}{2}[1+\dot{x}_{\rm cl}(\tau')^2]-eA(x_{\rm cl}(\tau'))\cdot\dot{x}_{\rm cl}(\tau')-i\epsilon\right]\right]. \tag{2.14}$$

Thus far, $\dot{x}_{\rm cl}(\tau)$ is arbitrary and the integration over τ cannot be carried out. A closed-form solution may be obtained if we assume that the electron moves on an effectively free trajectory without acceleration: $(x-y)^\mu=u^\mu\tau$ (u^μ=const). Then Eq. (2.14) yields

$$\begin{aligned}\tilde{G}(x,y|A)&=i\int_0^\infty d\tau\exp\left[-i\tau(m-i\epsilon)+ieu^\mu\int_0^\tau d\tau' A_\mu(x-u\tau')\right]\delta^4(x-y-u\tau)\\&=i\int_0^\infty d\tau\int\frac{d^4p}{(2\pi)^4}\exp\left[-ip\cdot(x-y)-i\tau(m-u\cdot p-i\epsilon)+ieu_\mu\int\frac{d^4k}{(2\pi)^2}A^\mu(k)e^{-ik\cdot x}\int_0^\tau d\tau' e^{iu\cdot k\tau'}\right].\end{aligned} \tag{2.15}$$

Precisely the last expression was derived in Ref. 7 by solving the first-order differential equation (2.7) using the substitution $\gamma_\mu\rightarrow u_\mu$. (Thus, as expected, spin does not matter for IR phenomena.) It is worth noting that our treatment of $\tilde{G}(x,y|A)$ not only effects the correspondence between Eq. (2.15) and the straight-line trajectory, but it also provides the possibility to calculate quantum path fluctuations via Eq. (2.13) in a systematic way.

Turn now to the electron Green's function [Eq. (2.4)] of the second-quantized theory. Performing the BN approximation,

$$\det[iG_E^{-1}(x,y|A)]=\exp\left[\int_0^e de'{\rm Tr}[G_E(x,x|A)\gamma_\mu A_\mu]\right]=1,$$

and the full photon propagator becomes identical with the free one. Then the full electron Green's function in momentum space takes the form

$$\begin{aligned}G_E(p)=N\int_0^\infty d\tau e^{-i\tau(m+u_\mu p_\mu)}\int[dA_\mu]\exp\Big[&-e\Big\langle u_\mu A_\mu(k)e^{ik_\nu x_\nu}\int_0^\infty d\tau' e^{-iu_\lambda k_\lambda\tau'}d\tau'\Big\rangle\\&-\tfrac{1}{2}\langle A_\mu(k)D^{-1}_{\mu\nu,E}(k,k')A_\nu(k')\rangle\Big],\end{aligned} \tag{2.16}$$

where

$$D^{-1}_{\mu\nu,E}(k,k')=\left[k^2\delta_{\mu\nu}-\frac{\lambda-1}{\lambda}k_\mu k_\nu\right]\delta^4(k+k').$$

By noticing that, for $\tau\rightarrow\infty$,

$$j_\mu(k,\tau)\equiv e\frac{1}{(2\pi)^2}u_\mu\int_0^\tau d\tau'\exp(-iu_\lambda k_\lambda\tau') \tag{2.17}$$

becomes the Fourier transform of the classical current, one observes that the functional integral over A_μ in (2.16) is entirely calculable by quadratures. The result of the integration continued back to Minkowski space is[13]

$$\begin{aligned}G(p)=i\int_0^\infty d\tau e^{-i\tau(m-u\cdot p-i\epsilon)}&\\\times\exp[-\tfrac{1}{2}\langle j^\mu(k,\tau)D_{\mu\nu}(k,k')&\\\times j^\nu(k',\tau)e^{i(k+k')\cdot x}\rangle]&\end{aligned} \tag{2.18}$$

with

$$D_{\mu\nu}(k,k')=-\frac{1}{k^2}\left[g_{\mu\nu}-(1-\lambda)\frac{k_\mu k_\nu}{k^2}+i\epsilon\right]\times\delta^4(k+k'). \tag{2.19}$$

The momentum integral

$$f(\tau)=-\frac{i}{2}\langle j^{\mu}(k,\tau)D_{\mu\nu}(k,k')\times j^{\nu}(k',\tau)\exp[i(k+k')\cdot x]\rangle \tag{2.20}$$

is IR finite but UV divergent. To deal with the UV singularities, we employ the dimensional regularization rather than the more traditional Pauli-Villars regularization used in Ref. 7. The virtue of the dimensional regularization is that it automatically preserves internal symmetries which do not involve the γ_5 matrix. The integration over the loop momentum k is shown in Appendix A. The result is[17]

$$f(\tau)=a\left[\frac{1}{\epsilon}+\ln(\tilde{m}\tau)+O(\epsilon)\right] \tag{2.21}$$

so that

$$G(p)=ie^{a/\epsilon}(2\pi^{3/2}\mu)^{a}e^{(2+3\gamma/2)a}\times\Gamma(a+1)(im-iu\cdot p-\tilde{\epsilon})^{-(1+a)}\ , \tag{2.22}$$

with $\epsilon=4-d$ and

$$\ln(\tilde{m}\tau)=\tfrac{1}{2}\ln(4\pi^{3}\mu^{2}\tau)+2+\tfrac{3}{2}\gamma\ , \tag{2.23}$$

$\gamma=0.5772\ldots$ being the Euler-Mascheroni constant. Here μ is the regularization scale parameter and $\mathrm{Re}(a+1)>0$. Multiplicative renormalization in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme[18] yields

$$G_{\mathrm{ren}}(p)=i\left[\frac{\bar{\mu}}{2}\right]^{a}e^{2a(\gamma+1)}\times\Gamma(a+1)(im-iu\cdot p-\tilde{\epsilon})^{-(1+a)}\ , \tag{2.24}$$

where the renormalization scale is $\bar{\mu}=(4\pi/e^{\gamma})^{1/2}\mu$. This solution has a singularity structure of the form (1.1), which, as mentioned in the Introduction, is characteristic of an intraparticle. It is equivalent to the solution calculated by Svidzinskii[7] within the Pauli-Villars regularization scheme, to Zwanziger's solution[6] obtained on the basis of the Kulish-Faddeev formalism,[5] and to a more recent solution derived by Harada and Kubo[19] by means of the improved perturbative technique of Grammer and Yennie.[20]

For the renormalization constants we obtain[21]

$$Z_2=e^{a/\epsilon},\quad Z_3=Z_\lambda=1,\quad Z_m=1 \tag{2.25}$$

(i.e., the coupling of a soft photon to the electron is finite) and thus the anomalous dimension of the electron is given by

$$\gamma_F(\alpha,\lambda)=\frac{\mu}{Z_2}\frac{\partial Z_2}{\partial\alpha}\frac{\partial\alpha}{\partial\mu}=-a=-\frac{\alpha}{2\pi}(3-\lambda)\ . \tag{2.26}$$

Note that in the Fried-Yennie gauge,[22] $\lambda=3$, $\gamma_F=0$ so that the soft-photon effect disappears and the full electron propagator (2.24) becomes equal to the free one.

III. n-FERMION GREEN'S FUNCTION

We now proceed to examine the general case of n fermions. It is a major virtue of our treatment that the one-particle analysis just completed can be expanded to an arbitrary number of fermions.

The n-fermion Green's function analogous to (2.4) is $(i,j=1,\ldots,n)$

$$G_E^{(n)}(x_i,y_i)=(-1)^n\left[\frac{\delta^{2n}Z_E[J,\bar{\eta}^{i},\eta^{i}]}{\prod_{i=1}^{n}\bar{\eta}^{i}(x_j)\eta^{i}(y_j)}\right]_{J=\bar{\eta}^{i}=\eta^{i}=0}$$
$$=N\int[dA_\mu]\prod_{i=1}^{n}\{\det[iG_E^{i}(x_j,y_j\,|\,A)]G_E^{i}(x_j,y_j\,|\,A)\}\exp[-\tfrac{1}{2}\langle A_\mu(x)D^{-1}_{\mu\nu,E}(x,y)A_\nu(y)\rangle] \tag{3.1}$$

because

$$G_E^{(n)}(x_j,y_j\,|\,A)=\prod_{i=1}^{n}G_E^{i}(x_j,y_j\,|\,A)\ .$$

Proceeding along similar lines as for the evaluation of Eq. (2.15), the n-fermion Green's function in a classical external field can be readily obtained:

$$G_E^{(i)}(x_j,y_j\,|\,A)=\int_0^\infty d\tau^{i}\int\frac{d^4p}{(2\pi)^4}\exp[-ip^{i}_{\mu}(x^{i}_{\mu}-y^{i}_{\mu})-i\tau^{i}(m^{i}+u_\mu p^{i}_{\mu})-\langle j_\mu(k,\tau^{i})A_\mu(k)e^{ik_\nu x^{i}_\nu}\rangle]\ , \tag{3.2}$$

where

$$j_\mu(k,\tau_i)=\frac{e}{(2\pi)^2}u_\mu\int_0^{\tau_i}d\tau' e^{-iu_\nu k_\nu\tau'} \tag{3.3}$$

and $\dot{x}^{i}_{\mu}=u_\mu=\mathrm{const}$. Inserting expression (3.2) into Eq. (3.1) and applying the same lines of argument as in the one-fermion analysis, the Fourier transform of $G_E^{(i)}(x_j,y_j)$ continued back to Minkowski space is given by

$$G^{(n)}(p_i)=i\int_0^\infty \prod_{i=1}^{n} d\tau_i e^{-i\tau_i(m_i-u\cdot p_i-i\epsilon)}\exp\left[-\frac{i}{2}\langle J_\mu^{(n)}(k,\tau_j,x_j)D^{\mu\nu}(k,k')J_\nu^{(n)}(k',\tau_k,x_k)\rangle\right], \tag{3.4}$$

where the total current is defined by

$$J_\mu^{(n)}(k,\tau_i,x_i)=\sum_{i=1}^{n} j_\mu(k,\tau_i)e^{ik\cdot x_i} . \tag{3.5}$$

Using now the dimensional regularization to control the UV behavior of Eq. (3.4), the momentum integral

$$f^{(n)}(\tau_i,\tau_j)=-\frac{i}{2}\langle J_\mu^{(n)}(k,\tau_i,x_i)D^{\mu\nu}(k,k')J_\nu^{(n)}(k',\tau_j,x_j)\rangle \tag{3.6}$$

is found to be (see Appendix B)

$$f^{(n)}(\tau_i,\tau_j)=a\left[\frac{n^2+n}{2}\frac{1}{\epsilon}+\ln\left[\frac{\prod_{i=1}^{n}\tau_i^n}{\prod_{i\neq j}^{n}|\tau_i-\tau_j|^{1/2}}\right]+\frac{n^2+n}{2}\ln\tilde{m}\right], \tag{3.7}$$

which inserted into (3.4) leads to

$$G^{(n)}(p)=i\exp\left[\frac{n^2+n}{2}\frac{a}{\epsilon}\right]\tilde{m}^{(n^2+n)a/2}\int_0^\infty\prod_{k=1}^{n} d\tau_k e^{-i\tau_k(m_k-u\cdot p_k-i\epsilon)}\left[\frac{\prod_{i=1}^{n}\tau_i^n}{\prod_{i\neq j}^{n}|\tau_i-\tau_j|^{1/2}}\right]^a . \tag{3.8}$$

Then the full n-fermion Green's function with renormalization carried out in the $\overline{\text{MS}}$ scheme is

$$G_{\overline{\text{MS}}}^{(n)}(p_i)=i\left[\frac{\bar{\mu}}{2}\right]^{(n^2+n)a/n}e^{(n^2+n)(\gamma+1)a}\int\prod_{k=1}^{n} d\tau_k e^{-i\tau_k(m_k-u\cdot p_k-i\epsilon)}\left[\frac{\prod_{i=1}^{n}\tau_i^n}{\prod_{i\neq j}^{n}|\tau_i-\tau_j|^{1/2}}\right]^a \tag{3.9}$$

with the renormalization constant

$$Z_2=\exp\left[\frac{n+1}{2}\frac{a}{\epsilon}\right] \tag{3.10}$$

and the anomalous dimension

$$\gamma_F^{(n)}(\alpha,\lambda)=-\frac{n+1}{2}\frac{\alpha}{2\pi}(3-\lambda) . \tag{3.11}$$

One notices that, analogously to the one-particle case, the choice of the Fried-Yennie gauge $\lambda=3$ leads to a renormalized Green's function which reduces to the free one. Otherwise ($\lambda\neq3$) $G_{\overline{\text{MS}}}^{(n)}(p_i)$ has no simple poles and, as we shall see below, it exhibits a complicated singularity behavior. On the other hand, keeping $|\lambda|<3$ the Green's function (3.9) falls off at short distances, i.e., for large momenta p_i, with an increasing number of fermions.

In the remainder of this section we shall specialize to the case of two fermions. This case is of particular interest because detailed knowledge of fermion-fermion scattering amplitudes has accumulated over the years.[23] However, most of these results have been derived by employing powerful but, in effect, perturbative techniques. Therefore, *nonperturbative* solutions for the two-fermion Green's function may help to accomplish these results.

As it is shown in Appendix C, for two fermions the proper-time integrations in (3.9) can be explicitly performed, and the result is[17]

$$G_{\overline{\text{MS}}}(p_1,p_2)=i\left[\frac{\bar{\mu}}{2}\right]^{3a}e^{6(\gamma+1)a}\frac{\Gamma(2a+1)\Gamma(1-a)\Gamma(3a+2)}{\Gamma(a+2)}$$
$$\times\left[q_1^{-(3a+2)}\,{}_2F_1\left[2a+1,3a+2;a+2;-\frac{q_2}{q_1}\right]q_2^{-(3a+2)}\,{}_2F_1\left[2a+1,3a+2;a+2;-\frac{q_1}{q_2}\right]\right], \tag{3.12}$$

where the quantities $q_i=i(m_i-u\cdot p_i-i\epsilon)$ $(i=1,2)$ measure fractional distance from the mass shell. The validity of Eq. (3.12) is gauge dependent. Convergence is preserved by restricting the gauge parameter to take values on the interval

$-857<\lambda<433$ ($\alpha=\frac{1}{137}$). Note that this is the largest continuous λ interval found to contain the Landau gauge $\lambda=0$. One checks that Eq. (3.12) amounts for $a=0$ to the free two-fermion propagator:

$$G_{\overline{\mathrm{MS}}}(p_1,p_2,a=0)=i\left[q_1^{-2}\,{}_2F_1\left[1,2;2;-\frac{q_2}{q_1}\right]+q_2^{-2}\,{}_2F_1\left[1,2;2;-\frac{q_1}{q_2}\right]\right]. \tag{3.13}$$

Noticing that ${}_2F_1(-a,b;b;-z)=(1+z)^a$ (see Ref. 24), this expression reduces to

$$G_{\overline{\mathrm{MS}}}(p_1,p_2,a=0)=\frac{i}{q_1q_2}=\frac{-i}{(m_1-u\cdot p_1-i\epsilon)(m_2-u\cdot p_2-i\epsilon)}. \tag{3.14}$$

Using now Euler's integral representation of the hypergeometric function,[25] Eq. (3.12) can be recast in the form

$$G_{\overline{\mathrm{MS}}}(p_1,p_2)=i\left[\frac{\mu}{2}\right]^{3a}e^{6(\gamma+1)a}\Gamma(3a+2)\left[q_1^{-(3a+2)}\int_0^1 dt\, t^{2a}(1-t)^{-a}\left[1+\frac{q_2}{q_1}t\right]^{-(3a+2)}\right.$$
$$\left.+q_2^{-(3a+2)}\int_0^1 dt\, t^{2a}(1-t)^{-a}\left[1+\frac{q_1}{q_2}t\right]^{-(3a+2)}\right], \tag{3.15}$$

which can be treated numerically, and which is more suitable to be expanded in terms of the coupling constant α:

$$G_{\overline{\mathrm{MS}}}(p_1,p_2)=\frac{1}{q_1q_2}\{1+a[3\ln(\bar{\mu}\pi)+3(\gamma+1)+2(q_1\ln q_1+q_2\ln q_2)-5(q_2\ln q_1+q_1\ln q_2)]+O(a^2)\}. \tag{3.16}$$

The physical content of the two-fermion Green's function may become more transparent by studying its behavior near the mass shell. We consider two cases.

(i) For $q_1\neq q_2$ we expand expression (3.15) in powers of q_1 to obtain

$$G_{\overline{\mathrm{MS}}}(p_1^2\to m_1^2,p_2)=i\left[\frac{\bar{\mu}}{2}\right]^{3a}e^{6(\gamma+1)a}\left[q_1^{-(1+a)}q_2^{-(2a+1)}\Gamma(1+a)\Gamma(2a+1)+q_1^{-a}q_2^{-(2a+2)}\Gamma(1+a)\Gamma(2a+2)\right.$$
$$+q_2^{-(3a+2)}\left[\frac{\Gamma(2a+1)\Gamma(1-a)\Gamma(3a+2)}{\Gamma(a+2)}+\frac{\Gamma(-1-a)\Gamma(1-a)\Gamma(3a+2)}{\Gamma(-2a)}\right]$$
$$\left.+O(\min(q_1,q_1^{1-a}))\right]. \tag{3.17}$$

(ii) For $q_1=q_2$ the particles have the same virtuality and, expanding (3.15) in the limit $p_1^2\to m_1^2$, the result is

$$G_{\overline{\mathrm{MS}}}(q_1=q_2)=i\left[\frac{\bar{\mu}}{2}\right]^{3a}e^{6(\gamma+1)a}2^{-a}\pi^{-1/2}\Gamma(2a+1)\times\Gamma\left[\frac{3a+2}{2}\right]\Gamma\left[\frac{1-a}{2}\right]q_1^{-(3a+2)}. \tag{3.18}$$

The asymmetry in the limits involved in Eq. (3.12) or Eq. (3.15) shows that the physical poles of the Green's function are interrelated. Hence, the mass-shell behavior of the two-fermion Green's function depends on the order in which the two particles are allowed to approach their mass shells. In addition, the mass-shell singularities in expressions (3.17) and (3.18) depend on the choice of gauge. For example, for gauges $3<\lambda<433$ the Green's function is *less* singular than in a theory without massless fields (photons).

The singularity behavior of the two-fermion Green's function for momentum values near their mass shells has been discussed previously by Kibble[26] in the context of asymptotic coherent states. Although our analysis is consistent with his and the results correspond to each other, there are also crucial differences. Kibble's solutions have been calculated through a leading-order summation of the perturbative series on the theoretical basis of a nonseparable Hilbert space. In particular, his treatment of the momentum integral in (3.4) extends only over a subregion Ω_{soft} which contains momenta $|k_0|\le K_0$ and $|k_i|\le K$ with $K\ll K_0\ll m$. This separation between hard and soft photons is not covariant.[4] In contrast, our method does not handle soft- and hard-photon contributions separately; i.e., no IR cutoff is involved in the calculations. Furthermore, careful comparison reveals[17] that solution (3.17) includes *additional* powers of $(m_i-u\cdot p_i-i\epsilon)^{-1}$ which are not accounted for in Kibble's formalism. To be specific, the term proportional to $q_1^{-(1+a)}q_2^{-(2a+1)}$ agrees with Kibble's result, while the extra term proportional to $q_1^{-a}q_2^{(2a+2)}$ and the one proportional to $q_2^{-(3a+2)}$ represent additional singularity contributions due to IR photons.

IV. SUMMARY

A set of new solutions for the fermionic Green's functions of the BN-approximated QED has been obtained. The problem has been solved in closed form to all orders in the coupling constant and for arbitrary gauges. For convenience, the results have been presented in the $\overline{\text{MS}}$ renormalization scheme.

From the calculational viewpoint, a method has been used which permits clear insights into the physical content of the model and works equally well for one as for more fermions. This method has the advantages of being independent of assumptions concerning the structure of asymptotic states and being able to incorporate contributions owing to quantum fluctuations around the classical trajectory in a systematic manner. An illustration of the method has been given with the calculation of the electron propagator.

In the case of n fermions a formal solution has been derived [Eq. (3.9)]. It has been shown that restricting the gauge parameter to values $|\lambda|<3$, the renormalized n-fermion Green's function gets, at short distances, more and more damped as the number of fermions increases.

In the case of one and two fermions explicit expressions have been obtained [Eqs. (2.24) and (3.12)]. They are consistent with solutions found previously by others[4,6–9] using different techniques. Analyzing the mass-shell behavior of the two-fermion Green's function, it has been shown that its physical singularities are interrelated and the result depends on whether the two particles are allowed to approach their mass shells sequentially [Eq. (3.17)] or simultaneously [Eq. (3.18)]. This interdependence has been discussed previously by Kibble[26] in connection with asymptotic coherent states and a nonseparable Hilbert space. Restricting the analysis to the IR sector, and recalling that spin does not matter for IR phenomena, Eqs. (3.17) and (3.18) correspond to Kibble's results, though his solution equivalent to (3.17) contains less powers of $(m_i - u\cdot p_i - i\epsilon)^{-1}$ $(i=1,2)$.

In conclusion we remark that our results could be used to check results obtained by other, less rigorous, nonperturbative methods, e.g., quenched QED on the lattice.

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APPENDIX A

In this appendix we derive (2.21) starting from (2.20). Applying dimensional regularization and substituting (2.17) and (2.19) into (2.20), the expression we have to evaluate reads

$$f_E(\tau,d) = -\frac{e^2\mu^{4-d}}{(2\pi)^d}\int d^dk \frac{1-\cos(u_\mu k_\mu \tau)}{(u_\nu k_\nu)^2 k^2} \times \left[1-(1-\lambda)\frac{(u_\sigma k_\sigma)^2}{k^2}\right], \quad \text{(A1)}$$

where the Euclidean metric has been used and μ is an arbitrary scalar with the dimension of mass. Using polar coordinates in d dimensions, the integration over the angles $\phi,\theta_1,\ldots,\theta_{d-2}$ can be carried out[27] so that

$$f_E(\tau,\epsilon) = -\frac{e^2\mu^\epsilon}{(2\pi)^{3-\epsilon}} \frac{\pi^{(1-\epsilon)/2}}{\Gamma\left[\frac{3-\epsilon}{2}\right]} \times \int_0^\pi d\theta \frac{\sin^{2-\epsilon}\theta}{\cos^2\theta}[1-(1-\lambda)\cos^2\theta] \times \int_0^\infty d|k||k|^{-1-\epsilon} \times [1-\cos(|u||k|\cos\theta\tau)] . \quad \text{(A2)}$$

Here we have introduced the defect of the dimension $\epsilon=4-d$, and the abbreviation $\theta=\theta_{d-2}$.

We proceed evaluating first the integral over $|k|$ (see Ref. 28):

$$f_E(\tau,\epsilon) = -\frac{e^2\mu^\epsilon}{(2\pi)^{3-\epsilon}} \frac{\pi^{(1-\epsilon)/2}}{\Gamma\left[\frac{3-\epsilon}{2}\right]} \Gamma(-\epsilon)\cos\left[\frac{\pi\epsilon}{2}\right]\tau^\epsilon \times \int_0^\pi d\theta \frac{\sin^{2-\epsilon}\theta}{|\cos\theta|^{2-\epsilon}}[1-(1-\lambda)\cos^2\theta] . \quad \text{(A3)}$$

The remaining θ integration will be performed making use of the integral

$$I_1 = \int_0^\pi d\theta \frac{\sin^{2-\epsilon}\theta}{|\cos\theta|^{2-\epsilon}} = 2\int_0^\infty \frac{x^{2-\epsilon}}{1+x^2}dx \quad \text{(A4)}$$

with $1<\epsilon<3$. To ensure the validity of the limiting process $\epsilon\to 0$, the range of convergence of I_1 has to be extended by analytical continuation:

$$I_1' = -\frac{4}{1-\epsilon}\int_0^\infty \frac{x^{2-\epsilon}}{(1+x^2)^2}dx \quad (-1<\epsilon<3) . \quad \text{(A5)}$$

Then, I_1' and the momentum integral in (A1) have a common range of convergence, namely, the interval $0<\epsilon<2$, and the x integration can be carried out:[29]

$$I_1' = -\frac{2}{1-\epsilon}\Gamma\left[\frac{3-\epsilon}{2}\right]\Gamma\left[\frac{1-\epsilon}{2}\right] . \quad \text{(A6)}$$

Similarly, one finds

$$I_2 = \int_0^\pi d\theta \sin^{2-\epsilon}\theta|\cos\theta|^\epsilon = 2\int_0^\infty \frac{x^{2-\epsilon}}{(1+x^2)^2}dx = \Gamma\left[\frac{3-\epsilon}{2}\right]\Gamma\left[\frac{1-\epsilon}{2}\right] \quad (0<\epsilon<2) . \quad \text{(A7)}$$

Hence, $f_E(\tau,\epsilon)$ is given by

$$f_E(\tau,\epsilon) = -\frac{e^2\mu^\epsilon}{(2\pi)^{3-\epsilon}}\pi^{(1-\epsilon)/2}\Gamma(-\epsilon)\Gamma\left[\frac{1-\epsilon}{2}\right] \times \cos\left[\frac{\pi\epsilon}{2}\right]\tau^\epsilon\left[\frac{2}{1-\epsilon}+(1-\lambda)\right] . \quad \text{(A8)}$$

Finally, by expanding (A8) in a Laurent series at $\epsilon=0$ and keeping only the leading terms, one arrives at (2.21).

APPENDIX B

We here mention the steps leading to Eq. (3.7). By substituting (2.19) and (3.5) into (3.6), we obtain

$$f^n(\tau_i,\tau_j)=\frac{i}{2}\frac{e^2}{(2\pi)^4}\int d^4k\frac{1-(1-\lambda)(u\cdot k)^2/k^2+i\epsilon}{k^2(u\cdot k)^2}\times\sum_{i,j}(e^{iu\cdot k\tau_i}-1)(e^{-iu\cdot k\tau_j}-1)\times e^{-ik\cdot(x_i-x_j)} . \tag{B1}$$

We consider two cases.

(i) $i=j$:

$$f_1^n(\tau_i,\tau_j)=\frac{ie^2}{(2\pi)^4}\int d^4k\frac{k^2-(1-\lambda)(u\cdot k)^2}{k^4(u\cdot k)^2}\times\sum_i[1-\cos(u\cdot k\tau)] =\sum_i f(\tau_i) . \tag{B2}$$

(ii) $i\neq j$: Here we assume that each fermion moves on a free trajectory with constant velocity u^μ so that $x_i^\mu(\tau_i)=u^\mu\tau_i$. Then (B1) takes on the form

$$f_2^n(\tau_i,\tau_j)=\frac{i}{2}\frac{e^2}{(2\pi)^4}\int d^4k\frac{k^2-(1-\lambda)(u\cdot k)^2}{k^4(u\cdot k)^2}\sum_{i\neq j}([1-\cos(u\cdot k\tau_i)]+[1-\cos(u\cdot k\tau_j)]-\{1-\cos[u\cdot k(\tau_i-\tau_j)]\}) \tag{B3}$$

$$=\tfrac{1}{2}\sum_{i\neq j}[f(\tau_i)+f(\tau_j)-f(|\tau_i-\tau_j|)] , \tag{B4}$$

where the symmetry properties of the momentum loop integral have been used to pass from (B3) to (B4).

By combining the two possibilities, we get

$$f^n(\tau_i,\tau_j)=f_1^n(\tau_i,\tau_j)+f_2^n(\tau_i,\tau_j)=n\sum_i f(\tau_i)-\tfrac{1}{2}\sum_{i\neq j}f(|\tau_i-\tau_j|) \tag{B5}$$

and with the use of (A8) we finally derive (3.7).

APPENDIX C

Equation (3.12) is obtained in the following manner. One starts from (3.9) evaluating for two fermions:

$$G_{\overline{\rm MS}}(p_1,p_2)=i\left[\frac{\bar\mu}{2}\right]^{3a}e^{6(\gamma+1)a}\int_0^\infty d\tau_1\int_0^\infty d\tau_2 e^{-i\tau_1(m_1-u\cdot p_1-i\epsilon)}e^{-i\tau_2(m_2-u\cdot p_2-i\epsilon)}\left[\frac{\tau_1^2\tau_2^2}{|\tau_1-\tau_2|}\right]^a . \tag{C1}$$

Consider now the integral

$$I=\int_0^\infty d\tau_1\int_0^\infty d\tau_2 e^{-\tau_1 q_1}e^{-\tau_2 q_2}\left[\frac{\tau_1^2\tau_2^2}{|\tau_1-\tau_2|}\right]^a , \tag{C2}$$

where $q_i=i(m_i-u\cdot p_i-i\epsilon)$ with $i=1,2$.

There are then two possibilities: $\tau_2<\tau_1$ or $\tau_1<\tau_2$. Hence

$$I=\int_0^\infty d\tau_1\int_0^\infty d\tau_2 e^{-\tau_1 q_1}e^{-\tau_2 q_2}\left[\frac{\tau_1^2\tau_2^2}{\tau_1-\tau_2}\right]^a\Theta(\tau_1-\tau_2)+\int_0^\infty d\tau_1\int_0^\infty d\tau_2 e^{-\tau_1 q_1}e^{-\tau_2 q_2}\left[\frac{\tau_1^2\tau_2^2}{\tau_2-\tau_1}\right]^a\Theta(\tau_2-\tau_1)$$
$$\equiv I_1+I_2 . \tag{C3}$$

It suffices to calculate I_1 because I_2 can be inferred from I_1 by interchanging q_1 and q_2. Upon performing the τ_2 integration,[28] one finds[17]

$$I_1=\frac{\Gamma(2a+1)\Gamma(1-a)}{\Gamma(a+2)}\times\int_0^\infty d\tau_1\tau_1^{3a+1}e^{-\tau_1 q_1}{}_1F_1(2a+1;a+2;-q_2\tau_1) , \tag{C4}$$

which holds for $-857<\lambda<433$. The remaining integration over τ_1 can be carried out by utilizing the relation[28]

$$\int_0^\infty f(t)e^{-pt}dt =\Gamma(\sigma)p^{-\sigma}{}_{m+n}F_n\left[a_1\cdots a_m;\sigma;\rho_1\cdots\rho_n;\frac{\lambda}{p}\right] \tag{C5}$$

with the inverse Laplace transformation

$$f(t)=t^{\sigma-1}\,{}_mF_n(a_1\cdots a_m;\rho_1\cdots\rho_n;\lambda)\ , \tag{C6}$$

provided $\mathrm{Re}\sigma>0$ and $\mathrm{Re}p>\mathrm{Re}\lambda$.

In our case these conditions are satisfied:

$$\sigma=3a+2>0 \quad \text{with} -\tfrac{1}{2}<a<1 \tag{C7}$$

and

$$\lambda=q_2 \to \mathrm{Re}\lambda=-\epsilon\ , \tag{C8}$$

$$p=q_1 \to \mathrm{Re}p=\epsilon\ . \tag{C9}$$

Thus, we obtain

$$I_1=\frac{\Gamma(2a+1)\Gamma(1-a)\Gamma(3a+2)}{\Gamma(a+2)}q_1^{-(3a+2)} \times {}_2F_1\left[2a+1,3a+2;a+2;-\frac{q_1}{q_2}\right] \tag{C10}$$

and a similar result for I_2 ($q_1 \leftrightarrow q_2$ in I_1). Inserting these expressions into (C1), one easily arrives at (3.12).

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