Accumulation-point approach to lattice field theory

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For a recently proposed novel action-variational approach refined criteria are worked out. Results of fourth-order computations for SU(2) gauge theory in four and five dimensions are presented and ones for U(1) theory reanalyzed. Universal features independent of gauge group and dimension become apparent.

Monte Carlo simulations of gauge theories have led to considerable insight. Nevertheless the development of other nonperturbative methods remains desirable, in particular, to get beyond the simulation region and for the investigation of basic mechanisms. This suggests a reconsideration of action-variational approaches which in lattice gauge theory started with mean-field methods.[1-3] The cumulant expansion of the free energy introduced next[4] suffers from the fact that the convexity inequality is the only true criterion available. Then using an expansion of correlation functions, which are the objects of actual interest, the Schwinger-Dyson equation has been imposed to fix the variational parameter in first-order calculations.[5,6] A more detailed investigation[7] with calculations to second order revealed, however, that this method has severe limitations since in many cases there is no solution and one can at best rely on a crude approximation.

In a novel approach, which I proposed recently,[8] the variational parameter in the expansion of the correlation function is adjusted to a value where the sequence of finite-order approximations has an accumulation point (i.e., approaches a limit point arbitrarily close). Having only a finite number of orders available it is understood that their behavior should conform with the expected general behavior. In practice parameter values within a small interval appear acceptable. The optimal evaluation then requires selecting the parameter value of fastest convergence. It is important to note that the existence of the indicated accumulation is necessary in order that the use of the series makes sense.

In the fourth-order calculations for U(1) gauge theory, which I performed to demonstrate the properties of the method,[8] it turned out that an accumulation exists above the crossover region from strong to weak coupling in the neighborhood of a parameter point roughly of order of magnitude of the mean-field estimate. The precise parameter value of this point depends on the particular function and is found to be crucial for quantitative results. When the crossover region is reached from the weak-coupling side the accumulation suddenly disappears. This provides a signal for the location of the phase transition (though not for its nature or existence). Proceeding further in the strong-coupling region an accumulation occurs again, now at zero trial action, i.e., at the parameter value giving a strong-coupling expansion. It is of interest to check if these features are general ones which hold for other gauge groups and dimensions, too.

Working at (relatively small) finite order the question arises how to determine the parameter value of fastest convergence in practice. In Ref. 8 the intersection of the zeroth and nth orders has been used as a technical prescription mainly on empirical grounds. This is not completely satisfactory because general applicability with optimal exploitation of the computations as well as the understanding of detailed properties needed for further analytical developments call for properly motivated rules. To establish such rules the systematic features of more cases are needed.

There has been a third-order calculation[9] for SU(2) using the prescriptions of Ref. 8. However, in contradiction to the basic idea of Ref. 8, in Ref. 9 intersections have been evaluated in the crossover region where no accumulation exists which unfortunately does not make sense and cannot lead to correct conclusions.

In this paper the results of fourth-order computations for a number of Wilson loops in SU(2) gauge theory in four and five dimensions are presented and together with corresponding U(1) results[8] used to obtain refined rules for the accumulation-point method satisfying the requirements indicated above. The evaluation confirms that the accumulation features first noticed for U(1) theory are of general nature.

For correlation functions $\langle \cdots \rangle$ with respect to the action S one has the expansion

$$\langle X\rangle=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\langle (S-S_0)^{\nu}X\rangle_0^c\,, \tag{1}$$

where $\langle \cdots \rangle_0^c$ are connected functions with respect to the trial action S_0. Equation (1) may be written in the form

$$\langle X\rangle=\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{1}{(\nu-\mu)!}\frac{1}{\mu!}(\beta/2)^{\nu-\mu}(-\alpha)^{\mu}F_{\nu-\mu,\mu}\,. \tag{2}$$

In the case of SU(2) with

$$S=(\beta/2)\sum_p \mathrm{tr}\,U_p$$

and

$$S_0=\sum_l \mathrm{tr}(\tilde{J}U_l+JU_l^\dagger)\ ,$$

p running over plaquettes and l over links, after evaluation of the link integrals one gets

$$F_{p\mu}=\sum_j c_{p\mu j}\alpha^{-E_{p\mu j0}}V_1^{E_{p\mu j1}}\cdots V_m^{E_{p\mu jm}}\ , \tag{3}$$

where j numbers different configurations and V_s is the ratio of modified Bessel functions $I_{1+s}(2\alpha)/I_1(2\alpha)$. The functions at a particular link follow from the related partition function by suitable differentiations with respect to J and $\tilde{J}$. After this a major simplification is possible because, without restricting generality, one can put $J=\tilde{J}=(\alpha/2)\mathbb{1}$ (which has not been noticed in Refs. 6 and 9). Then the link contribution becomes a sum of terms which only contain factors of type δ_{ab}, ϵ_{ab}^{cd}, V_s, and α^{-1} (where $\epsilon_{ab}^{cd}=1,-1,0$ if ab even, odd, no permutation of cd). Therefore, noting that $\epsilon_{ab}^{cd}\epsilon_{ce}^{fg}=\epsilon_{ab}^{fg}\delta_{de}$, the matrix multiplications can be replaced by (fast) integer algorithms which involve the numbering of the links in the occurring loops. The integer constants $c_{p\mu j},E_{p\mu j0},\ldots,E_{p\mu jm}$ are determined by the possible geometric configurations. They were, similarly as the ones of the U(1) case, obtained by computer. The fourth-order computations became possible using the CDC CYBER 205 at Karlsruhe.

Considering rectangular Wilson loops, $\langle X\rangle$ in (1) is denoted by W and the approximation where the sum in (1) runs up to n by W_n. Figure 1 shows typical results for W_0 to W_4 as functions of $r=\alpha/[(d-1)\beta]$. Figure 1(a) gives an example of the accumulation in the weak-coupling region which occurs in the neighborhood of an r value of order of magnitude one. For lower β this accumulation disappears as is seen in the example of Fig. 1(b). In the strong-coupling region accumulation then forms at $r=0$. These features are observed for SU(2) in four and five dimensions for all Wilson loops calculated. Thus, as in the case of U(1), a definite expansion which makes sense exists in the weak-coupling region and in the strong-coupling region, however, not in the crossover region.

It turns out that in all cases the accumulation structure in the weak-coupling region follows two typical patterns related to the upper part and to the lower part of this region. In the upper part illustrated by Fig. 2(a) it is seen that instead of using the series of type $a_0+a_1+a_2+\cdots$ (with $a_0=W_0$, $a_1=W_1-W_0$, $a_2=W_2-W_1,\ldots$) better convergence is obtained by turning to the even-order series $a_0+\bar{a}_2+\bar{a}_4+\cdots$ where $\bar{a}_n=a_n+a_{n-1}$. In the lower part illustrated by Fig. 2(b) the original form of the series applies with the orders 2,3,4 being of dominating importance. Thus, if one wishes to exploit the information available at finite order as much as possible, for the evaluation the structures given schematically in Fig. 3 remain to be considered.

The transition from the upper to the lower part of the weak-coupling region occurs by W_3 going down and W_2 going up on the left-hand side of the patterns in Fig. 2. Therefore the separation of these parts can be character-

TABLE I. Characteristic values of accumulation regions.

	Loop	β_{234}	$\beta_{\min}$
SU(2)	1×1	2.18	2.05
$d=4$	2×1	3.67	2.15
	2×2	5.64	2.45
	3×1	3.81	2.55
	3×2	5.59	2.60
	3×3	5.55	2.85
SU(2)	1×1	1.79	1.70
$d=5$	2×1	2.86	1.75
	2×2	4.19	2.00
U(1)	1×1		0.51
$d=4$	2×1	0.73	0.50
	2×2	1.19	0.54

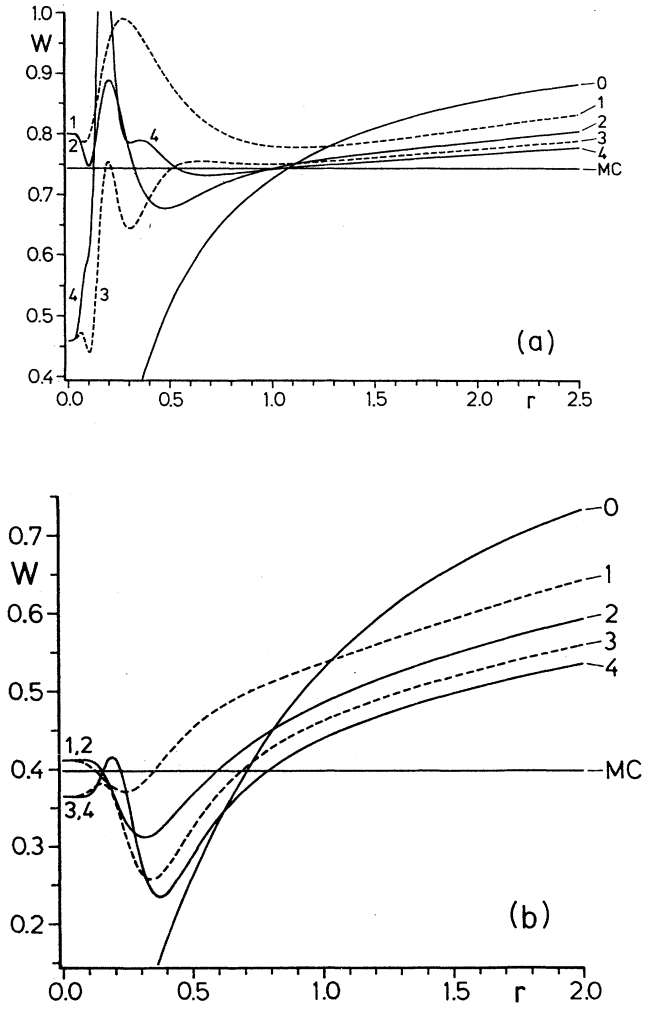
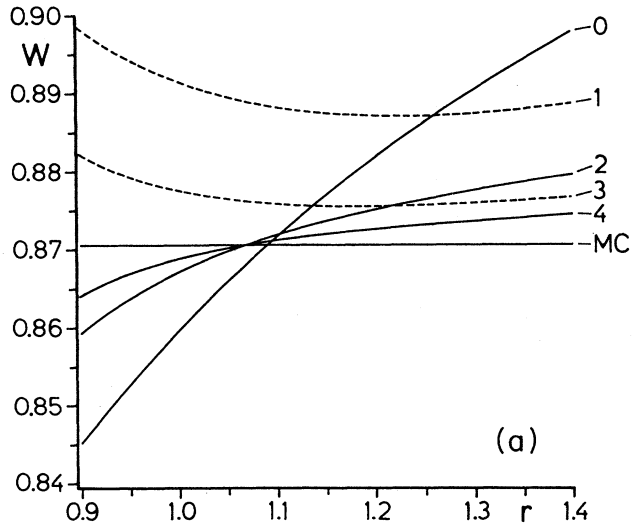


FIG. 1. Typical behavior of W_0 to W_4 as functions of r and Monte Carlo value. (a) Weak-coupling region, (b) lower region [shown for a 1×1 loop, SU(2), and $d=4$ at $\beta=3.2$ and 1.6445 with MC values from Refs. 11 and 10].

ized by the value β_{234} at which the orders 2,3,4 meet at one point. The value $\beta_{\min}$ at the lower end of the weak-coupling region, where the series breaks down, is determined by the disappearance of the structure in Fig. 3(b). Because of the very rapid change, inspection in steps of 0.05 for SU(2) and of 0.01 for U(1) allows us to get a reasonable approximation for $\beta_{\min}$. Table I gives an overview of the values β_{234} and $\beta_{\min}$ obtained. For the 1×1 loop in U(1) the upper type of structure extends down to the breakdown of the series. [The definition of β for U(1) used[8] is such that the phase transition is expected at 0.505.]

Looking for the parameter value of fastest convergence



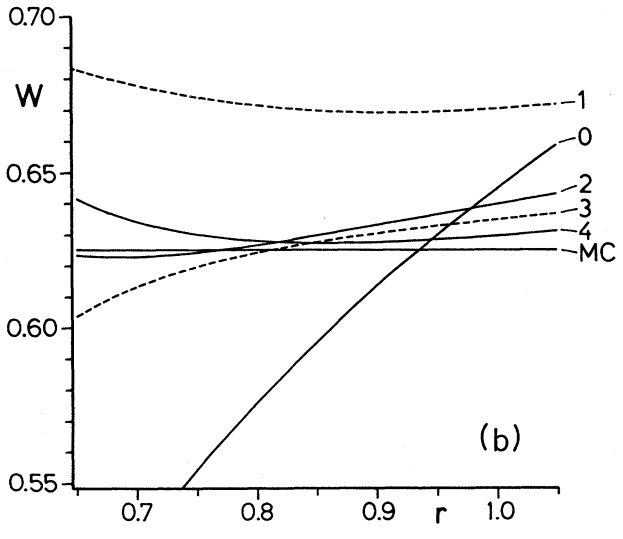


FIG. 2. Typical accumulation structure of the W_n in the weak-coupling region and Monte Carlo value. (a) Upper part, (b) lower part of region [shown for a 2×1 loop, SU(2), and $d=4$ at $\beta=10$ and 3.5 with MC values from Ref. 11].

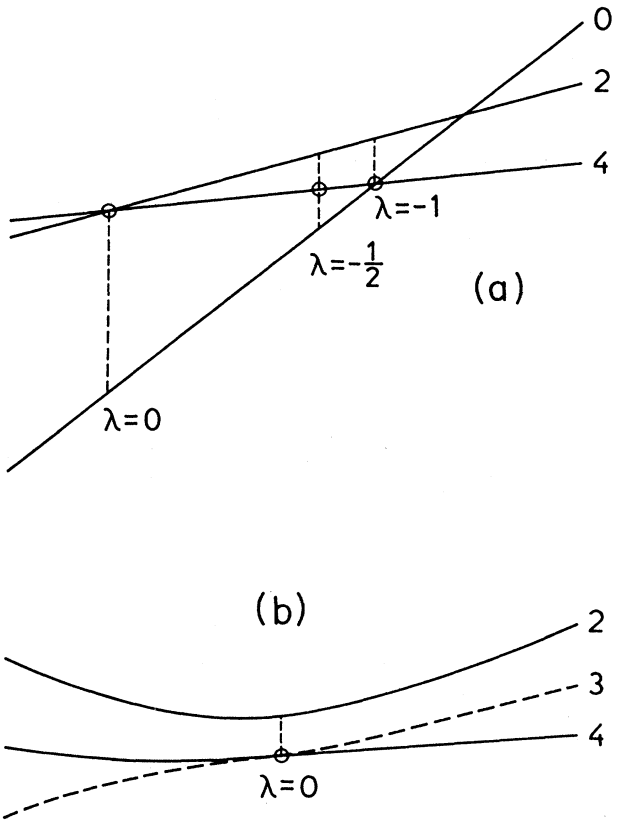


FIG. 3. Evaluation of accumulation structure (a) in upper part, (b) in lower part of weak-coupling region.

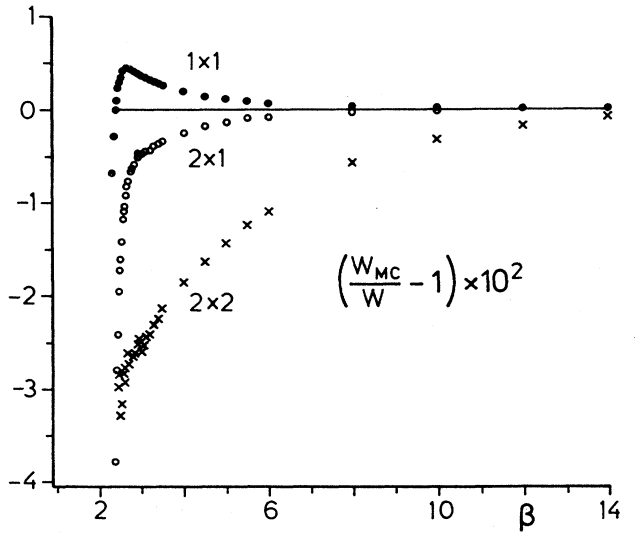


FIG. 4. Comparison of results W for 1×1, 2×1, and 2×2 loops for SU(2) and $d=4$ with Monte Carlo data (Ref. 11) (dots, 1×1; circles, 2×1; crosses, 2×2).

in the upper weak-coupling region the possibilities may be described by $\tilde{a}_4=\lambda\tilde{a}_2$. In Fig. 3(a) superficially $\lambda=-1$ shows minimal fluctuation; however, decreasing $\tilde{a}_4$ at the cost of $\tilde{a}_2$ appears more appropriate. Then, for example, $\lambda=-\frac{1}{2}$ with $W_4=(W_2+W_0)/2$ or $\lambda=0$ with $W_4=W_2$ may be considered. In the lower region, now with $a_4=\lambda a_3$, from Fig. 3(b) it is seen that in any case $\lambda=0$ is the most advantageous value since a_3 is also minimal there.

Figure 4 compares W, determined as W_4 at $\lambda=0$ in both regions, with Monte Carlo data.[11] For larger loops this ($\lambda=0$) prescription is seen to be all that can be motivated without further information. For the 1×1 loop Fig. 5 reveals that actually the case $\lambda=-\frac{1}{2}$ is approached with high precision. The corresponding comparisons for U(1) show the same situation there. In the following, special considerations for the 1×1 loop would give rise only to negligible changes. Therefore, for the present the ($\lambda=0$) prescription is adopted generally.

As in the case of U(1) also for SU(2) the α value at the point of optimal convergence depends on the particular function and its individual adjustment is important for getting precise results. To extract the nontrivial contributions of this α the quantity Δ, defined by $2\alpha/(V_1^3\beta)-2(d-1)$ for SU(2) and similarly for U(1), can be used. Δ is constructed to vanish in the simple mean-field case and to exhibit minimal d dependence. It turns out that Δ depends mainly on the loop size and on β in the lower region. It is in the range from 2.0 to -0.9 for the cases considered with larger values for smaller loops. There is only little dependence on gauge group and dimension. It should be realized here that the choice of α is β dependent, which, as generally in mean-field-type methods, is what makes the description of the weak-coupling region possible.

In the strong-coupling region only a small fraction of terms still contributes to (1) and much higher orders become easily accessible. For increasing β the series breaks down when the crossover region is reached. The details of this can be readily studied up to 15th order for the 1×1 loop using the results of Ref. 2. While in the range of convergence the magnitude of fluctuations decreases with order, it increases above this range. The first indication of this reversal occurs for 1×1, 2×1, and 2×2 loops for $d=4$ at $\beta=1.51$, 1.48, and 1.41 and for $d=5$ at $\beta=1.26$, 1.24, and 1.21 (which to see needs at least 5th, 6th, and 8th orders). These values signal the lower end of the crossover region.

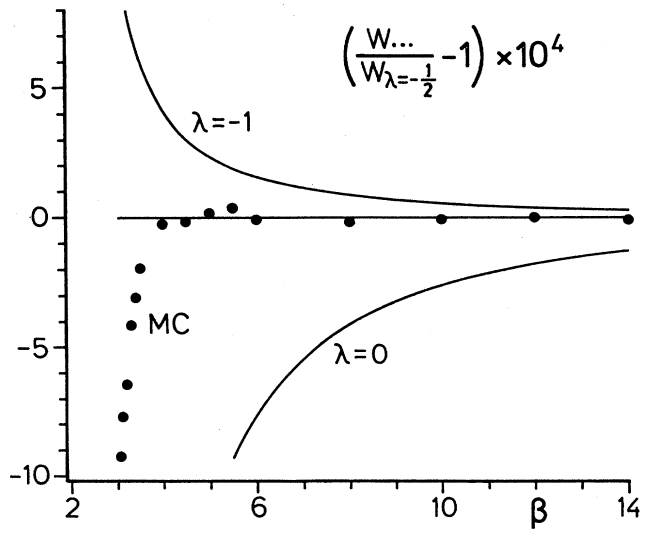


FIG. 5. Detailed behavior of W for a 1×1 loop, SU(2), and $d=4$ determined as W_4 at $\lambda=-1$, $-\frac{1}{2}$, 0, compared with Monte Carlo data (Ref. 11) (with $W...$ standing for $W_{\lambda=0}$, $W_{\lambda=-1/2}$, $W_{\lambda=-1}$, and W_{MC}).

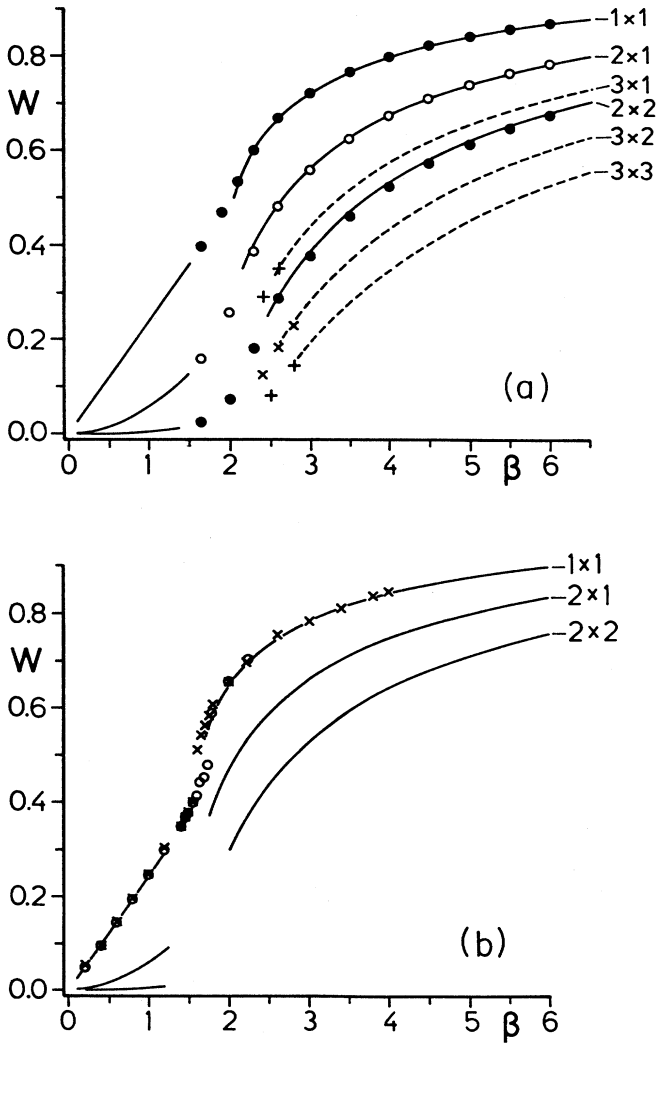


FIG. 6. Results W for SU(2) and various Wilson loops, (a) for $d=4$ compared with Monte Carlo data (Refs. 10–12) (dots, 1×1 and 2×2; circles, 2×1; pluses 3×1 and 3×3; crosses, 3×2), (b) for $d=5$ compared with Monte Carlo data (Ref. 13) (crosses, heating; circles, cooling; for 1×1).

In Fig. 6 the results for W are collected and compared with Monte Carlo data.[10–13] Obviously the agreement is good at this scale. In more detail Fig. 4 shows that to compete with high-statistics data some work still remains to be done. The thermal cycle in Fig. 6(b), indicating the phase transition for $d=5$, fits well to the beginning of the crossover region determined from the expansion.

Figure 7 shows results for the quantities

$$\chi(I,J)=-\ln\{W(I,J)W(I-1,J-1)\times[W(I,J-1)W(I-1,J)]^{-1}\}$$

introduced by Creutz[14] and $\chi(I,J)$ obtained from Monte Carlo data.[11,12] It is seen that the ending of the curves caused by the breakdown of the series occurs approximately where one would get the enveloping curve of the string-tension analysis.[14] From Fig. 4 it is apparent that the deviations of $\chi(2,2)$ are caused by the faster increase of accuracy of $W(2,1)$ as compared to $W(2,2)$ and that these deviations become small again for still larger β [as is observed for U(1), too[8]]. It should be noted here that $\chi(I,J)$ rely on the area sensitivity which for larger loops requires larger orders.

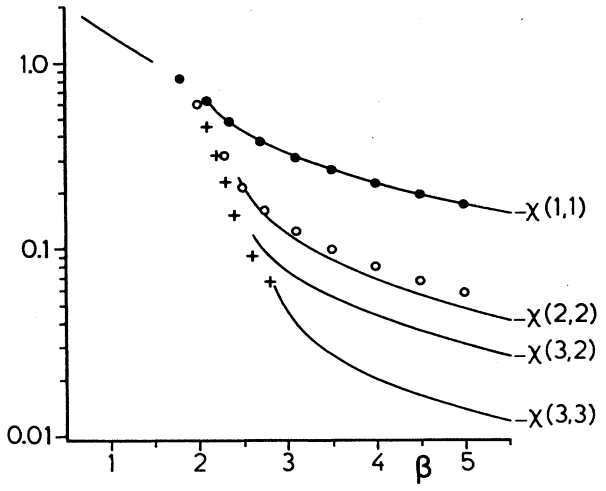


FIG. 7. $\chi(I,J)$ for SU(2) and $d=4$ compared with Monte Carlo results (Refs. 11 and 12) (dots, $I=J=1$; circles, $I=J=2$; pluses $I=J=3$).

It has turned out that the principles of accumulation and fastest convergence give rules to exploit the expansions in detail. A rather sharp signal for the beginning of the crossover region and the location of possible phase transitions is obtained. The occurring features are found to be largely independent of gauge group and dimension. The present results and experiences provide a basis to work on analytical modifications which increase the accuracy within a given order and on changes of the computational strategy reducing the CPU times for larger loops and orders. They also suggest investigating some properties rigorously.

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