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Discrete Bogomolny equations for the nonlinear O(3) σ model in $2+1$ dimensions

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Discrete analogues of the topological charge and of the Bogomolny equations are constructed for the nonlinear O(3) σ model in $2+1$ dimensions, subject to the restriction that the energy density be radially symmetric. These are then incorporated into a discretized version of the evolution equations. Using the discrete Bogomolny relations to construct the initial data for numerical simulations removes the "lattice wobble" sometimes observed at low kinetic energies. This feature is very important for the delicate question of instanton stability.

I. INTRODUCTION

The nonlinear O(3) σ model in $2+1$ dimensions has long been of interest; it has, for example, application as the continuum limit of the (classical) two-dimensional Heisenberg ferromagnet. Its minimum-energy static solutions (sometimes called instantons) were first obtained explicitly by Belavin and Polyakov.[1] The instantons have no fixed size and so their stability is a central question. There is a possibility that under small perturbations they could shrink toward infinitely tall spikes of zero width. General time-dependent solutions cannot be constructed explicitly, and so it is natural to investigate numerical evolution techniques which discretize the partial differential equations.

A study of instanton stability in the lattice Heisenberg model is hampered by the absence of explicit static solutions on the lattice, although some progress can still be made: for example, Voruganti[2] has recently investigated whether the inclusion of a Chern-Simons term can stabilize the instanton size. This paper will get around the problem by discretizing the (unmodified) O(3) model in such a way that there *are* explicit instantons on the lattice, which may then be used as the basis for a numerical study of instanton stability.

An intrinsic feature of the O(3) σ model is its nontrivial topology.[3] Briefly, all finite-energy field configurations belong to a particular topological sector, labeled by an integer-valued topological charge N. The charge N is conserved as the configuration evolves in time. The instanton solutions satisfy a set of first-order differential equations, namely, the Bogomolny (self-duality) equations, which themselves imply the full second-order field equations. Within each topological sector, the energy of these solutions saturates the so-called Bogomolny bound (a lower bound on the energy).

It is not clear how best to reconcile the nontrivial topological aspects with a lattice formulation, although there have been several proposals.[4–6] This paper will show that, by constructing a discrete analogue of the Bogomolny bound, one can devise a numerical evolution which incorporates the topology in a natural way and which possesses explicit instantons. In other words, the central idea is to find a set of first-order difference relations to play the role of the Bogomolny equations on a lattice. Solutions of these relations saturate the topological lower bound on the (discretized) energy.

Only field configurations for which the energy density (but not necessarily the fields themselves) is radially symmetric will be considered here. In principle this restriction could be lifted, but the construction of the discrete Bogomolny relations is then more complicated. However, for the problem of instanton stability, the restriction to axial symmetry is not a serious constraint, since nonaxial modes are unlikely to lead to instabilities. It is worth remarking that Mikhailov and Yaremchuk have considered imposing radial symmetry on the fields themselves.[7] Then the model is integrable via an inverse scattering method, but all the solutions obtained in this way have topological charge zero.

This paper is arranged as follows. The next section reviews the familiar Bogomolny bound and then reparametrizes the fields to impose radial symmetry. Only then is the model discretized and, in Sec. III, the discrete Bogomolny relations constructed. Section IV discusses their properties (with some of the more mathematical details gathered in an appendix), and in Sec. V they are incorporated into a full evolution scheme. Some preliminary results, indicating that the numerical model is working well, are presented in Sec. VI.

II. RADIAL SYMMETRY IN THE O(3) σ MODEL

To establish notation, recall that the O(3) σ model contains three real scalar fields, $\boldsymbol{\phi}\equiv(\phi_1,\phi_2,\phi_3)$, which are functions of the spacetime coordinates (t,x,y), sometimes also written (x_0,x_1,x_2). Imposing the constraint $\boldsymbol{\phi}\cdot\boldsymbol{\phi}=1$ on the free field theory gives rise to the field equations

$$\partial^\mu\partial_\mu\boldsymbol{\phi}+(\partial^\mu\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})\boldsymbol{\phi}=\mathbf{0}\ . \tag{2.1}$$

Here the greek indices take values 0,1,2 and label spacetime coordinates, and ∂_μ denotes partial differentiation with respect to x^μ. The kinetic and potential energies T and V, respectively, are given by

$$T=\tfrac{1}{4}\int(\partial_0\boldsymbol{\phi})\cdot(\partial_0\boldsymbol{\phi})d^2x\ , \tag{2.2}$$

$$V=\tfrac{1}{4}\int(\partial_i\boldsymbol{\phi})\cdot(\partial_i\boldsymbol{\phi})d^2x\ , \tag{2.3}$$

with the convention that latin indices only take the values

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1,2. Summation over repeated indices is implied throughout. The (integer-valued) topological charge is given by

$$N=\pm\frac{1}{8\pi}\int \epsilon_{ij}\boldsymbol{\phi}\cdot(\partial_i\boldsymbol{\phi}\times\partial_j\boldsymbol{\phi})d^2x \ , \tag{2.4}$$

where ϵ_{ij} is the antisymmetric symbol on two indices such that $\epsilon_{12}=-\epsilon_{21}=1$. The sign of N is determined by the signs of the individual components of $\boldsymbol{\phi}$, and also by the choice of the leading sign in Eq. (2.4). We shall assume that things are always arranged so that $N\geq 0$. [However, one should consistently choose either the upper or lower signs in Eqs. (2.4), (2.5), and (2.7).]

The Bogomolny bound arises from consideration of the identity

$$\int[(\partial_i\boldsymbol{\phi}\pm\epsilon_{ij}\boldsymbol{\phi}\times\partial_j\boldsymbol{\phi})\cdot(\partial_i\boldsymbol{\phi}\pm\epsilon_{ik}\boldsymbol{\phi}\times\partial_k\boldsymbol{\phi})]d^2x\geq 0 \ , \tag{2.5}$$

which may be recast using Eqs. (2.3) and (2.4) into the form

$$V\geq 2\pi N \ . \tag{2.6}$$

Equality in (2.6) holds if and only if

$$\partial_i\boldsymbol{\phi}\pm\epsilon_{ij}\boldsymbol{\phi}\times\partial_j\boldsymbol{\phi}=\mathbf{0} \ . \tag{2.7}$$

These are the Bogomolny equations. The instantons are the time-independent solutions of (2.7). They have zero kinetic energy; so V is equal to the total energy E and (2.6) becomes $E=2\pi N$.

The fields take values on a sphere S of unit radius, and it is often useful to relate $\boldsymbol{\phi}$ to a complex scalar field u by means of a stereographic projection from the north pole of S onto the complex u plane:

$$u=\frac{\phi_1+i\phi_2}{1-\phi_3} \ . \tag{2.8}$$

The radially symmetric instantons of charge N are given by $u=\lambda/z^N$ where $z=x+iy$ and λ is a real constant. Note that the global O(3) invariance of the model has now been removed by implicitly choosing $u\to\infty$ [equivalently $\boldsymbol{\phi}\to(0,0,1)$] as $z\to 0$, and by taking λ real. When $N=0$ the field is constant and the energy density is zero everywhere. For $N=1$ the instanton looks like a lump peaked at the origin, and for $N>1$ it is a ring centered on the origin and peaked at

$$r=\left[\frac{N-1}{N+1}\right]^{1/2N}\lambda^{1/N} \ , \tag{2.9}$$

where r is the polar radius in the xy plane. Physically, λ may be interpreted as the instanton size.

With these observations in mind, one may write down a more general family of radially symmetric configurations, namely,

$$u=\frac{\lambda(r,t)}{z^N} \ , \tag{2.10}$$

where now λ is allowed to be a (possibly complex) function of r and t. The remainder of this paper deals exclusively with these configurations. It seems that all radially symmetric energy densities can be derived from a field of the form (2.10), at least in some global gauge.

The field u provides a concise description of the radial configurations. However, it is not suited to numerical implementation, since it possesses a singularity (at the origin). Instead, one reverts to the $\boldsymbol{\phi}$ picture, in which (2.10) is equivalent to

$$\begin{aligned}\phi_1&=f(r,t)\cos N\theta+g(r,t)\sin N\theta \ ,\\ \phi_2&=-f(r,t)\sin N\theta+g(r,t)\cos N\theta \ ,\\ \phi_3&=h(r,t) \ ,\end{aligned} \tag{2.11}$$

where f, g, and h satisfy the constraint

$$f^2+g^2+h^2=1 \ , \tag{2.12}$$

and θ is the polar angle in the xy plane. Roughly speaking, f comes from the real part of λ and g from the imaginary part (so the instantons have $g=0$). The boundary conditions are $h(0)=1$ and, for $N\neq 0$, $h(\infty)=-1$.

Now one is in a position to reformulate the Bogomolny bound in terms of the single spatial coordinate r. Set $g\equiv 0$ in (2.11) (since for the moment we are concerned only with instanton solutions), choose f to be positive, and choose the lower signs in Eqs. (2.4)–(2.7). Then substituting for $\boldsymbol{\phi}$ in Eqs. (2.2) and (2.3) yields

$$T=\tfrac{1}{4}\int_0^\infty\left[\frac{h_t^2}{1-h^2}\right](2\pi r\,dr) \ , \tag{2.13}$$

$$V=\tfrac{1}{4}\int_0^\infty\left[\frac{h_r^2}{1-h^2}+\frac{N^2(1-h^2)}{r^2}\right](2\pi r\,dr) \ , \tag{2.14}$$

where the subscripts r and t denote partial differentiation. The topological charge density becomes simply

$$\rho\equiv-\frac{1}{8\pi}\epsilon_{ij}\boldsymbol{\phi}\cdot(\partial_i\boldsymbol{\phi}\times\partial_j\boldsymbol{\phi})=-\frac{Nh_r}{4\pi r} \ . \tag{2.15}$$

The Bogomolny bound $V\geq 2\pi N$ is essentially the identity

$$\tfrac{1}{4}\int_0^\infty\left[\frac{h_r}{\sqrt{1-h^2}}+\frac{N\sqrt{1-h^2}}{r}\right]^2(2\pi r\,dr)\geq 0 \tag{2.16}$$

and the Bogomolny equations are

$$rh_r+N(1-h^2)=0 \ . \tag{2.17}$$

Explicitly, the instanton solutions are given by

$$(f,g,h)=\left[\frac{2\lambda r^N}{\lambda^2+r^{2N}},0,\frac{\lambda^2-r^{2N}}{\lambda^2+r^{2N}}\right] \ , \tag{2.18}$$

for arbitrary real constant λ. Note that (provided $N\neq 0$) h decreases monotonically as a function of r, from $h=1$ at $r=0$, to $h=-1$ as $r\to\infty$.

III. BOGOMOLNY RELATIONS IN A DISCRETE FORMULATION

So far, all we have done is to reexpress the (radially symmetric) instanton solutions in terms of a single real field h, which is a function only of the polar radius r. It will now be seen how this description is useful in constructing discrete analogues of the topological charge and of the Bogomolny equations.

Consider a discrete set of values h_n ($n \in \mathbb{Z}$, $n \geq 0$) with the properties that $h_0 = 1$ and $h_n \to -1$ as $n \to \infty$. (In both this section and the next, the special case $N = 0$ will be avoided, in order to ensure $h_n \to -1$.) One expects the Bogomolny bound to take the general form

$$\sum_{n=0}^{\infty} (\alpha_n + \beta_n)^2 = V - 2\pi N \geq 0 \ . \qquad (3.1)$$

By analogy with Eqs. (2.5) and (2.16), the cross terms of the infinite sum in (3.1) should yield the topological charge; the remaining terms give the potential energy. So to fix up the charge one may take

$$\alpha_n \beta_n = \tfrac{1}{2}\pi N (h_{n+1} - h_n) \ . \qquad (3.2)$$

Turning to the energy, the form of (2.14) suggests the choices

$$\alpha_n = \left[\frac{\pi}{2}\right]^{1/2} N \frac{\sqrt{1-h_n^2}}{\sqrt{n}} \ , \qquad (3.3)$$

$$\beta_n = \left[\frac{\pi n}{2}\right]^{1/2} \frac{h_{n+1} - h_n}{\sqrt{1-h_n^2}} \ . \qquad (3.4)$$

The only problem with this is that (3.3) and (3.4) are undefined when $n = 0$. Clearly the origin must be treated in a special way. One solution is to arrange that $\alpha_0 + \beta_0 = 0$ identically, while still being consistent with (3.2). So choose

$$\alpha_0 = \left[\frac{\pi N}{2}\right]^{1/2} \sqrt{1-h_1} \ , \qquad (3.5)$$

$$\beta_0 = -\left[\frac{\pi N}{2}\right]^{1/2} \sqrt{1-h_1} \ . \qquad (3.6)$$

Putting the pieces together gives the following discretized potential energy:

$$V = \pi N (1-h_1) + \tfrac{1}{4} \sum_{n=1}^{\infty} (2\pi n) \left[\frac{(h_{n+1}-h_n)^2}{1-h_n^2} + \frac{N^2(1-h_n^2)}{n^2} \right] . \qquad (3.7)$$

Apart from the leading term, which comes from $\alpha_0^2 + \beta_0^2$, this is perhaps what one would have written down immediately as an analogue of (2.14). The advantage of the above approach lies in the appearance of the associated Bogomolny relations. There is equality in (3.1) if and only if $\alpha_n + \beta_n = 0$ for $n \geq 1$. (Recall that $\alpha_0 + \beta_0 = 0$ identically.) Substituting for α_n and β_n from (3.3) and (3.4) one finds

$$N(1-h_n^2) + n(h_{n+1} - h_n) = 0 \ . \qquad (3.8)$$

This equation is the central topic of the paper. The proposal is that instantons on the lattice should satisfy (3.8). This discrete Bogomolny relation is a nonlinear first-order difference equation for h_n, and so its solutions contain one degree of freedom, which specifies the instanton size in some way. It is simplest to think of h_1 as the free parameter: roughly speaking, the closer h_1 is chosen to 1 then the larger the width of the corresponding instanton. One should also be clear about the role of N. There is no analogue of Eq. (2.4), giving the topological charge in terms of a specified set of h_n. Rather, N is now also a (positive integer) parameter to be specified.

Given h_1 and N, it is clearly very simple to generate all other h_n by repeated application of (3.8). So far, (3.8) has resisted all attempts to write down the general solution in a closed form. But despite this lack of explicit solutions, one can still make considerable progress in investigating the properties of lattice instantons: this is the subject of the next section.

As a final remark, one might ask how things would differ if the restriction of radial symmetry were dropped. The answer is that one would now need two scalar fields, labeled with two indices each. Moreover, to get the topological charge appearing in the Bogomolny bound correctly, the right-hand side of (3.2) would look like the area of a spherical triangle (see, for example, the paper by Berg and Lüscher[4]).

IV. INSTANTONS ON THE LATTICE

All lattice instantons satisfy the Bogomolny relations, but does the converse hold, i.e., are all solutions of (3.8) lattice instantons? The answer is no: the requirement that $h_n > -1$ for all n puts a lower bound on the allowed values of h_1, as the following argument shows. From (3.8),

$$h_{n+1} = h_n - \frac{N}{n}(1-h_n^2) \qquad (4.1)$$

and so, assuming $h_n > -1$,

$$\begin{aligned} h_{n+1} > -1 &\Longleftrightarrow h_n + 1 > \frac{N}{n}(1-h_n^2) \\ &\Longleftrightarrow 1 > \frac{N}{n}(1-h_n) \\ &\Longleftrightarrow h_n > 1 - \frac{n}{N} \ . \end{aligned} \qquad (4.2)$$

The condition (4.2) is automatically satisfied if $n \geq 2N$, but for $n < 2N$ it gives a set of $2N-1$ inequalities, which are equivalent to putting a lower bound on h_1. When $N=1$ one requires simply that $h_1 > 0$ (and, of course, $h_1 < 1$). For $N = 2$, (4.2) becomes

$$h_1 > \tfrac{1}{2}, \quad h_2 > 0, \quad h_3 > -\tfrac{1}{2} \ . \qquad (4.3)$$

Using the explicit Bogomolny relations, namely,

$$h_2 = 2h_1^2 + h_1 - 2, \quad h_3 = h_2^2 + h_2 - 1 \ , \qquad (4.4)$$

one finds that the three conditions in (4.3) are all satisfied if and only if $h_1 > \sqrt{3}/2$.

These results are a little curious. One possible interpretation is that the lattice "wants" to support only those instantons larger than a certain width. This is somehow in keeping with the intuitive notion that a good discrete representation will put several lattice points inside the instanton. From now on it is assumed that h_1 is always large enough to satisfy (4.2).

One further check must be made. If $|h_n|<1$ for some n then from (4.1) $h_{n+1}<h_n<1$. Hence, subject to the above provisos on h_1, $\{h_n\}$ is a monotone decreasing sequence bounded below by -1, and so must tend to some limit l, where $l\geq -1$. For an instanton solution, one requires $l=-1$, and it is easy to show that this is indeed always the case: since $\{h_n\}$ tends to a limit, the series

$$\sum_{n=1}^{\infty}(h_n-h_{n+1})$$

converges. By (4.1) this series is equal to

$$\sum_{n=1}^{\infty}\frac{N}{n}(1-h_n^2)\ , \tag{4.5}$$

and if $l>-1$ then (4.5) diverges. So $l=-1$.

One could argue that the classification of lattice instantons is now complete. However, h_1 is not a convenient parameter to work with in practice. It would be much nicer to have a lattice analogue of the instanton width λ, which appeared in (2.18). One possibility is to set

$$\lambda_n^2\equiv\frac{1+h_n}{1-h_n}n^{2N} \tag{4.6}$$

and then to define

$$\lambda\equiv\lim_{n\to\infty}\lambda_n\ , \tag{4.7}$$

provided this limit exists. The idea is that one could specify λ and somehow relate it back to h_1. It turns out that λ is indeed well defined: the proof of this is given in the Appendix. However, it seems difficult to relate λ to h_1. Instead, it may be related to h_{n_0} for large n_0:

$$\lambda^2=\frac{(n_0+2N-1)!}{(n_0-1)!}\left[\frac{1+h_{n_0}}{1-h_{n_0}}\right]\exp(-4N^2S_0)+O(n_0^{-3})\ , \tag{4.8}$$

where

$$S_0\equiv\frac{\pi^2}{6}-\sum_{n=1}^{n_0-1}\frac{1}{n^2}\ . \tag{4.9}$$

This result is also derived in the Appendix. It suggests the following procedure for constructing instantons on the lattice.

(1) Specify λ and choose n_0 large enough so that $O(n_0^{-3})$ may be neglected.

(2) Calculate h_{n_0} using (4.8).

(3) Calculate h_n for $n>n_0$ using (4.1).

(4) Calculate h_n for $0<n<n_0$ by solving (4.1) as a quadratic for h_n:

$$h_n=\frac{n}{2N}\left\{-1+\left[1+\frac{4N}{n}\left[\frac{N}{n}+h_{n+1}\right]\right]^{1/2}\right\}\ . \tag{4.10}$$

Consider (4.10) for a moment. As h_{n+1} varies between -1 and 1 so does h_n, and if $|h_{n+1}|<1$ then $h_n>h_{n+1}$; i.e., step (4) is perfectly well behaved. The only problem will occur in step (3) if $n_0<2N$ and the value of h_{n_0} causes (4.2) to be violated [in which case the derivation of (4.8) breaks down anyway]. So to be completely safe, one should always choose $n_0>2N$.

To sum up, lattice instantons may be generated by choosing h_1 and then repeatedly using (4.1), but there is a lower bound on the allowed h_1, which is dependent on N. It is much better to fix h_{n_0} for some large n_0; first because one need not worry about the allowed values of h_{n_0}, and second because h_{n_0} has been related, albeit approximately, to the width λ.

V. THE FULL DISCRETE EVOLUTION SCHEME

We shall now address the question of incorporating the notion of lattice instantons into a full numerical evolution scheme. One possibility is to write down the general time-dependent equations of motion for the continuum model and then to discretize them in some way, but the lattice instantons of Sec. IV will not, in general, be static solutions of these full discretized equations.

A better plan is to construct a discrete action in which the potential energy (3.7) of the lattice instantons appears, and then to vary the action with respect to the fields at each site on the spacetime lattice. This method guarantees that solutions of the Bogomolny relations (i.e., the lattice instantons) are automatically solutions of the full evolution scheme.

In the continuum model, substituting (2.11) into (2.2) and (2.3) gives the kinetic and potential energy densities, ϵ_T and ϵ_V:

$$\epsilon_T=\tfrac{1}{4}(f_t^2+g_t^2+h_t^2)\ ,$$
$$\epsilon_V=\tfrac{1}{4}\left[f_r^2+g_r^2+h_r^2+\frac{N^2}{r^2}(f^2+g^2)\right]\ . \tag{5.1}$$

It is now convenient to write

$$f(r,t)+ig(r,t)=R(r,t)e^{i\psi(r,t)}\ , \tag{5.2}$$

so replacing f and g with R and ψ. (Note the instantons have $g=0$, i.e., $R=f$, $\psi=0$.) We shall always choose $R>0$. The constraint on the fields becomes $R^2+h^2=1$, and so it is straightforward to eliminate R in favor of h, eventually having a theory containing just h and ψ. From (5.1) the action density may be written

$$\mathcal{L}=\tfrac{1}{4}\left[h_r^2+R_r^2+\frac{N^2R^2}{r^2}+R^2\psi_r^2-h_t^2-R_t^2-R^2\psi_t^2\right]$$
$$=\frac{1}{4}\left[\frac{h_r^2}{1-h^2}+\frac{N^2(1-h^2)}{r^2}+(1-h^2)(\psi_r^2-\psi_t^2)-\frac{h_t^2}{1-h^2}\right]\ . \tag{5.3}$$

The first two terms are just the potential energy density (2.14) of instanton solutions; they will be discretized according to (3.7). In the full lattice formulation, the fields acquire a superscript to label the time slices. The discrete action derived from (5.3) is

$$S=\pi N\sum_m (1-h_1^m)+\tfrac{1}{2}\pi\sum_m\sum_{n=1}^{\infty} n\left[\frac{(h_{n+1}^m-h_n^m)^2}{1-h_n^{m2}}+\frac{N^2(1-h_n^{m2})}{n^2}-\frac{(h_n^{m+1}-h_n^m)^2}{\delta^2(1-h_n^{m2})}\right.$$
$$\left.+(1-h_n^{m2})\left[(\psi_{n+1}^m-\psi_n^m)^2-\frac{1}{\delta^2}(\psi_n^{m+1}-\psi_n^m)^2\right]\right], \quad (5.4)$$

where δ is the time interval between successive time slices. The evolution equations arise from varying S with respect to h_n^m and ψ_n^m separately. For convenience, introduce the shorthand notation

$$h\equiv h_n^m,\quad h_L\equiv h_{n-1}^m,\quad h_U\equiv h_n^{m+1},\quad h_R\equiv h_{n+1}^m,\quad h_D\equiv h_n^{m-1}, \quad (5.5)$$

and similarly for ψ. (Think of L, R, U, D as meaning left, right, up, and down.) Then varying with respect to h_n^m gives

$$(h-h_U)(hh_U-1)=A(1-h^2)^2, \quad (5.6)$$

where

$$A=\begin{cases}\delta^2\left[\dfrac{(h-h_R)(hh_R-1)}{(1-h^2)^2}+N(Nh+1)\right]-\left[\dfrac{h-h_D}{h_D^2-1}\right]+h[\delta^2(\psi-\psi_R)^2-(\psi-\psi_U)^2] & (n=1),\\ \delta^2\left[\dfrac{(h-h_R)(hh_R-1)}{(1-h^2)^2}+\dfrac{N^2h}{n^2}+\dfrac{n-1}{n}\left[\dfrac{h-h_L}{h_L^2-1}\right]\right]-\left[\dfrac{h-h_D}{h_D^2-1}\right]+h[\delta^2(\psi-\psi_R)^2-(\psi-\psi_U)^2] & (n>1).\end{cases}$$

Varying with respect to ψ_n^m gives

$$n(h^2-1)[\delta^2(\psi-\psi_R)-(\psi-\psi_U)]-n(h_D^2-1)(\psi-\psi_D)+(n-1)\delta^2(h_L^2-1)(\psi-\psi_L)=0\quad (n\geq 1). \quad (5.7)$$

Equations (5.6) and (5.7) together form the evolution scheme for h and ψ. Note that (5.7) is linear in ψ_U and so is easily rearranged to give ψ_U explicitly. But (5.6) is quadratic in h_U (unless $h=0$, in which case it is linear). The choice of solution is fixed by requiring that as $\delta\to 0$ (and hence $A\to 0$) then $h_U\to h$:

$$h_U=\begin{cases}A & (h=0),\\ \dfrac{1}{2h}(1+h^2-(1-h^2)\sqrt{1-4hA}) & (h\neq 0).\end{cases} \quad (5.8)$$

It should be emphasized that if one sets $h=h_U=h_D$ (i.e., looks for time-independent solutions on the lattice) then the Bogomolny relations (3.8) imply (5.6). It is this feature that distinguishes the scheme from a more naive approach; in other words, one gets exact static solutions on the lattice.

To be completely rigorous, one should carry out some sort of stability analysis for the discrete model. However, (5.6) and (5.7) are sufficiently complicated to make this a far from trivial task. For the moment, one must be content with the fact that extensive use of these difference equations has not revealed any instabilities.

VI. PRELIMINARY RESULTS

In this section we shall compare the results of the numerical evolution with the continuum behavior, in the cases of a charge-one instanton and a slowly moving charge-two ring.

The lattice formulation necessarily has a spatial boundary at $n=n_{\max}$, say. It is therefore useful to calculate the kinetic and potential energies within some radius R_0 ($\leq n_{\max}$), together with a measure W of the configuration size. In the continuum model one has

$$T(R_0,t)\equiv 2\pi\int_0^{R_0} r\epsilon_T(r,t)\,dr,$$
$$V(R_0,t)\equiv 2\pi\int_0^{R_0} r\epsilon_V(r,t)\,dr, \quad (6.1)$$
$$W(R_0,t)\equiv\frac{2\pi\int_0^{R_0} r^2[\epsilon_T(r,t)+\epsilon_V(r,t)]\,dr}{T(R_0,t)+V(R_0,t)}.$$

The discrete versions of T, V, W come from the discrete action (5.4):

$$T(R_0,m)\equiv 2\pi\sum_{n=1}^{R_0} n\epsilon_T(n,m),$$
$$V(R_0,m)\equiv 2\pi\sum_{n=1}^{R_0} n\epsilon_V(n,m), \quad (6.2)$$
$$W(R_0,m)\equiv\frac{2\pi\sum_{n=1}^{R_0} n^2[\epsilon_T(n,m)+\epsilon_V(n,m)]}{T(R_0,m)+V(R_0,m)},$$

where R_0 is assumed to be an integer and

$$\epsilon_T(n,m)\equiv\frac{1}{4\delta^2}\left[\frac{(h_n^{m+1}-h_n^m)^2}{1-h_n^{m2}}+(1-h_n^{m2})(\psi_n^{m+1}-\psi_n^m)^2\right],$$
$$\epsilon_V(n,m)\equiv\frac{1}{4}\left[\frac{(h_{n+1}^m-h_n^m)^2}{1-h_n^{m2}}+\frac{N^2(1-h_n^{m2})}{n^2}+(1-h_n^{m2})(\psi_{n+1}^m-\psi_n^m)^2\right]. \quad (6.3)$$

Figure 1 shows T, V, and W for the charge-one instanton given by $\lambda=30$ over the range $0\leq t\leq 100$ and for $R_0=100$. The continuum model (finely broken line) has $T=0$ (not plotted) and, from (2.18),

$$V(R_0,t)=\frac{2\pi R_0^2}{\lambda^2+R_0^2}\,,$$
$$W(R_0,t)=\frac{\lambda}{R_0^2}\left[(\lambda^2+R_0^2)\arctan\frac{R_0}{\lambda}-\lambda R_0\right]. \tag{6.4}$$

The other curves are results of the numerical model with $n_{\max}=100$ and $\delta=0.1$: first for initial data taken directly from the continuum solution (2.18) (solid line); and second for initial data derived from the Bogomolny relations taking $n_0=100$ (coarsely broken line). On the boundary ($n=n_{\max}$) the fields are taken to be fixed in time. The important feature is that the initial data taken from the continuum model lead to "lattice wobble," which is eliminated when the discrete Bogomolny relations are used instead. Of course, this is precisely what the discrete Bogomolny formalism was designed to do.

Note from Fig. 1 that the Bogomolny relations lead to a value of W which is slightly larger than that predicted by the continuum analysis (a feature caused by λ being redefined on the lattice to be the limit of $\{\lambda_n\}$). The absence of a natural scale in the problem means that this small difference is unimportant; what matters is that taking initial data from the Bogomolny relations leads to numerical results which are qualitatively close to the continuum model.

Turning now to time-dependent configurations, one could attempt to reproduce some of the solutions which have been obtained analytically for $N=0$. (Included in these is the special case $h=0$; then ψ satisfies the radial wave equation.) However, taking $N=0$ does not test the ability of the model to handle nontrivial topologies. We shall instead consider a "slow-motion" approximation, originally proposed by Manton in connection with monopole scattering.[8] The idea is that the manifold of static configurations possesses a natural metric given by the kinetic part of the action; in the limit of small kinetic energies, the evolution is approximated by geodesic motion on this manifold. In essence, one is letting λ depend on t, but not on r in Eq. (2.10).

For $N=1$, the requirement of finite kinetic energy means that λ must be independent of time as $r\to\infty$, so ruling out a slow-motion approximation. In other words, taking λ to be a function only of t leads to a divergent kinetic energy. But when $N=2$ there are sufficient powers of r on the denominator of (2.10) to keep the energy finite. For this case the slow-motion approximation has been considered by Ward:[9] one solution is

$$\lambda(t)=a(b+it)^2\,, \tag{6.5}$$

where a and b are real constants. The kinetic energy is $\pi^2 a$ (so the approximation is expected to be best when a is small) and the potential energy is 4π. Note that because the evolution is being approximated by a sequence of instanton configurations, the kinetic and potential energies are conserved separately.

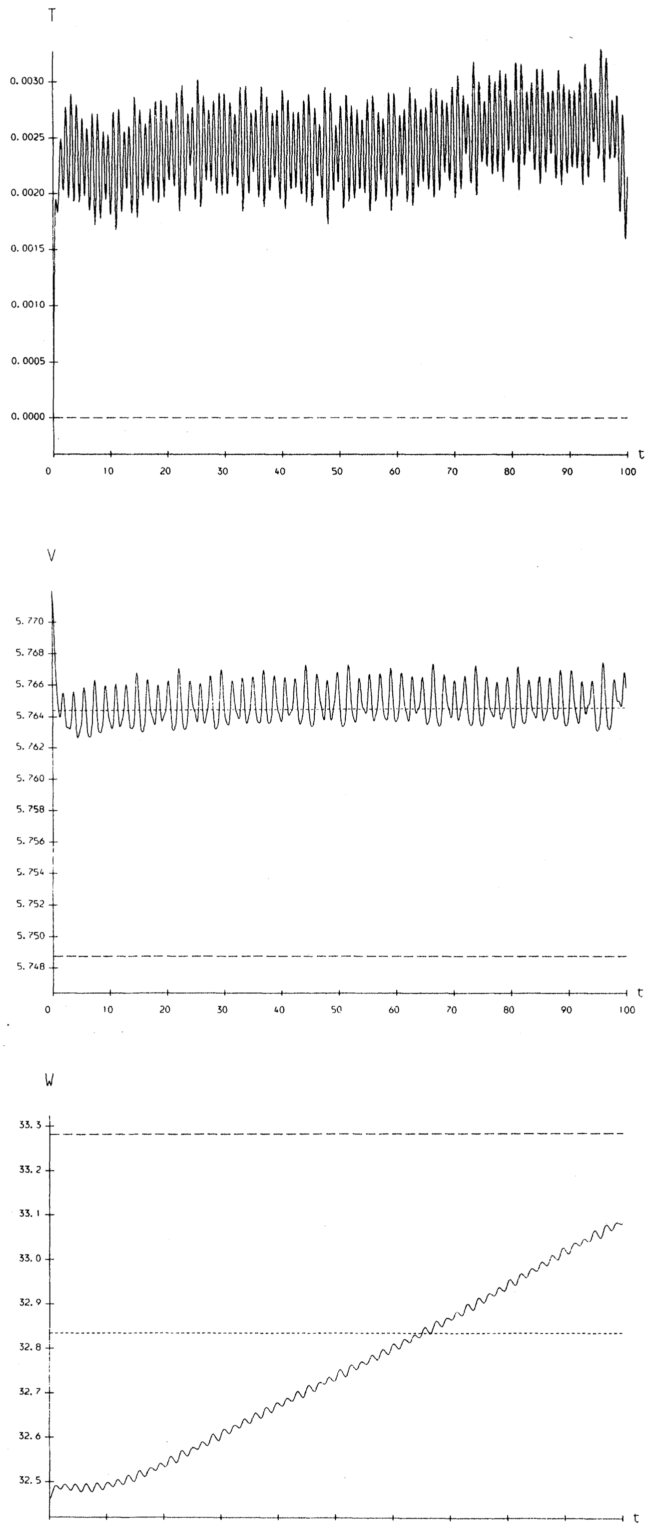

FIG. 1. The variations of T, V, and W over the range $0\leq t\leq 100$ for a charge-one instanton in the continuum model (finely broken line), and also for the numerical evolution. In the latter case, initial data are taken both directly from the continuum model (solid line) and from the discrete Bogomolny relations (coarsely broken line).

In terms of the (h,R,ψ) parametrization, (6.5) corresponds to

$$h(r,t)=\frac{\gamma^4-r^4}{\gamma^4+r^4}\,,$$
$$R(r,t)=\frac{2\gamma^2r^2}{\gamma^4+r^4}\,, \tag{6.6}$$
$$\psi(r,t)=\arctan\left[\frac{2bt}{b^2-t^2}\right],$$

where

$$\gamma\equiv\sqrt{a(b^2+t^2)}\,. \tag{6.7}$$

The potential energy is peaked at $r=3^{-1/4}\gamma$: physically, one has a ring which contracts to a minimum radius at $t=0$ and then expands again. From (6.6), one obtains

$$T(R_0,t)=2a\pi\left[\frac{\pi}{2}-\arctan\frac{\gamma^2}{R_0^2}-\frac{\gamma^2R_0^2}{\gamma^4+R_0^4}\right], \tag{6.8}$$

$$V(R_0,t)=\frac{4\pi R_0^4}{\gamma^4+R_0^4}\,, \tag{6.9}$$

$$W(R_0,t)=\tfrac{1}{4}\sqrt{2}\,\gamma\left[1+\frac{\gamma^4}{R_0^4}\right]\left[\operatorname{arctanh}\frac{\sqrt{2}\,\gamma R_0}{\gamma^2+R_0^2}+\arctan\frac{\sqrt{2}\,\gamma R_0}{\gamma^2-R_0^2}\right]-\frac{\gamma^4}{R_0^3}\,, \tag{6.10}$$

where in (6.10) the principal range of arctan is taken to be $[0,\pi)$.

Figure 2 shows these quantities (finely broken lines) for $R_0=150$, $a=0.001$, $b=1000$ over the range $0\le t\le 1000$. Also shown are numerical results for $n_{\max}=150$ and $\delta=0.1$, first for initial data taken directly from (6.6) (solid lines) and second for initial data derived from the discrete Bogomolny equations with $n_0=150$ (coarsely broken lines). The boundary conditions on ψ and h are

$$\psi_{n_{\max}+1}=\psi_{n_{\max}}\,,$$
$$h_{n_{\max}+1}=h_{n_{\max}}-\frac{2}{n_{\max}}(1-h_{n_{\max}}^2)\,. \tag{6.11}$$

In other words, ψ has zero gradient on the boundary and h falls off as a charge-two instanton field. This choice means that energy may flow off the edge of the lattice (or alternatively into the lattice from outside): in Fig. 2 approximately 0.75% of the original energy is "lost" as the ring expands up to $t=1000$. Note that once again the solid lines are affected by lattice wobble. On the other hand, using the Bogomolny relations gives a much smoother numerical evolution.

One may ask to what extent the boundary conditions affect the numerical results. This may be studied by taking a larger $n_{\max}$ (i.e., putting the boundary further away), but keeping R_0 fixed. Figure 3 is the same as Fig. 2 but with $n_{\max}=300$. The greatest change is in T: the

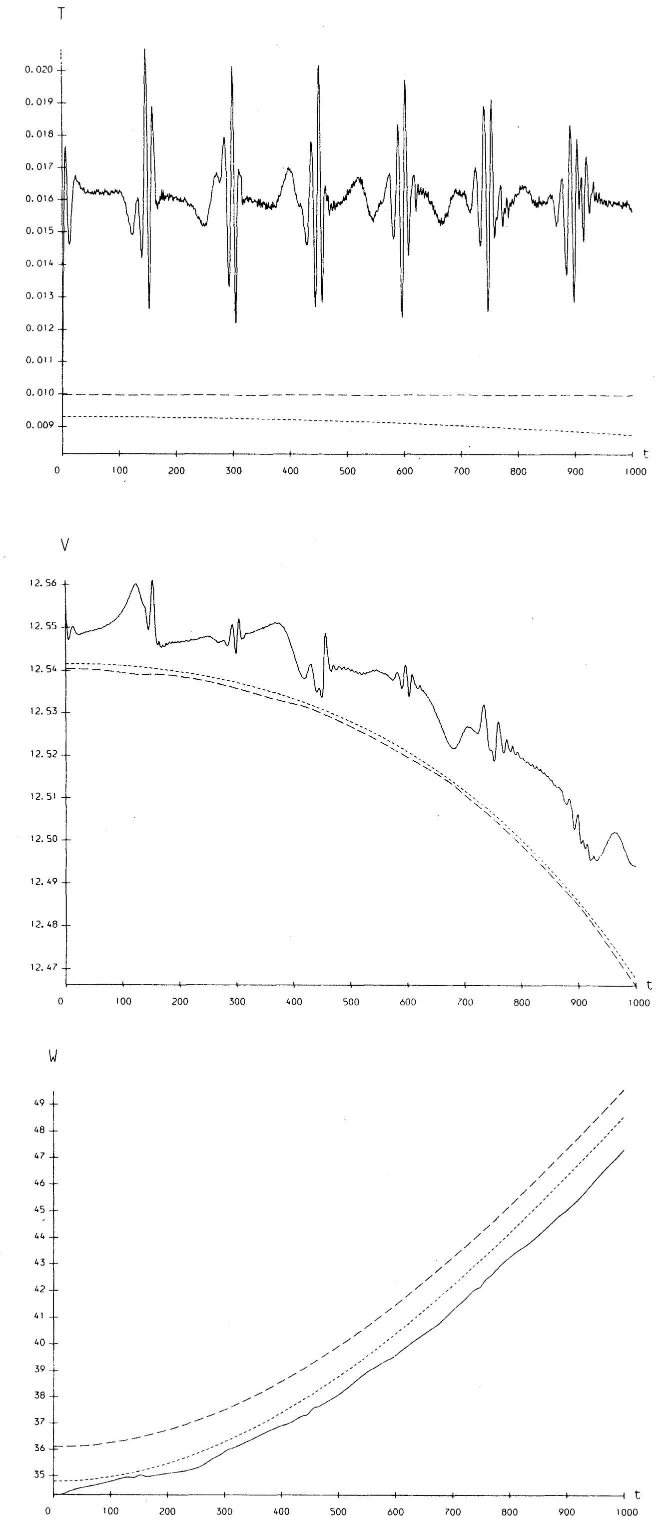

FIG. 2. The variations of T, V, and W over the range $0\le t\le 1000$ for a slowly expanding charge-two ring in the analytic slow-motion approximation (finely broken line), and also for the numerical evolution. In the latter case, initial data are taken both directly from the slow-motion analysis (solid line) and from the discrete Bogomolny relations (coarsely broken line).

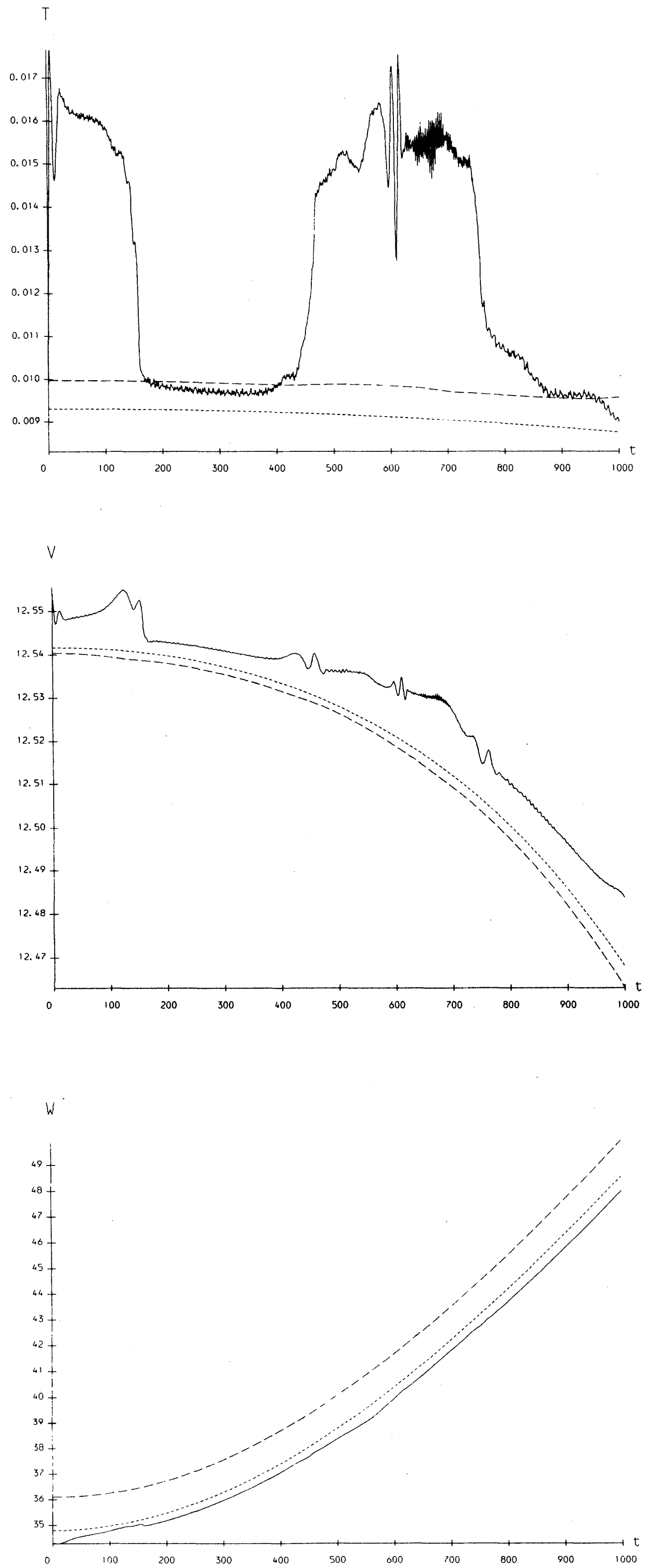

FIG. 3. The variations of T, V, and W for a slowly expanding charge-two ring. The parameters are the same as in Fig. 2, except that, in the numerical evolution, $n_{\max}$ is set to 300, instead of 150.

solid line still fluctuates wildly, but the coarsely broken line now exhibits the gradual decrease predicted by the slow-motion approximation. The evolutions of V and W are virtually unchanged. So it seems that to a large extent the boundary conditions (6.11) are transparent to the flow of energy, but they are not completely invisible. However, since we are using a one-dimensional lattice, it is usually computationally feasible to make the boundary effectively invisible by taking $n_{\max} \gg R_0$.

VII. CONCLUDING REMARKS

The development of a set of discrete Bogomolny relations removes the lattice wobble, which is observed in numerical simulations at low kinetic energies, by providing explicit instantons on the lattice. Two particular cases have been studied in some detail and the numerical scheme appears to be working very well. Now one can return to the original question of instanton stability. It may be that the absence of a natural scale in the model means that if an instanton is "squashed" then it eventually becomes an infinitely tall spike; on the other hand, the condition that the fields be fixed at infinity (to ensure finite energy) may rescue the situation. Clearly it is essential that the lattice wobble be eliminated if one is to study small perturbations.

Work on this question is currently in progress and detailed results will be presented elsewhere. Early results indicate that a perturbed instanton loses kinetic energy by emitting a pulse of radiation, which travels outward at the speed of light. But it does not completely overcome its tendency to shrink (or expand, depending on the sign of the initial perturbation).

Finally, the technique of discretizing the Bogomolny bound, in order to obtain static solutions on the lattice, may be applied to other models which have nontrivial topologies. A few possibilities are the sine-Gordon equation in 1+1 dimensions, the Maxwell-Higgs model in 2+1 dimensions (which can describe vortices in superconductors), and the Skyrme model in 3+1 dimensions. Although there have been many numerical studies (see, for example, Refs. 10–14), this approach to a discrete evolution scheme does not seem to have been considered.

ACKNOWLEDGMENTS

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APPENDIX

The proof that λ, as given by (4.7), is well defined involves several intermediate results (lemmas 1–3) and also a further difference equation, closely related to the discrete Bogomolny equations, and derived as follows. Take ϵ small and look for a solution to the Bogomolny relations of the form

$$h_n = \frac{1-a_n\epsilon}{1+a_n\epsilon} = 1-2a_n\epsilon + O(\epsilon^2) \; . \tag{A1}$$

To leading order in ϵ, the Bogomolny relations become

$$a_{n+1}=a_n\left[1+\frac{2N}{n}\right]\quad (n\geq 1)\ , \tag{A2}$$

which has general solution

$$a_n=\frac{a_1}{(2N)!}\frac{(n+2N-1)!}{(n-1)!}\ . \tag{A3}$$

This provides the motivation for defining a new set of lattice quantities:

$$h_n'\equiv\frac{1-a_n}{1+a_n}\ , \tag{A4}$$

which, it is easily checked, are the general solution of the difference equation

$$h_{n+1}'=\frac{h_n'-(N/n)(1-h_n')}{1+(N/n)(1-h_n')}\ . \tag{A5}$$

Now assume that a_1 is chosen to be positive. Then $a_n>0$ for all $n\geq 1$; moreover $a_{n+1}>a_n$ and $a_n\to\infty$ as $n\to\infty$. So from (A4) it is seen that $\{h_n'\}$ has the following properties in common with $\{h_n\}$:

(1) $|h_n'|<1$,

(2) $h_{n+1}'<h_n'$,

(3) $h_n'\to -1$ as $n\to\infty$.

It is useful to define counterparts to the λ_n^2 for the new quantities h_n':

$$\lambda_n'^2\equiv\frac{1+h_n'}{1-h_n'}n^{2N}=\frac{n^{2N}}{a_n}\ , \tag{A6}$$

where the last equality follows from (A4). Using the explicit form of a_n given in (A3),

$$\lambda_n'^2\to\lambda'^2=\frac{(2N)!}{a_1}\quad\text{as } n\to\infty\ . \tag{A7}$$

The relevant sequence of lemmas may now be constructed.

Lemma 1. Suppose that $\{h_n'\}$ and $\{h_n''\}$ both satisfy (A5) and that $h_n'>h_n''$ for some n. Then $h_{n+1}'>h_{n+1}''$.

Proof. Let $D=[1+(N/n)(1-h_n')][1+(N/n)(1-h_n'')]>0$; then (A5) implies

$$h_{n+1}'-h_{n+1}''=\frac{1}{D}\left[\left[h_n'-\frac{N}{n}(1-h_n')\right]\left[1+\frac{N}{n}(1-h_n'')\right]-(h_n'\leftrightarrow h_n'')\right]$$

$$=\frac{1}{D}(h_n'-h_n'')\left[1+\frac{2N}{n}\right]>0\ .$$

Lemma 2. Suppose that $\{h_n\}$ satisfies the Bogomolny relations (4.1), that $\{h_n'\}$ satisfies (A5), and further that $h_n'=h_n$ for some n. Then $h_{n+1}'>h_{n+1}$.

Proof. Let $D=1+(N/n)(1-h_n)>0$; then (4.1) and (A5) imply

$$h_{n+1}'-h_{n+1}=\frac{1}{D}\left[h_n-\frac{N}{n}(1-h_n)-\left[1+\frac{N}{n}(1-h_n)\right]\left[h_n-\frac{N}{n}(1-h_n^2)\right]\right]$$

$$=\frac{1}{D}\frac{N^2}{n^2}(1-h_n)(1-h_n^2)>0\ .$$

Lemma 3. $\{\lambda_n^2\}$ is bounded.

Proof. Lemmas 1 and 2 together show that if $h_{n_0}'=h_{n_0}$ for some n_0 then $h_n'>h_n$ for all $n>n_0$. Therefore $\lambda_n'^2>\lambda_n^2$ for $n>n_0$. But, given $\delta>0$, $\lambda_n'^2<\lambda'^2+\delta$ for sufficiently large n. Clearly $\{\lambda_n^2\}$ is bounded below by zero, and so

$$0\leq\lambda_n^2\leq\lambda'^2$$

for sufficiently large n.

The final stages of the proof now follow. From (4.1),

$$1-h_{n+1}=(1-h_n)\left[1+\frac{N}{n}(1+h_n)\right]\ ,$$

$$1+h_{n+1}=(1+h_n)\left[1-\frac{N}{n}(1-h_n)\right]\ .$$

Dividing one of these equations by the other, and using (4.6),

$$\frac{\lambda_{n+1}^2}{\lambda_n^2}=\left[1+\frac{1}{n}\right]^{2N}\frac{1-\frac{N}{n}(1-h_n)}{1+\frac{N}{n}(1+h_n)}\ . \tag{A8}$$

The fact that $\{\lambda_n^2\}$ is bounded, together with the form of (4.6), means that h_n must approach -1 at least as fast as $1/n^{2N}$, i.e., for sufficiently large n, $h_n=-1+O(n^{-2N})$. Therefore, expanding (A8) in powers of $1/n$:

$$\frac{\lambda_{n+1}^2}{\lambda_n^2}=\left[1+\frac{2N}{n}+\frac{N(2N-1)}{n^2}\right]\left[1-\frac{2N}{n}\right]+O(n^{-3})$$

$$=1-\frac{N(2N+1)}{n^2}+O(n^{-3})\ , \tag{A9}$$

which shows that $\{\lambda_n^2\}$ is monotonically decreasing for sufficiently large n. Since $\{\lambda_n^2\}$ is bounded below by zero, it must tend to a finite limit λ^2.

Equation (A9) may be used to provide an approximate value for λ. By definition

$$\lambda^2=\lambda_{n_0}^2 \prod_{n=n_0}^{\infty} \frac{\lambda_{n+1}^2}{\lambda_n^2} . \tag{A10}$$

Take logarithms and choose n_0 sufficiently large so that for $n \geq n_0$ the $O(n^{-3})$ terms in (A9) are small compared with $O(n^{-2})$. Then

$$\begin{aligned}\ln\lambda^2 &\approx \ln\lambda_{n_0}^2 - \sum_{n=n_0}^{\infty} \frac{N(2N+1)}{n^2} \\ &\approx \ln\lambda_{n_0}^2 - N(2N+1)\left[\frac{\pi^2}{6} - \sum_{n=1}^{n_0-1}\frac{1}{n^2}\right] ,\end{aligned} \tag{A11}$$

where in the last step we have used the result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} . \tag{A12}$$

Introducing the notation

$$S_0 \equiv \frac{\pi^2}{6} - \sum_{n=1}^{n_0-1} \frac{1}{n^2} , \tag{A13}$$

we get

$$\lambda^2 \approx \lambda_{n_0}^2 \exp[-S_0 N(2N+1)] . \tag{A14}$$

However, as it stands, (A14) is of limited use because it still involves $\lambda_{n_0}^2$. It is much better to rewrite it by relating $\lambda_{n_0}^2$ to λ'^2. Using (A6) and then (A3) one finds

$$\begin{aligned}\frac{\lambda_{n+1}'^2}{\lambda_n'^2} &= \left[1+\frac{1}{n}\right]^{2N} \frac{a_n}{a_{n+1}} \\ &= \left[1+\frac{1}{n}\right]^{2N} \left[1+\frac{2N}{n}\right]^{-1} \\ &= 1+\frac{N(2N-1)}{n^2} + O(n^{-3}) .\end{aligned} \tag{A15}$$

If we again choose n_0 sufficiently large so that the $O(n^{-3})$ terms are small then

$$\lambda'^2 \approx \lambda_{n_0}'^2 \exp[S_0 N(2N-1)] . \tag{A16}$$

But now recall that $h_{n_0}=h'_{n_0}$ and so $\lambda_{n_0}=\lambda'_{n_0}$. Hence comparison of (A14) and (A16) yields

$$\lambda^2 \approx \lambda'^2 \exp(-4N^2 S_0) . \tag{A17}$$

In Eq. (A7), λ'^2 was given in terms of a_1, but it may equally be expressed in terms of a_{n_0} (or h_{n_0}) by using (A3), (A4), and the fact that $h_{n_0}=h'_{n_0}$:

$$\begin{aligned}\lambda'^2 &= \frac{(n_0+2N-1)!}{(n_0-1)!} \frac{1}{a_{n_0}} \\ &= \frac{(n_0+2N-1)!}{(n_0-1)!} \left[\frac{1+h_{n_0}}{1-h_{n_0}}\right] .\end{aligned} \tag{A18}$$

So finally we obtain Eq. (4.8):

$$\lambda^2 \approx \frac{(n_0+2N-1)!}{(n_0-1)!} \left[\frac{1+h_{n_0}}{1-h_{n_0}}\right] \exp(-4N^2 S_0) . \tag{A19}$$

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