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Correlation function for $N=1$ superconformal models on the supertorus

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We calculate the partition function for $N=1$ superconformal models on the supertorus with $(+,+)$ boundary conditions. Also the two-point correlation function of the $N=1, c=\frac{3}{2}$ superconformal model on the odd spin structure of the supertorus is constructed using a supersymmetric Coulomb-gas formalism.

I. INTRODUCTION

The constructions of correlation functions on higher-genus (super) Riemann surfaces are important not only from the viewpoint of (super)conformal field theories but also in the perturbation theory of string models, which was proposed by Gepner. Beginning with Verlinde[1] on the relation between modular transformations and fusion rules, a great deal of papers have dealt with the properties of conformal field theories on the torus.[2] The Gepner models,[3] which describe string propagation in Calabi-Yau space, are built from the summation of nontrivial conformal field theories. The study of string perturbation theory along this line requires the explicit construction of the correlation functions on higher-genus Riemann surfaces.

In this paper we wish to construct the two-point function of the $N=1$, $c=\frac{3}{2}$ superconformal model on the odd spin structure of the supertorus. The construction of a correlation function for these models on even spin structures of the supertorus is essentially equivalent to the construction of a correlation function on even spin structures of an ordinary torus. On the other hand, the essential idea of a Coulomb-gas representation[4] is to express all primary fields in terms of vertex operators of a bosonic scalar field. In this formalism one can introduce a background charge e at infinity on the complex plane. However, we have to remark that the metric $dz\,d\bar{z}$ on the torus has no poles or zeros, which are necessary for implementing a background charge. Therefore, on the (super)torus, one can introduce the floating charge instead of the background charge on the complex plane.[5]

The organization of this paper is as follows. In Sec. II we review the $N=1$ supertorus, and we derive the partition function for the $N=1$, $c=\frac{3}{2}$ superconformal model on the supertorus in Sec. III. Section IV is devoted to find the partition function of the $N=1$ superminimal model on the supertorus. We construct correlation functions for the $N=1$, $c=\frac{3}{2}$ superconformal model in Sec. V and we summarize our results in Sec. IV. Finally we derive the properties of supertheta functions in the Appendix.

II. $N=1$ SUPERTORUS

We begin by reviewing the results on the uniformization theorem of genus-1 super Riemann surfaces. A supertorus is obtained as the quotient of the complex superplane (CSP) with coordinates (z,θ) by a supergroup $G=\mathrm{Osp}(1,2)$ of superconformal transformations of the form[6]

$$z'=\frac{az+b}{cz+d}+\theta\frac{\gamma z+\delta}{(cz+d)^2}\,,\qquad \theta'=\frac{\gamma z+\delta}{cz+d}+\frac{\theta}{cz+d}\left(1+\tfrac{1}{2}\delta\gamma\right) \tag{1}$$

$$\text{with}\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathrm{SL}(2,Z)\,.$$

Because a subgroup of G on the supertorus $[=\mathrm{SPL}(2,C)]$ is isomorphic to a fundamental group of a torus, it must be Abelian and has precisely two commuting generators. Furthermore, it can be chosen to preserve the flat supergeometry on the CSP characterized by the complete frame fields and its dual fields:

$$E^{\theta}=d\theta,\quad E^{\bar{\theta}}=d\bar{\theta},\quad E^{+}=dz+\theta\,d\theta,\quad E^{-}=d\bar{z}-\bar{\theta}\,d\bar{\theta}$$

and

$$E_{\theta}=D_{\theta}^2=\partial_z,\quad E_{\bar{\theta}}=E_{\bar{\theta}}^2=-\partial_{\bar{z}}\,,\qquad E_{+}=D_{\theta}=\partial_{\theta}+\theta\partial_z,\quad E_{-}=D_{\bar{\theta}}=\partial_{\bar{\theta}}-\bar{\theta}\partial_{\bar{z}}. \tag{2}$$

The generators of this subgroup can be given by

$$z'=z+1,\quad \theta'=\theta\,,\qquad z'=z+\tau+\theta\delta,\quad \theta'=\theta+\delta \tag{3}$$

for the odd spin structure [$(+,+)$ boundary conditions] and

$$z'=z+1,\quad \theta'=\theta\,,\qquad z'=z+\tau,\quad \theta'=-\theta \tag{4}$$

for the even spin structure with $(+,-)$ boundary conditions. The other even spin structures are obtained by replacing signs $(-,-)$ or $(-,+)$ in the transformations of θ in (4). The even spin structures are just the superspace version of a torus and lead to no difficulty. However, the periodicity of (3) for the odd spin structure induces some problems. For example, the requirement of the periodicity (3) on a scalar superfield gives boundary conditions that mix the component fields. Avoiding this difficulty, we introduce new coordinates $W=(\omega,\phi)$ related to $Z=(z,\theta)$ as[7]

$$z=\omega+\phi\delta\frac{\omega_I}{\tau_I}, \quad \theta=\phi+\delta\frac{\omega_I}{\tau_I} , \tag{5}$$

where ω_I and τ_I denote the imaginary part of ω and τ. Rewriting ω and ϕ in terms of $(z,\bar z,\theta,\bar\theta,\delta,\bar\delta,\tau,\bar\tau)$, we have

$$\begin{aligned}\omega&=z-\frac{z_I}{\tau_I}\theta\delta\left[1+\frac{i\bar\theta\bar\delta}{2\tau_I}\right] ,\\ \phi&=\theta-\frac{z_I}{\tau_I}\delta\left[1+\frac{i\bar\theta\bar\delta}{2\tau_I}\right] .\end{aligned} \tag{6}$$

In this system the periodicity (3) reduces to

$$\omega'=\omega+1, \quad \phi'=\phi, \quad \omega'=\omega+\tau, \quad \phi'=\phi . \tag{7}$$

However, supercovariant derivatives D_θ and $D_{\bar\theta}$ take the more complicated forms

$$\begin{aligned}D_\theta&=\left[1-\frac{i\delta}{2\tau_I}\phi+\frac{\bar\delta\delta}{4\tau_I^2}\bar\phi\phi\right]D_\phi-\frac{i\bar\delta}{2\tau_I}\phi D_{\bar\phi}+\frac{i\bar\delta}{\tau_I}\bar\phi\phi\partial_{\bar\omega} ,\\ D_{\bar\theta}&=\left[1-\frac{i\bar\delta}{2\tau_I}\bar\phi+\frac{\bar\delta\delta}{4\tau_I^2}\bar\phi\phi\right]D_{\bar\phi}-\frac{i\delta}{2\tau_I}\bar\phi D_\phi+\frac{i\delta}{\tau_I}\bar\phi\phi\partial_\omega .\end{aligned} \tag{8}$$

The presence of $\delta,\bar\delta$-dependent terms in (8) shows a gravitino zero mode on the torus in (ω,ϕ) coordinates system.

Let us define the integration on the odd supertorus as

$$\int_{\mathrm{ST}}d^4Z\,f(Z,\bar Z)=\int_B d^2\phi\int_T d^2\omega\,E'^{-1}\times f(Z(W,\bar W),\bar Z(W,\bar W)) , \tag{9}$$

where $Z(W,\bar W)$ is the transformation in (5) and $E'^{-1}=(\tau+\phi\delta)_I/\tau_I$. Here B and T mean that the ϕ integration is the ordinary Berezin integration and the ω integral is over a torus with $(0,1,\tau,\tau+1)$. On the other hand, the supermodular transformations on the odd supertorus are given by[8]

$$\tau'=\frac{a\tau+b}{c\tau+d}, \quad \delta'=\frac{\delta}{(c\tau+d)^{3/2}} , \tag{10}$$

where $(\tau,\delta)\in$ supertorus and

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in \mathrm{SL}(2,Z) .$$

In terms of a new parameter $T\equiv\tau+\theta\delta$, these can be rewritten as

$$T'=\frac{aT+b}{cT+d} \tag{11}$$

which is exactly the same form as the modular transformation law in the torus. Using this, we will construct the supertheta functions in the Appendix.

III. PARTITION FUNCTION FOR $N=1$, $c=\frac{3}{2}$ SUPERCONFORMAL MODEL ON THE SUPERTORUS

The action for a free scalar superfield S on the odd spin structure of the supertorus is given by[7]

$$A=\frac{g}{2\pi}\int_{\mathrm{ST}}d^4Z\,D_\theta S\,D_{\bar\theta}S . \tag{12}$$

In order to obtain a simple situation on the boundary conditions, using (9), we can transform (12) into

$$\begin{aligned}A=\frac{g}{2\pi}\int_T d^2\omega\Big[&\partial_\omega\Phi_1\partial_{\bar\omega}\Phi_1+\bar\psi_1\partial_\omega\bar\psi_1-\psi_1\partial_{\bar\omega}\psi_1+\frac{\bar\delta\delta}{2\tau_I^2}\bar\psi_1\psi_1\\ &-F_1^2-\frac{i\delta}{\tau_I}\psi_1\partial_\omega\Phi_1-\frac{i\bar\delta}{\tau_I}\bar\psi_1\partial_{\bar\omega}\Phi_1\Big]\\ &+g\frac{\bar\delta\delta}{4\pi\tau_I}\bar\psi_0\psi_0-\frac{g\tau_I F_0^2}{2\pi} ,\end{aligned} \tag{13}$$

where we have split S $(=S_0+S_1)$ into a zero-mode part S_0 and a nonzero-mode part S_1. The component expansion of S is defined by

$$S(W,\bar W)=\Phi(\omega,\bar\omega)+\phi\psi(\omega,\bar\omega)-\bar\phi\bar\psi(\omega,\bar\omega)+\bar\phi\phi F(\omega,\bar\omega) . \tag{14}$$

Here we can eliminate the auxiliary field F by its equation of motion. The partition function for Φ_1 leads to

$$Z_0=\int D\Phi_1 e^{-A_{\Phi_1}}=\left[\frac{g}{2\tau_I}\right]^{1/2}\frac{1}{|\eta(\tau)|^2} , \tag{15}$$

where

$$A_{\Phi_1}=\frac{g}{2\pi}\int_T d^2\omega\,\partial_\omega\Phi_1\partial_{\bar\omega}\Phi_1$$

and η is the Dedekind function. To calculate the fermionic part, let us define the measure for the zero modes as

$$DS_0=C\,dF_0 d\psi_0 d\bar\psi_0, \quad C=\text{const} . \tag{16}$$

For convenience, we choose C such that the partition function takes the form

$$Z_1=\int DS_0 D[\psi_1,\bar\psi_1]e^{-A_r}=\frac{\bar\delta\delta}{\tau_I}|\eta(\tau)|^2 , \tag{17}$$

where $A_r=A-A_{\Phi_1}$. Note that there is no contribution from the $\delta,\bar\delta$-dependent terms because of the presence of the $\bar\psi_0\psi_0$ term. Also we remind the reader that there is no contribution to the partition function from the double periodic free fermion action (Ising model) on the torus, due to the absence of the $\bar\psi_0\psi_0$ term in action. However, on the supertorus there is a direct contribution from the fermion action to the partition function.

As is shown in Refs. 5 and 9, for the case of the (super)torus, one has to account also for the classical part of action because in a finite geometry the boundary conditions generate various constraints. This comes from the classical solutions and their winding on the homology cycles of the torus. Then, for variations $\delta\Phi=2\pi m$, $\delta\Phi'=2\pi m'$ along two generators of the superspace version of the torus in (7), the corresponding continuum limit is the frustrated partition function

$$Z'_{m,m'}=\int_{\substack{\delta\Phi=2\pi m\\ \delta\Phi'=2\pi m'}} DS\, e^{-A} . \tag{18}$$

This is evaluated writing $S\to S+S_{\rm cl}$, where S is now a periodic quantum field and $S_{\rm cl}$ is given by

$$S_{\rm cl}=2\pi\,{\rm Im}\left[\frac{m'-m\bar\tau}{\tau_I}z\right]+\phi\frac{\pi\delta(m\bar\tau-m')}{2\tau_I^2}\bar\omega -\bar\phi\frac{\pi\bar\delta(m\tau-m')}{2\tau_I^2}\omega , \tag{19}$$

as a solution to the equation

$$\left[1-\frac{i(\phi\delta+\bar\phi\bar\delta)}{2\tau_I}+\frac{\bar\delta\delta\bar\phi\phi}{\tau_I^2}\right]D_\phi D_{\bar\phi}S-\frac{i}{2\tau_I}(\delta\bar\phi\partial_\omega+\bar\delta\phi\partial_{\bar\omega})S +\frac{i\bar\phi\phi}{\tau_I}(\delta D_\phi\partial_\omega-\bar\delta D_{\bar\phi}\partial_{\bar\omega})S=0 . \tag{20}$$

Then $Z'_{m,m'}$ factorizes as

$$Z'_{m,m'}=Z_0Z_1\exp(-A_{\rm cl})\equiv Z_1Z_{m,m'} . \tag{21}$$

Considering (13) and (19), $A_{\rm cl}$ leads to

$$A_{\rm cl}=\frac{\pi g}{2}\frac{|m'-m\tau|^2}{\tau_I} . \tag{22}$$

Note that there is no contribution to $A_{\rm cl}$ from the fermion-dependent terms in (13). Under these supermodular transformations in (10), $Z'_{m,m'}$ transforms in the same way as the frustrations

$$Z'_{m,m'}\left[\frac{a\tau+b}{c\tau+d},\frac{\delta}{(c\tau+d)^{3/2}}\right]=Z'_{cm'+dm,am'+bm}(\tau,\delta) . \tag{23}$$

A simple supermodular-invariant object is then obtained by summing over m,m'. After a Poisson transformation on m' one obtains

$$Z_{\rm odd}=\sum_{m,m'\in Z}Z'_{m,m'}=\sum_{m,e\in Z}\frac{\bar\delta\delta}{\tau_I}q^{h_{em}}\bar q^{\bar h_{em}},\quad q=e^{2\pi i\tau} , \tag{24}$$

where conformal weights x and spin s are given by

$$x=h_{em}+\bar h_{em},\quad s=h_{em}-\bar h_{em} \tag{25}$$

with

$$h_{em}=\frac14\left[e\left[\frac{2}{g}\right]^{1/2}+m\left[\frac{g}{2}\right]^{1/2}\right]^2 ,$$

$$\bar h_{em}=\frac14\left[-e\left[\frac{2}{g}\right]^{1/2}+m\left[\frac{g}{2}\right]^{1/2}\right]^2 .$$

In order to obtain the full partition function for the $N=1$, $c=\frac32$ superconformal model on the odd supertorus, we have to consider the coupling between Φ and ψ, which is purely induced by the boundary conditions. Through the SU(2) $k=2$ Wess-Zumino-Witten (WZW) model and requiring supermodular invariance, this coupling contribution is just the partition function for the $N=1$, $c=\frac32$ superconformal model on odd $(+,+)$ spin structure of the torus as[9]

$$Z_{\rm even}(g/2)=\sum_{r,s=0,1}\mathcal{L}_2(r,s)\sum_{\substack{m=r[2]\\ m'=s[2]}}Z_{m,m'}(g/2) , \tag{26}$$

where $\mathcal{L}_2(r,s)$ denotes the partition function of an Ising model with twisted boundary conditions $e^{i\pi r}$ ($e^{i\pi s}$) on the spin variables. Here (k) stands for "modulo k." Also $Z_{m,m'}$ is already defined in (21).

Finally we obtain the partition function for the $N=1$, $c=\frac32$ superconformal model on the odd supertorus:

$$\begin{aligned}Z_{c=3/2}=Z_{\rm odd}+Z_{\rm even}=&\sum_{m,m'\in Z}Z_1Z_{m,m'}(g/2)+Z_2\left[\sum_{m,m'\in e}-\sum_{m\in e,m'\in o}+\sum_{m\in o,m'\in e}+\sum_{m,m'\in o}\right]Z_{m,m'}(g/2)\\ &+Z_3\left[\sum_{m,m'\in e}+\sum_{m\in e,m'\in o}+\sum_{m\in o,m'\in e}-\sum_{m,m'\in o}\right]Z_{m,m'}(g/2)\\ &+Z_4\left[\sum_{m,m'\in e}+\sum_{m\in e,m'\in o}-\sum_{m\in o,m'\in e}+\sum_{m,m'\in o}\right]Z_{m,m'}(g/2)\end{aligned} \tag{27}$$

with

$$Z_\nu=\frac12\left|\frac{\theta_\nu(0)}{\eta(\tau)}\right| ,\quad \nu=2,3,4 .$$

Here e (o) denotes even (odd) integers.

IV. PARTITION FUNCTION FOR THE $N=1$, $c<\frac32$ SUPERCONFORMAL MODELS ON SUPERTORUS

Using the super Coulombic partition function in (27), one can reproduce the superminimal partition function on the supertorus.

Before we proceed, we notice that on the supertorus there is no room for implementing the background charge e because the metric d^4Z on the supertorus has no poles or zeros, which are necessary for the background charge. However, instead of the background charge, we can introduce a floating charge e_n on the supertorus. Starting from a free superfield in the complex superplane and adding a charge e at infinity to $c=\frac{3}{2}$ leads to

$$c=\frac{3}{2}-\frac{6e^2}{g} \tag{28}$$

with $g=p/2p'$, $e=(p'-p)/2p'$. The dimensions of the operators read

$$h_{rs}=\frac{(rp-sp')^2-(p-p')^2}{8pp'}+\frac{t(2-t)}{16} \tag{29}$$

with the constraints

$$1\le r\le p'-1,\quad 1\le s\le p-1\ ,\quad t=|r-s| \mod 2\ .$$

Here t is $O(1)$ for the Neveu-Schwarz (NS) [Ramond (R)] sector. Unitary theories correspond to $|p-p'|=2$; the simplest realization of this model is the tricritical Ising model.[10] According to Ref. 11, minimal superconformal partition functions are still classified by a pair of simply laced algebras (G,G') of Coxeter numbers p and p'. For the $G=A$ case, modular invariance requires dimensions (29) with $rp-sp'=2n$, $n\in N$ (exponents of G'). This implies that one should supplement the free superfield by a set of $N-1$ fractional electric charges $e_n=n/p'$.

Turning to the supertorus, a way of introducing an electric floating charge to (27) respecting supermodular invariance is to include an interaction term between the shifts m and m', $\cos(2\pi e_n m\Lambda m')$. Here Λ denotes the greatest common divisor of m and m'. As a result, we can build the partition function for the $N=1$ superminimal model on the supertorus with $(+,+)$ boundary conditions:

$$Z=\left[\sum_{m,m'\in Z} Z_1 Z_{m,m'}(g) + \sum_{r,s=0,1}\mathcal{L}_2(r,s)\sum_{\substack{m=r[2]\\ m'=s[2]}} Z_{m,m'}(g)\right] \times\sum_{n\in N}\cos\left[2\pi\frac{n}{p'}m\Lambda m'\right] \tag{30}$$

with $g=p/2p'$. Here, for $p=p'-2$, we have $g=(p'-2)/2p'$, $c=\frac{3}{2}-12/p'(p'-2)$. For a tricritical branch ($p=p'+2$), we have $g=(p'+2)/2p'$, $c=\frac{3}{2}-12/p'(p'+2)$.

V. CORRELATION FUNCTIONS ON THE ODD SUPERTORUS

Let us start with the theory of a Gaussian free superfield on a complex superplane:

$$A=\frac{g}{2\pi}\int_{\text{CSP}} d^4Z\, D_\theta S\, D_{\bar\theta} S\ . \tag{31}$$

In this theory, the basic operators are the vertex operator of the field S, $V_E=\exp(iES)$, satisfying

$$\langle V_E(z_1,\theta_1)V_{-E}(z_2,\theta_2)\rangle\sim|Z_{12}|^{-2E^2/g} \tag{32}$$

with $Z_{12}=z_1-z_2-\theta_1\theta_2$ and the dual operator V_M, its correlation functions of which can be obtained by imposing a discontinuity of $2\pi M$ on the field S when one crosses a line connecting (z_1,θ_1) to (z_2,θ_2):

$$\langle V_M(z_1,\theta_1)V_{-M}(z_2,\theta_2)\rangle\sim|Z_{12}|^{-gM^2/2}\ . \tag{33}$$

Combining (32) and (33), one gets a more general object V_{EM}:

$$\langle V_{EM}(z_1,\theta_1)V_{-E-M}(z_2,\theta_2)\rangle\sim|Z_{12}|^{-(2E^2/g)-(gM^2/2)} \times\exp[-2iEM\alpha(Z_{12})]\ , \tag{34}$$

where $\alpha(Z_{12})$ is the angle of the Z_{12} vector with arbitrary direction. Then V_{EM} has a dimension x and spin s given by (25).

On the supertorus, to take into account the discontinuity of $2\pi M$ on a cut relating (ω_1,ϕ_1) to (ω_2,ϕ_2) we introduce the classical field S_{cl} which satisfies the following: (i) doubly periodic, $S_{\text{cl}}(\omega+1)=S_{\text{cl}}(\omega)$, $S_{\text{cl}}(\omega+\tau)=S_{\text{cl}}(\omega)$; (ii) singular at (ω_1,ϕ_1) and (ω_2,ϕ_2) and (iii) satisfies the classical equation in (20), at $(\omega,\phi)\neq(\omega_1,\phi_1)$, (ω_2,ϕ_2). One candidate is given by

$$S_{\text{cl}}=\Phi_{\text{cl}}+\phi\psi_{\text{cl}}-\bar\phi\bar\psi_{\text{cl}}\ , \tag{35}$$

where

$$\Phi_{\text{cl}}=M\left[\text{Im}\ln\left[\frac{\theta_1(\omega-\omega_1)}{\theta_1(\omega-\omega_2)}\right]-\frac{2\pi}{\tau_I}\text{Im}\omega\,\text{Re}(\omega_1-\omega_2)\right]\ ,$$

$$\psi_{\text{cl}}=\left[-\frac{i\delta}{2\tau_I}\partial_\omega\Phi_{\text{cl}}\right]\bar\omega\ ,\quad \bar\psi_{\text{cl}}=\left[\frac{i\bar\delta}{2\tau_I}\partial_{\bar\omega}\Phi_{\text{cl}}\right]\omega\ .$$

Although ψ_{cl} ($\bar\psi_{\text{cl}}$) is not periodic in the τ direction, these terms do not contribute the final expression of correlation functions. The reason comes from the fact that ψ_{cl} ($\bar\psi_{\text{cl}}$) is dependent on odd Teichmüller parameter δ ($\bar\delta$). Since the final expression of the correlation functions takes

$$\sum_{m,m'}\frac{\bar\delta\delta}{\tau_I}|\eta(\tau)|^2 Z_{m,m'}\langle V_{EM}V_{-E-M}\rangle_{m,m'}\ ,$$

the δ- ($\bar\delta$-)dependent terms cannot contribute to the correlation functions. In other words, we are unable to accommodate the fermionic propagators with this solution.

Another one is given by

$$S_{\text{cl}}=M\left[\text{Im}\ln\left[\frac{\theta_1(\omega-\omega_1-\phi\phi_1|\tau)}{\theta_1(\omega-\omega_2-\phi\phi_2|\tau)}\right] -\frac{2\pi}{\tau_I}\text{Im}[\omega+\tfrac{1}{2}(\phi_1+\phi_2)\phi]\text{Re}\omega_{12} -2\pi\,\text{Re}[\phi(\phi_1-\phi_2)+\phi_1\phi_2]\right] \tag{36}$$

with $\omega_{12}=\omega_1-\omega_2-\phi_1\phi_2$.

This is doubly periodic and singular at (ω_1,ϕ_1) and (ω_2,ϕ_2). Note that (36) satisfies $D_\phi D_{\bar\phi} S_{\rm cl}=0$ instead of (20). In the strict sense, $S_{\rm cl}$ should satisfy (20). However, considering the fact that the zero-mode-dependent term (δ- or $\bar\delta$-dependent term) does not contribute to the correlation function, this case is not un-natural. The desired discontinuities around $(\omega,\phi)=(\omega_1,\phi_1)$ and (ω_2,ϕ_2) appear as

$$\oint_{(\omega_1,\phi_1)}(D_\phi S_{\rm cl}d\omega\, d\phi+D_{\bar\phi}S_{\rm cl}d\bar\omega\, d\bar\phi)=2\pi M\ ,$$
$$\oint_{(\omega_2,\phi_2)}(D_\phi S_{\rm cl}d\omega\, d\phi+D_{\bar\phi}S_{\rm cl}d\bar\omega\, d\bar\phi)=-2\pi M\ . \qquad (37)$$

Then, writing $S\to S_{\rm cl}+S$, the calculation of the functional integral leads to, as in (21),

$$\langle V_{EM}(\omega_1,\phi_1)V_{-E-M}(\omega_2,\phi_2)\rangle_{pp} \propto \exp\Big[iE[S_{\rm cl}(\omega_1,\phi_1)-S_{\rm cl}(\omega_2,\phi_2)] +E^2\langle S(\omega_1,\phi_1)S(\omega_2,\phi_2)\rangle -\frac{g}{2\pi}\int_T d^2\omega\int_B d^2\phi\, D_\phi S_{\rm cl}D_{\bar\phi}S_{\rm cl}\Big]\ , \qquad (38)$$

where $\langle S(\omega_1,\phi_1)S(\omega_2,\phi_2)\rangle$ is the propagator of superfield S on the superspace version of the $(+,+)$ torus, which satisfies

$$D_{\phi_1}D_{\bar\phi_1}\langle S(\omega_1,\phi_1)S(\omega_2,\phi_2)\rangle =\frac{\pi}{g}\left[\delta^2(\omega_1-\omega_2)-\frac{1}{\tau_I}\right]\delta^2(\phi_1-\phi_2)\ . \qquad (39)$$

The explicit form of this propagator is given by

$$\langle S(\omega_1,\phi_1)S(\omega_2,\phi_2)\rangle=-\frac{2}{g}\ln\left[\left|\frac{\theta_1(\omega_{12}|\tau)}{\theta_1'(0|\tau)}\right|\times\exp\left[-\pi\frac{{\rm Im}^2\omega_{12}}{\tau_I}\right]\right]. \qquad (40)$$

If one expands θ_1, $\bar\theta_1$, ω_{12}, and $\bar\omega_{12}$ in terms of $\phi_{\omega_1}\phi_{\omega_2}$ and $\bar\phi_{\bar\omega_1}\bar\phi_{\bar\omega_2}$, then one can easily find $\langle\Phi\Phi\rangle$, $\langle\psi\psi\rangle$, and $\langle\bar\psi\bar\psi\rangle$, which are all orthogonal to the zero modes.

In order to calculate the classical action, we should introduce the other field such as $\bar S_{\rm cl}$:

$$\bar S_{\rm cl}=M\left[\ln\left|\frac{\theta_1(\omega-\omega_1-\phi\phi_1|\tau)}{\theta_1(\omega-\omega_2-\phi\phi_2|\tau)}\right| -\frac{2\pi}{\tau_I}{\rm Re}\omega\,{\rm Re}\omega_{12}\right], \qquad (41)$$

since this leads to

$$D_\phi D_{\bar\phi}\bar S_{\rm cl}=-\frac{\pi M}{2}[\delta^2(\omega-\omega_1)\delta^2(\phi-\phi_1) -\delta^2(\omega-\omega_2)\delta^2(\phi-\phi_2)] \qquad (42)$$

instead of $D_\phi D_{\bar\phi}S_{\rm cl}=0$. Considering these together, (38) takes the form

$$\langle V_{EM}(\omega_1,\phi_1)V_{-E-M}(\omega_2,\phi_2)\rangle_{pp} =\left[\frac{\theta_1'(0|\tau)}{\theta_1(\omega_{12}|\tau)}\right]^{2h_{EM}}\left[\frac{\bar\theta_1'(0|\tau)}{\bar\theta_1(\omega_{12}|\tau)}\right]^{2\bar h_{EM}} \times\exp\left[-\frac{\pi}{\tau_I}(\delta_{EM}\omega_{12}+\bar\delta_{EM}\bar\omega_{12})^2\right] \qquad (43)$$

with

$$\delta_{EM}=\frac{1}{2}\left[\frac{\sqrt2}{\sqrt g}E+\frac{\sqrt g}{\sqrt2}M\right],\quad h_{EM}=\delta_{EM}^2\ ,$$
$$\bar\delta_{EM}=\frac{1}{2}\left[-\frac{\sqrt2}{\sqrt g}E+\frac{\sqrt g}{\sqrt2}M\right],\quad \bar h_{EM}=\bar\delta_{EM}^2\ .$$

This is not the product of an analytic function by an anti-analytic one due to zero modes. For $M\neq0$, it is not doubly periodic, since shifting ω_{12} by 1, τ, or $1+\tau$ is equivalent to adding a new frustration line wrapping around the superspace version of the torus. Further, considering the corresponding contribution from a soliton sector (m,m'), we get

$$\langle V_{EM}(\omega_1,\phi_1)V_{-E-M}(\omega_2,\phi_2)\rangle_{m,m'} =\langle V_{EM}V_{-E-M}\rangle_{pp} \times\exp\left[\pi\sqrt{2g}\left[\frac{m'-m\bar\tau}{\tau_I}\delta_{EM}\omega_{12} +\frac{m'-m\tau}{\tau_I}\bar\delta_{EM}\bar\omega_{12}\right]\right]\ . \qquad (44)$$

The correlation function for a free superfield is given by

$$\langle V_{EM}(\omega_1,\phi_1)V_{-E-M}(\omega_2,\phi_2)\rangle =\frac{1}{Z_{c=3/2}}\frac{\bar\delta\delta}{\tau_I}\sum_{m,m'\in Z}Z_{m,m'} \times\langle V_{EM}(\omega_1,\phi_1) \times V_{-E-M}(\omega_2,\phi_2)\rangle_{m,m'}\ . \qquad (45)$$

Finally, let us transform the (ω,ϕ) coordinates into (z,θ) coordinates. The relation between ω_{12} and Z_{12} is given by

$$\omega_{12}=Z_{12}-\frac{({\rm Im}z_1)(\theta_1+\theta_2)\delta}{\tau_I}\left[1+\frac{i\bar\theta_1\bar\delta}{2\tau_I}\right] +\frac{({\rm Im}z_2)(\theta_1+\theta_2)\delta}{\tau_I}\left[1+\frac{i\bar\theta_2\bar\delta}{2\tau_I}\right]\ . \qquad (46)$$

Considering the presence of the $\bar\delta\delta$ term in (45), we can substitute τ into $T_{12}\equiv\tau+(\theta_1+\theta_2)\delta$ in the odd sector. After a calculation, we arrive at the correlation function of the $N=1$, $c=\frac{3}{2}$ superconformal model on the odd spin structure of the supertorus:

$$\langle V_{EM}(z_1,\theta_1)V_{-E-M}(z_2,\theta_2)\rangle$$
$$=\frac{1}{Z'_{c=3/2}}\frac{\bar{\delta}\delta}{(\mathrm{Im}T_{12})^{3/2}}\sum_{m,m'\in Z}\exp\left[-\pi\frac{|m'-mT_{12}|^2}{\mathrm{Im}T_{12}}\right]\left[\frac{\theta_1'(0|T_{12})}{\theta_1(Z_{12}|T_{12})}\right]^{2h_{EM}}\left[\frac{\bar{\theta}_1'(0|T_{12})}{\bar{\theta}_1(Z_{12}|T_{12})}\right]^{2\bar{h}_{EM}}$$
$$\times\exp\left[-\frac{\pi}{\mathrm{Im}T_{12}}(\delta_{EM}Z_{12}+\bar{\delta}_{EM}\bar{Z}_{12})^2+\pi\sqrt{2g}\left[\frac{m'-m\bar{T}_{12}}{\mathrm{Im}T_{12}}\delta_{EM}Z_{12}+\frac{m'-mT_{12}}{\mathrm{Im}T_{12}}\bar{\delta}_{EM}\bar{Z}_{12}\right]\right] \quad (47)$$

with

$$Z'_{c=3/2}=\sum_{e,m\in Z}\frac{\bar{\delta}\delta}{\mathrm{Im}T_{12}}q_{T_{12}}^{h_{em}}\bar{q}_{T_{12}}^{\bar{h}_{em}}+\sum_{r,s=0,1}\mathcal{L}_2(r,s)\sum_{\substack{m=r[2]\\ m'=s[2]}}Z_{m,m'}(g)\ ,$$
$$q_{T_{12}}=e^{2\pi i T_{12}}\ . \quad (48)$$

Here $\theta_1(Z_{12}|T_{12})$ is the supertheta function, which is defined in the Appendix. Using the properties of the supertheta function, we can check that

$$Z'_{m,m'}\langle V_{EM}V_{-E-M}\rangle_{m,m'}(Z_{12}+1)=Z'_{m,m'-M}\exp[2i\pi E(m-M)]\times\langle V_{EM}V_{-E-M}\rangle_{m,m'-M}(Z_{12})\ , \quad (49)$$

$$Z'_{m,m'}\langle V_{EM}V_{-E-M}\rangle_{m,m'}(Z_{12}+T_{12})=Z'_{m+M,m'}\exp[2i\pi E(m'-M)]\times\langle V_{EM}V_{-E-M}\rangle_{m+M,m'}(Z_{12}). \quad (50)$$

Hence, the correlation function of (47) is doubly periodic only for $E,M=$integers. Also this correlation function is supermodular invariant.

VI. SUMMARY

We derived the partition function for the $N=1$, $c=\frac{3}{2}$ superconformal model on the odd $(+,+)$ spin structure of the supertorus. The partition function of this model on the even spin structures is essentially equivalent to the partition function on the even spin structures of the ordinary torus. Also, introducing the floating charge instead of the background charge, we have built the partition function for the $N=1$ superminimal model on the odd supertorus. Further, using the supersymmetric Coulomb-gas representation, we have constructed the correlation function (two-point function) on the odd supertorus. Although we can derive the multipoint correlators without any further difficulty on the odd torus, the multipoint correlators on the odd supertorus may not appear as a straightforward extension of the two-point correlator because of the presence of the nontrivial boundary conditions (3) and the complicated zero modes on the odd supertorus.

Finally we note that the supertheta functions play an important role in constructing the correlation function on the odd spin structure of the supertorus.

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APPENDIX: THE SUPERTHETA FUNCTIONS

The following definition of the supertheta function depends critically on the fact that the numbers of bosonic moduli and fermionic supermoduli on the $N=1$ supertorus with the odd spin structure are equal. Thus the combination of $T=\tau+\theta\delta$ can be formed, and this will not generalize to higher-genus super Riemann surfaces because for genus-g super Teichmüller spaces there exist $(6g-6)$ even and $(4g-4)$ odd moduli parameters. Under the supermodular transformations, the coordinates transform as

$$z'=\frac{z}{c\tau+d}-\frac{c\theta\delta z}{(c\tau+d)^2},\quad \theta'=\frac{\theta}{(c\tau+d)^{1/2}}-\frac{\delta}{(c\tau+d)^{3/2}}\ , \quad (A1)$$

where $(z,\theta)\in\mathrm{CSP}$ and

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\mathrm{SL}(2,Z)\ .$$

Using this fact and (11), we can also obtain the following supermodular transformations:

$$Z'_{12}=\frac{Z_{12}}{cT_{12}+d},\quad T'_{12}=\frac{aT_{12}+b}{cT_{12}+d}\ . \quad (A2)$$

Under the supermodular transformations, the frame fields and their dual fields transform like

$$E^{\theta'}=(D_\theta\theta')E^\theta+(D_\theta^2\theta')E^z\ ,$$
$$E^{\bar{\theta}'}=(D_{\bar{\theta}}\bar{\theta}')E^{\bar{\theta}}+(D_{\bar{\theta}}^2\bar{\theta}')E^{\bar{z}}\ , \quad (A3)$$
$$E^{z'}=(D_\theta\theta')^2E^z,\quad E^{\bar{z}'}=(D_{\bar{\theta}}\bar{\theta}')^2E^{\bar{z}}\ ,$$

and

$$E'_\theta=(D_\theta\theta')^{-2}E_\theta+(D_\theta\theta')^{-1}[D_\theta(D_\theta\theta')^{-1}]E_+ ,$$
$$E'_{\bar\theta}=(D_{\bar\theta}\bar\theta')^{-2}E_{\bar\theta}+(D_{\bar\theta}\bar\theta')^{-1}[D_{\bar\theta}(D_{\bar\theta}\bar\theta')^{-1}]E_- , \quad (A4)$$
$$E'_+=(D_\theta\theta')^{-1}E_+,\quad E'_-=(D_{\bar\theta}\bar\theta')^{-1}E_-$$

with $D_\theta\theta'=(cT+d)^{1/2}$, $D_{\bar\theta}\bar\theta'=(c\bar T+d)^{1/2}$, $T=\tau+2\theta\delta$, and $\bar T=\bar\tau-2\bar\theta\bar\delta$.

Let us define the supertheta functions as

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(Z_{12}|T_{12})=\sum_n\exp\left\{\pi i\left[T_{12}\left[n+\frac{\epsilon}{2}\right]^2+2\left[n+\frac{\epsilon}{2}\right]\times\left[Z_{12}+\frac{\epsilon'}{2}\right]\right]\right\} \quad (A5)$$

with

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-Z_{12}|T_{12})=(-1)^{\epsilon\epsilon'}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(Z_{12}|T_{12}) ,$$
$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(Z_{12}+1|T_{12})=(-1)^{\epsilon}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(Z_{12}|T_{12}) , \quad (A6)$$
$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(Z_{12}+T_{12}|T_{12})=(-1)^{\epsilon}e^{-\pi i(T_{12}+2Z_{12})}\times\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(Z_{12}|T_{12}) .$$

Under the supermodular transformations in (A2), $\theta_1\equiv\theta[{}^1_1]$ the supertheta function leads to

$$T:\tilde\theta_1\left[\frac{Z_{12}}{T_{12}}\middle|-\frac{1}{T_{12}}\right]=-i\left[\frac{T_{12}}{i}\right]^{1/2}\exp\left[\pi i\frac{Z_{12}^2}{T_{12}}\right]\times\theta_1(Z_{12}|T_{12}) ,$$
$$S:\tilde\theta_1(Z_{12}|T_{12}+1)=\exp\left[\frac{\pi i}{4}\right]\theta_1(Z_{12}|T_{12}) . \quad (A7)$$

Also the supertheta constants transform as

$$T:\tilde\theta'_1\left[0\middle|-\frac{1}{T_{12}}\right]=-i\left[\frac{T_{12}^3}{i}\right]^{1/2}\theta'_1(0|T_{12}) ,$$
$$S:\tilde\theta'_1(0|T_{12}+1)=\exp\left[\frac{i\pi}{4}\right]\theta'_1(0|T_{12}) . \quad (A8)$$

Here a tilde (prime) means the supermodular transformations (differentiation with respect to Z_{12}).

The nontrivial example, which is expressed by this supertheta function, is the two-point function of a scalar superfield on the $(+,+)$ supertorus:

$$\langle S(z_1,\theta_1)S(z_2,\theta_2)\rangle=\frac{\mathrm{Im}Z_{12}^2}{2\,\mathrm{Im}T_{12}}-\frac{1}{4\pi}\ln\left|\frac{\theta_1(Z_{12}|T_{12})}{\theta'_1(0|T_{12})}\right|^2 \quad (A9)$$

which satisfies the following equation:

$$4D_{\theta_1}D_{\bar\theta_1}\langle S(z_1,\theta_1)S(z_2,\theta_2)\rangle=\delta^2(z_1-z_2)\delta^2(\theta_1-\theta_2)-\frac{1}{\mathrm{Im}T_{12}}\left[(\theta_1-\theta_2)-\frac{\mathrm{Im}Z_{12}}{\mathrm{Im}T_{12}}\delta\right]\times\left[(\bar\theta_1-\bar\theta_2)-\frac{\mathrm{Im}Z_{12}}{\mathrm{Im}\,T_{12}}\bar\delta\right] . \quad (A10)$$

$\langle S(z_1,\theta_1)S(z_2,\theta_2)\rangle$ is also doubly periodic under $Z_{12}\to Z_{12}+1$ and $Z_{12}+T_{12}$, and is supermodular invariant. Furthermore, (A10) satisfies

$$\int_{\mathrm{ST}}d^4Z[D_{\theta_1}D_{\bar\theta_1}\langle S(z_1,\theta_1)S(z_2,\theta_2)\rangle]=0 . \quad (A11)$$

This means that two-point function $\langle S(z_1,\theta_1)S(z_2,\theta_2)\rangle$ is orthogonal to zero modes.

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