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Zamolodchikov's *c*-theorem reexamined

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We present an alternative proof of Zamolodchikov's *c*-theorem using finite-size field-theory methods. We find that although the existence of a *c* function which decreases along renormalization-group flows and coincides with the conformal anomaly at the fixed points is independent of the renormalization scheme, the relationship between the *c* function and the beta function is scheme dependent. We discuss some of the consequences of our proof.

Zamolodchikov's *c*-theorem[1] has rapidly become an important tool in the study of two-dimensional field theories. It has been applied to statistical-mechanical models,[2] to the study of the topology of the space of all conformal field theories,[3] and to the generation of the effective action of the bosonic string theory.[4] For this reason we should try to understand this theorem from as many points of view as possible. In this paper we provide an alternative to Zamolodchikov's original proof which uses only basic concepts of the renormalization group and finite-temperature or finite-size field theory. A Wilsonian approach using an infinite-dimensional space of coupling constants is certainly mathematically appealing, but for computational purposes of great relevance, for example, to statistical mechanics, one is usually restricted to considering only a finite number of operators and a *controlled* perturbative approach, since more sophisticated tools are unavailable at present.

Our reasons for presenting this proof are twofold. Firstly, as mentioned above, we hope this proof will lead to new insights about the *c*-theorem. Secondly, we would like to address some subtle but important questions in the original proof. For example, the fact that the stress tensor requires a subtraction is completely glossed over in Ref. 1 (although Polchinski[5] has noticed this and filled in the gap in the proof). Furthermore, one of the main ingredients in the construction of the *c* function is the expression

$$\Theta=\beta^i O_i \ , \tag{1}$$

where Θ is the trace of the energy-momentum tensor, O_i are operators with couplings g_i, and β^i are the corresponding beta functions. As far as we know this expression is usually proven in a particular renormalization scheme, usually dimensional regularization with minimal subtraction. This then brings up the question of whether the relation between the *c* function and the beta functions is renormalization-scheme independent. In particular, Θ must always be subtracted and this procedure depends on the renormalization scheme used. This then raises questions about the original proof since Zamolodchikov makes critical use of the fact that the two-point correlation function of Θ is positive definite. If Θ requires subtractions, there is, in principle, no guarantee of this positivity. The formal proof makes no mention of these subtleties.

Let us first recall the statement of the *c*-theorem. If the space of all two-dimensional (2D) renormalizable and unitary quantum field theories is represented by the space (infinite dimensional) of all coupling constants g_i of these theories,[6] then the theorem claims the existence of a function $c(g_i)$ such that (1) $c(g_i)$ is non-negative and nonincreasing on infrared renormalization-group flows, (2) the stationary points of $c(g_i)$ correspond to critical fixed points of the field theory, and (3) at these fixed points, $c(g_i^*)$ corresponds to the central charge of the Virasoro algebra (the conformal anomaly).

Before we proceed with our proof we must set down some preliminary results. For simplicity we consider a theory with only one bare coupling g_B and an action of the form

$$S=S_0+g_B\int d^2r \ O(\mathbf{r}) \ . \tag{2}$$

The action S_0 describes a critical theory and the operator $O(\mathbf{r})$ is a scalar scaling operator with dimension X in the theory described by S_0. The operator O will be relevant for $2-X>0$, irrelevant for $2-X<0$, and marginal for $X=2$. We are ultimately interested in the relevant case since in this case, there exists the possibility of an IR stable fixed point at $g=g^*$ at which the long-distance properties of the theory will be described by a conformally invariant theory given by $S(g^*)$ and a new value of the conformal anomaly $c(g^*)$.

In our proof we essentially follow the approach advocated by Ludwig and Cardy.[7] The main point is that conformal invariance predicts that the finite-size (or finite-temperature) corrections to the free energy density in a conformally invariant theory are given by[8]

$$\mathcal{F}(L)=\mathcal{F}_b-\frac{\pi c}{6L^2} \ . \tag{3}$$

In the above expression, L is the width of the semi-infinite strip (or inverse temperature), and $\mathcal{F}_b$ is the bulk free energy (calculated in the $L \to \infty$ limit). The constant c is the conformal anomaly. This expression is valid in the critical (fixed-point) theory in flat space. The crucial observation is that the free energy density, like the stress tensor, does *not* acquire an anomalous dimension under renormalization and scales with scaling dimension $X=2$. This fact allows us to write the finite-size corrections to the free energy density in a form similar to that in Eq. (3), *even away from the critical point:*

$$\mathcal{F}(g_B,L)=\mathcal{F}_b(g_B,a)-\frac{\pi c(g_B,L,a)}{6L^2}\ , \tag{4}$$

where the function $c(g_B,L,a)$ is a dimensionless function of its arguments and a is an ultraviolet cutoff introduced to regularize the theory.

Now, all the required subtractions can be absorbed in the bulk contribution $\mathcal{F}_b$ in Eq. (4) since the short-distance singularities are insensitive to finite-size (-temperature) effects and are subtracted at $L=\infty$ $(T=0)$. The free energy density $\mathcal{F}(g_B,L)$ may be written as

$$\mathcal{F}(g_B,L)=-\frac{1}{V}\ln Z_0-\frac{1}{V}\ln\left\langle \exp\left[-g_B\int d^2w\, O\right]\right\rangle_0^L\ , \tag{5}$$

where V is the strip volume, Z_0 is the partition function for the action S_0, and the angular brackets denote an expectation value taken in the finite-size S_0 theory. From Eq. (5) we may read off the expressions

$$\frac{\partial c(g_B,L,a)}{\partial g_B}=-\frac{6L^2}{\pi V}\left\langle\int d^2w\, O(w)\right\rangle_{g_B}^L\ ,$$
$$\frac{\partial^2 c(g_B,L,a)}{\partial g_B^2}=\frac{6L^2}{\pi V}\left\langle\int d^2w_1\int d^2w_2 O(w_1)O(w_2)\right\rangle_{g_B,\mathrm{conn}}^L\ , \tag{6}$$

with $\langle\cdots\rangle_{g_B}$ being the expectation value in the full theory with action $S(g_B)$. Note that $c(g_B,L,a)$ does not require any IR cutoffs because the correlation functions fall off exponentially at large distances. If there is a nontrivial fixed point at $g=g^*$, then at this point the operator O scales with anomalous dimension $X+\gamma(g^*)$ [where $\gamma(g=0)=0$]. At the fixed points the theory is scale invariant, so we expect that operators with nonzero scaling dimensions will have vanishing ground-state expectation values. However, care must be taken in making this claim. The operator O may require normal ordering (subtractions) as well as multiplicative renormalizations. However, we will now argue that the required subtractions are done in the bulk geometry ($L=\infty$, $T=0$) and are taken care of by $\mathcal{F}_b$ in Eq. (4). Computing $\langle O(r)\rangle$ in the plane with an UV cutoff a and an IR cutoff μ^{-1} yields, to lowest order,

$$\langle O(r)\rangle=\frac{g_B}{2-2X}(\mu^{2(X-1)}-a^{2(1-X)})\ .$$

We may also calculate this on the strip, using Eq. (6) and a result from Ref. 7. We find

$$\frac{\partial c}{\partial g_B}=\langle O(r)\rangle_S$$
$$=g_B\left[\frac{2\pi}{L}\right]^{2(X-1)} I_2(X,a)\ ,$$

where $I_2(X,a)$ is defined by[7]

$$I_2(X,a)=-\frac{\pi}{1-X}\left[\frac{2\pi a}{L}\right]^{2(1-X)}+I_2(X,0)\ .$$

We see that the a-dependent piece is L independent and cancels against the a-dependent piece computed on the plane. This argument is in fact quite general. The UV singularities at finite temperature or finite size are the same as at zero temperature or infinite size and they are canceled by the zero-temperature counterterms to all orders. This is a well-known result from work on finite-temperature field theory.

Now at the fixed points $g=0$, g^*, O scales with definite conformal weight and is covariant under conformal transformations. Having been subtracted in the plane, its expectation value on the strip must vanish at these points. This implies that

$$\frac{\partial c}{\partial g}=0 \tag{7}$$

at the fixed points. We are now ready to begin our proof of the c-theorem.

Define the renormalized dimensionless coupling constant λ_R by

$$g_B=\lambda_R\mu^{-\epsilon}Z[\lambda_R,\mu a]\ ,$$
$$Z[\lambda_R,\mu a]=1+z_1(\mu a)\lambda_R+\cdots\ , \tag{8}$$

where μ is a renormalization scale (with the units of mass) and $\epsilon=X-2$. Note that there is no L dependence in this equation since the renormalization constants and counterterms are those of the $L=\infty$ theory.

The fact that the free energy density does not pick up an anomalous dimension, and that $c(g_B,L,a)$ needs no subtractions implies

$$c(g_B,L,a)=c(\lambda_R,L\mu)\ , \tag{9}$$

where $c(\lambda_R,L\mu)$ is finite when all cutoffs are removed keeping λ_R fixed. The μ and L dependence must be as shown since c is a dimensionless object, as is λ_R. From the above equation we may deduce the renormalization-group (RG) equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}\right]c(\lambda_R,L\mu)=0 \tag{10}$$

with

$$\beta(\lambda_R)\equiv\mu\frac{\partial}{\partial\mu}\bigg|_{g_B,a}\lambda_R\ .$$

Writing $L\mu/2\pi$ as e^t, the above RG equation becomes

$$\left[\frac{\partial}{\partial t}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}\right]c(\lambda_R,e^t)=0 \; . \tag{11}$$

This equation can be solved as usual to yield $c=c(\lambda(t))$, where the running coupling constant $\lambda(t)$ is defined via

$$\frac{\partial\lambda(t)}{\partial t}=-\beta(\lambda(t)) \tag{12}$$

with the boundary conditions $\lambda(t=0)=\lambda_R$. Our next task is to calculate the derivative $\partial c(\lambda_R,L\mu)/\partial\lambda_R$ in terms of the beta function. The first step in doing this is to note the following result of Ludwig and Cardy:[7]

$$\left.\frac{\partial^2 c}{\partial g^2}\right|_{\text{fixed pts}}=24\pi^2\frac{\Gamma^2(D/2)\Gamma(1-D)}{\Gamma^2(1-D/2)\Gamma(D)}\left[\frac{2\pi}{L}\right]^{2(D-2)} , \tag{13}$$

where at $g=0$, $D=X$ while at $g=g^*$, $D=X+\gamma(g^*)$. At these fixed points the coupling has dimension $D-2$. Now, at the trivial fixed point $g=0$ the product of gamma functions in Eq. (13) can be written as

$$\frac{\Gamma^2(X/2)\Gamma(1-X)}{\Gamma^2(1-X/2)\Gamma(X)}\simeq\frac{\epsilon}{4}+O(\epsilon^2)+\cdots \; , \tag{14}$$

where we have taken $\epsilon=X-2$ to be much less than one in absolute value. The reason for this expansion in ϵ is that this is the only way to study the nonrenormalizable $(2<X)$ and superrenormalizable $(X<2)$ regimes in a controlled manner. In particular, any perturbative approach will face severe ultraviolet divergences in the irrelevant case and infrared ones in the relevant case so that the perturbation expansion must be controlled in some manner. We choose to do this via an expansion in ϵ. This is completely analogous to the ϵ expansion in the theory of critical phenomena[9] and is the usual way in which one enters the nonrenormalizable and superrenormalizable regimes with a finite number of couplings to all orders. Note that the expression in Eq. (17) has a pole at $X=1$. This pole occurs if the operator O is highly relevant; this case is outside the regime we are equipped to explore via perturbation theory. Let us now write $c=c_0+\bar{c}$ where c_0 is the conformal anomaly of the S_0 theory. Then $\bar{c}(g=0)=0$ and we can expand $\bar{c}$ in a power series in the renormalized, dimensionless coupling λ_R:

$$\bar{c}(\lambda_R)=c_0(\epsilon)\left[\frac{\lambda_R^2}{2}\epsilon+\frac{\lambda_R^3}{3}c_1(\epsilon)+\frac{\lambda_R^4}{4}c_2(\epsilon)+\cdots\right] , \tag{15}$$

where $c_0(\epsilon)$ is a constant given by

$$c_0(\epsilon)=24\pi^2[\tfrac{1}{4}+O(\epsilon)+\cdots] \; . \tag{16}$$

Note that since $c_0(\epsilon)$ is finite as is $c(\lambda_R,L\mu)$, the coefficients c_1,c_2 are finite functions of ϵ (i.e., they have no poles at $\epsilon=0$). We have written $\bar{c}$ in this way so as to be able to compare $\partial c/\partial\lambda_R=\partial\bar{c}/\partial\lambda_R$ with the beta function. We may write

$$\frac{\partial c(\lambda_R)}{\partial\lambda_R}=c_0(\epsilon)[\lambda_R\epsilon+\lambda_R^2c_1(\epsilon)+\lambda_R^3c_2(\epsilon)+\cdots] \tag{17}$$

and we know that this must vanish at the fixed point λ_R^*.

On the other hand, we can compute the beta function in any renormalization scheme to find

$$\beta_\lambda=\lambda_R\epsilon+\beta_1(\epsilon)\lambda_R^2+\beta_2(\epsilon)\lambda_R^3+\cdots \; . \tag{18}$$

The coefficients c_n,β_n in Eqs. (15) and (18) may be expanded in a power series in ϵ:

$$c_n=\sum_{m=0}c_{n,m}\epsilon^m, \quad \beta_n=\sum_{m=0}\beta_{n,m}\epsilon^m \; . \tag{19}$$

For example, in minimal subtraction, the β_n's have no ϵ dependence so that $\beta_n=\beta_{n,0}$. In particular *the only universal coefficient* is given by

$$\beta_{1,0}=\pi d \; , \tag{20}$$

where d is the coefficient of O in the operator product of O with itself. Thus the result in Eq. (20) is completely determined by conformal invariance. In the case of the c_n's, they have contributions from two sources: a universal one and a nonuniversal one. The universal one is completely determined by the correlation functions of the bare operators on the strip. The second contribution arises from coupling-constant renormalization and is nonuniversal. Both these contributions can generate an ϵ dependence for the c_n's which may be quite complicated in general, despite the fact that the ϵ dependence of the nonuniversal piece may be quite simple in some schemes.

The fixed point of the system can be found by expanding λ^* in a power series in ϵ,

$$\lambda^*=\sum_{n=1}\lambda_n\epsilon^n \; , \tag{21}$$

inserting this into the beta function above and setting this equal to zero order by order in ϵ. Doing this we find the relations

$$1+\lambda_1\beta_{1,0}=0, \quad \lambda_2\beta_{1,0}+\lambda_1\beta_{1,1}+\lambda_1^2\beta_{2,0}=0 \; , \tag{22}$$

as well as higher-order ones. However, the fact that $\partial c/\partial\lambda$ vanishes at the fixed points implies that the following equations (together with others arising from higher orders in ϵ) must hold:

$$1+\lambda_1c_{1,0}=0, \quad \lambda_2c_{1,0}+\lambda_1c_{1,1}+\lambda_1^2c_{2,0}=0 \; . \tag{23}$$

By comparing the above equations we see that $\beta_{1,0}=c_{1,0}$ and to second order in λ and ϵ, $\partial c/\partial\lambda=6\pi^2\beta_\lambda$. This equality, however, is a feature of the universal terms in the expansions of c and β to second order, and does *not* hold to higher order. Regardless, we can write the following relation between the beta function and $\partial c/\partial\lambda$:

$$\frac{\partial c}{\partial\lambda}=c_0(\epsilon)\beta_\lambda F(\lambda) \; . \tag{24}$$

The function $F(\lambda)$ can be obtained as a power series in λ, ϵ, and is analytic in ϵ between the two fixed points under consideration. This follows since both β_λ and $\partial c/\partial\lambda$ are both analytic. We can determine the first few terms in the power-series expansion of $F(\lambda)$:

$$F(\lambda)=1+A(\epsilon)\lambda+B(\epsilon)\lambda^2+\cdots\,,$$
$$A(\epsilon)=(c_{1,1}-\beta_{1,1})+O(\epsilon)+\cdots\,, \qquad (25)$$
$$B(\epsilon)=(c_{2,1}-\beta_{2,1})+O(\epsilon)+\cdots\,,$$

where the ellipses in the expressions for $A(\epsilon)$ and $B(\epsilon)$ represent higher-order terms in ϵ. The important thing to note is that the function $F(\lambda)$ is *positive definite* within the ϵ expansion between the two fixed points. $F(\lambda)$ has no zeros between the fixed points since it is analytic in ϵ and β_λ coincides with $\partial c/\partial\lambda$ to second order in ϵ (up to an overall factor). Another way to say this is that if we try to set $F(\lambda)$ equal to zero within the ϵ expansion, we would have to try to balance a term of order one with one of order ϵ or smaller. This clearly cannot work. The fact that $F(\lambda)$ is positive definite essentially yields the c-theorem as stated in the beginning of this paper. If we insert Eq. (24) into Eq. (12) we arrive at

$$\frac{\partial c}{\partial t}=-\beta^2[c_0(\epsilon)F(\lambda)]\,. \qquad (26)$$

The term in square brackets is positive definite since the order-one term in the ϵ expansion of $c_0(\epsilon)$ is positive due to the unitarity of the S_0 theory. In fact, unitarity determines the sign of the two-point correlation function from which $c_0(\epsilon)$ is determined.

Thus, we see that our c function satisfies the conditions of the c-theorem stated above.

In Zamolodchikov's original work, it was suggested that the c function could be evaluated by using the following relation between the gradient of the c function and the beta function:

$$\frac{\partial c}{\partial\lambda}=(\text{const})\beta(\lambda)\,. \qquad (27)$$

Our aim is to show that a relation of this sort is only valid in *one* renormalization scheme. To do this we will make use of the fact that the c function transforms as a scalar under coupling-constant redefinitions, and that the beta functions are the components of a vector field defined over coupling constant space (again for simplicity we consider only the one coupling case). Under a reparametrization $\lambda\rightarrow g(\lambda)$ we have

$$c(\lambda)=c(g(\lambda)),\quad \beta_\lambda\frac{\partial c}{\partial\lambda}=\beta_{g(\lambda)}\frac{\partial c(g)}{\partial g}\,. \qquad (28)$$

We can use this freedom in the choice of parametrization to eliminate the function $F(\lambda)$ from Eq. (24). Furthermore, because $F(\lambda)$ is analytic in ϵ and has no zeros between the fixed points, we can find a reparametrization that is analytic in ϵ to do the job:

$$g(\lambda)=\lambda(1+g_1\lambda+g_2\lambda^2+\cdots)\,,$$
$$\left[\frac{\partial g(\lambda)}{\partial\lambda}\right]^2=F(\lambda)\,. \qquad (29)$$

If we make this reparametrization we have

$$\frac{\partial c(g)}{\partial g}=c_0(\epsilon)\beta_g\,. \qquad (30)$$

This coupling-constant redefinition is equivalent to the choice of a renormalization scheme, so that the function $F(\lambda)$ can be eliminated *only in one scheme.* Thus the beta function must be computed in the scheme in which the coefficients β_n are the same as the c_n in Eq. (15).

The next question that arises is can we find a function $G(\lambda)$ that satisfies the results of the c-theorem *and* also satisfies Eq. (27) in *any* scheme? We now show that the answer to this is a resounding no. For suppose that such a function $G(\lambda)$ did exist. If we set $h(\lambda)=G(\lambda)-c(\lambda)$ then we require that, in any scheme,

$$\frac{\partial c}{\partial\lambda}-c_0(\epsilon)\beta_\lambda=-\frac{\partial h}{\partial\lambda}\,. \qquad (31)$$

From Eqs. (24) and (25) we see that this equation can be integrated from $\lambda=0$ to $\lambda=\lambda^*$. Comparing the result with the fixed-point conditions of Eq. (22) we can easily convince ourselves that to all orders in a double expansion in λ and ϵ that $h(\lambda^*)$ vanishes only if $\beta_n(\epsilon)=c_n(\epsilon)$, i.e., only in the renormalization scheme in which Eq. (30) holds. Thus, no such function $G(\lambda)$ exists. This fact is of great importance since it means that if the beta function is used to calculate the c function, we must be sure to be using the correct renormalization prescription.

In summary, we have offered an alternative proof of the c-theorem within the framework of a double expansion in the couplings and ϵ. Although our proof was within the framework of perturbation theory, it is valid to *all* orders in both λ and ϵ. Admittedly, at large orders of perturbation theory, convergence problems in the ϵ expansion may arise (involving Borel summability and/or renormalons). If this occurs a nonperturbative analysis of the underlying model must be carried out, a fact that will cast doubt on *any* perturbative approach to understanding the theory. We have explicitly shown that some of the relations involving the c function and the beta functions are renormalization-scheme dependent and have given a way to find this scheme. We are in the process of extending this work and applying it to a variety of systems such as the Coulomb gas[10] and perturbations with vertex operators. These will appear in a later work.[11] Finally, we say some words on the case where many couplings are involved. Whereas it might seem that our result is rather restricted since only one coupling constant was considered, the many coupling case can be obtained from our work by noticing the following fact. If we have a system with n couplings and two nearby fixed points, the beta functions of the theory will determine a particular trajectory in coupling-constant space (a one-dimensional submanifold) along which $n-1$ of the couplings can be parametrized in terms of only one linear combination of the n original couplings. This restricts the theory along this fixed path to depend on only one coupling and thus we are back at the case treated in this paper. An explicit realization of this scenario will be found in our treatment of the Coulomb gas in Ref. 11.

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[1]A. B.Zamolodchikov, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 565 (1986) [JETP Lett. **43**, 730 (1986)]; Yad. Fiz. **46**, 1819 (1987) [Sov. J. Nucl. Phys. **46**, 1090 (1987)].

[2]J. L. Cardy, Les Houches Lectures on Fields, Strings and Statistical Mechanics, 1988 (unpublished); Phys. Rev. Lett. **60**, 2709 (1988); Phys. Lett. B **215**, 749 (1988).

[3]C. Vafa, Phys. Lett. B **212**, 28 (1988); S. R. Das, G. Mandal, and S. Wadia, Mod. Phys. Lett. A **4**, 745 (1989). V. Periwal, Commun. Math. Phys. **120**, 71 (1988).

[4]A. A. Tseytlin, Phys. Lett. B **194**, 63 (1987); N. E. Mavromatos and J. L. Miramontes, *ibid.* **212**, 33 (1988); N. E. Mavromatos, Phys. Rev. D **39**, 1659 (1989); S. P. de Alwis, Phys. Lett. B **217**, 467 (1989).

[5]J. Polchinski, Nucl. Phys. **B303**, 226 (1988).

[6]T. Banks and E. Martinec, Nucl. Phys. **B294**, 733 (1987).

[7]A. W. W. Ludwig and J. L. Cardy, Nucl. Phys. **B285** [FS19], 687 (1987).

[8]H. W. Blote, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, *ibid.* **56**, 746 (1986); J. L. Cardy, J. Phys. A **19**, L1093 (1986).

[9]See, for example, D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978), and references therein.

[10]D. Boyanovsky, J. Phys. A **22**, 2601 (1989).

[11]D. Boyanovsky and R. Holman, Nucl. Phys. B (to be published).