Global gauge anomalies for theories with the Green-Schwarz local-anomaly-cancellation mechanism

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We show that theories with the Green-Schwarz mechanism of canceling local anomalies in even dimensions have the possibility of the worst type of global Yang-Mills chiral anomalies given by the homotopy group of the Yang-Mills gauge group. The examples are given for SU(3) gauge theories in six dimensions $[\Pi_6(\mathrm{SU}(3))=Z_{3!}]$ whose global anomalies are of types $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$.

One of the miracles of superstring theories is a fantastic cancellation of chiral local anomalies in both gravitational and Yang-Mills gauge sectors, using the Green-Schwarz mechanism. Without the Green-Schwarz mechanism, we would not have interesting consistent theories in ten dimensions. Later the connection between heterotic string structures and the Green-Schwarz mechanism was clarified:[1] Once the modular invariance is imposed, heterotic string theories in *any* dimensions will have the Green-Schwarz structure of canceling local anomalies. Furthermore, once local anomalies are canceled by the Green-Schwarz mechanism in high dimensions, that theory remains locally anomaly-free in lower dimensions.[2] However, we have so far found no connections between string structures and global anomalies, except that usually the Yang-Mills group in string theories happens to have vanishing homotopy groups and thereby no global anomalies to worry about for the Yang-Mills sector. On the contrary, as Witten pointed out, the absence of gravitational *global* anomalies puts severe constraints on the possible compactification of strings. That is, the use of strings does not lead to the absence of global anomalies. Then, we wonder whether or not the global Yang-Mills anomalies put severe constraints in higher dimensions. (In four dimensions, the global Yang-Mills anomalies do not put strong constraints, because the possible global anomaly is only $\mathbb{Z}_2$ [for SU(2), Sp(N) with any N, SO(N), Spin(N) with $N=3,5$], $\mathbb{Z}_2\oplus\mathbb{Z}_2$ [for SO(4) and Spin(4)], and no global anomalies for other groups and thus just doubling fermion multiplets always cancels the global anomaly). Furthermore, seeing that even in string theories many possibilities exist, we should not limit ourselves to just strings. Although the search for alternatives seems very difficult, they must be anomaly-free in a local and global sense. Looking for clues, many people have searched for local-anomaly-free configurations in both gravitational and Yang-Mills sectors in higher dimensions, since critical dimensions for alternatives again may not be in four dimensions. The use of the Green-Schwarz mechanism greatly helped them to find solutions. However, in this paper we show that *global Yang-Mills anomalies are worse with the Green-Schwarz mechanism* than with the complete cancellation of local anomalies. Thus, as in the case of global gravitational anomalies, the absence of global Yang-Mills anomalies will put severe constraints for model construction with Yang-Mills gauge groups with nonvanishing homotopy groups in higher dimensions.

A number of papers[3–9] have discussed the chiral global Yang-Mills (YM) gauge anomalies when the local YM gauge anomaly is *completely* canceled, i.e.,

$$\mathrm{Tr}X^{n+1}=0 \tag{1}$$

in $(D=2n)$-dimensional space-time where the gauge group in H and $X\in\mathcal{H}$ (Lie algebra of H). In particular, we have proved that the global anomaly is at most of type $\mathbb{Z}_2$ when Eq. (1) is satisfied.[4] There are many cases where the global anomalies do not exist, even though $\Pi_{2n}(H)\neq 0$ (Ref. 3–9). In this paper, we discuss the global anomalies for theories with the Green-Schwarz mechanism of canceling the local gauge anomaly,[10] which do not satisfy Eq. (1) but the weaker factorization condition explained below. Although the general formula for global chiral anomalies exists, including both the Yang-Mills and the gravitational ones, it is expressed in terms of topological indices, which are unfortunately hard to estimate, except for simple cases.[11] Here, we develop a simple formula which is expressed in terms of a purely topological number and a purely group-theoretical quantity—a Dynkin index. Thus, it is easy to estimate the global anomalies. Using this formula, we show that the global anomalies with the Green-Schwarz mechanism can be worse than $\mathbb{Z}_2$. The catch is that this formula can be applied only to the global chiral YM gauge anomalies, but not the gravitational ones. Because of this catch, we disregard the dependence on the gravitational curvature two-forms hereafter in this paper. This is the generalization of the formula we have for theories with complete cancellation of local anomalies.[3–9] Note that superstring theories in $D=10$ do not have global gauge anomalies, since $\Pi_{10}(\mathrm{E}_8)=\Pi_{10}(\mathrm{SO}(32))=0$. However, there may exist yet unknown theories which use different Yang-Mills gauge groups, whose local anomalies are canceled by the Green-Schwarz mechanism[10] or its generalization.[12] Note that there are many local anomaly-free solutions with the Green-Schwarz mechanism[13,14] or its generalization.[12] Note also that the modular invariance of heterotic string theories requires the Green-Schwarz mechanism.[14] We note here that a similar approach must have been taken by Harvey,[15] although his results are not known to us.

The (generalized) Green-Schwarz mechanism of canceling the local anomalies proceeds as follows.[10,12] For the group H, we demand that the anomaly form factorize, which is the $2n+2$ form made of traces of curvatures R and F in dimension $2n$:

$$I_{2n+2}=X_{2l}X_{2n+2-2l}\ , \tag{2}$$

where X_{2m} is an invariant polynomial of $2m$-form, made of curvature two-forms F and R. Then, the anomaly for this theory, ω^1_{2n}, becomes

$$\delta S_0=\omega^1_{2l-2}X_{2n+2-2l}\ . \tag{3}$$

As explained in Ref. 12, this is not equivalent to the so-called Bose-symmetrized one given by Green and Schwarz.[10] However, the difference between them is given by exact forms and/or gauge variations of some forms. Thus, the ambiguities of defining anomalies allows us to choose this form. Note that both satisfy the Wess-Zumino consistency condition. We add the counterterm of the following, using the antisymmetric tensor B_{2l-2},

$$\Delta S=B_{2l-2}X_{2n+2-2l} \tag{4}$$

with the transformation property

$$\delta B_{2l-2}=-\omega^1_{2l-2}\ . \tag{5}$$

Then the local anomaly is completely canceled. This means that our new action, which does not have local anomalies, is of the form

$$S=S_0+\Delta S\ . \tag{6}$$

Now, we would like to calculate the global gauge anomalies with the additional term ΔS. (We postpone the argument on the representation dependence until the last moment.) In order to do so, we embed the gauge group H into a larger group G, whose homotopy group does vanish in $D=2n$ (i.e., no global gauge anomalies for G). However, since the group is now G, the Green-Schwarz mechanism does not work for G and thus the local anomalies exist for G. Then, we can calculate the anomalies for a finite gauge transformation by *integrating* the local anomalies of G caused by infinitesimal gauge transformations. Thus, the local anomaly form for G is of the form

$$\begin{aligned}\delta S&=\delta S_0+\delta B_{2l-2}X_{2n+2-2l}\\&=\omega^1_{2n}-\omega^1_{2l-2}X_{2n+2-2l}\equiv\Omega^1_{2n}\end{aligned} \tag{7}$$

since $\delta X_{2n+2-2l}=0$. We calculate the anomaly for this particular Ω^1_{2n}. For a finite gauge transformation $g(x)$, we can find its extension $\tilde g(x,t)$ such that $\tilde g(x,0)=1$ and $\tilde g(x,1)=g(x)$, because $\Pi_{2n}(G)=0$. Then, the $\tilde g$ gauge-transformed A and F are given by

$$\begin{aligned}A^{\tilde g}&=\tilde g^{-1}A\tilde g+\tilde g^{-1}d_x\tilde g+\tilde g^{-1}d_t\tilde g=\mathcal{A}+v\ ,\\F^{\tilde g}&=dA^{\tilde g}+(A^{\tilde g})^2=\tilde g^{-1}F\tilde g=\mathcal{F}\ ,\end{aligned} \tag{8}$$

where

$$\begin{aligned}\mathcal{A}&=\tilde g^{-1}A\tilde g+\tilde g^{-1}d_x\tilde g\ ,\\v&=\tilde g^{-1}d_t\tilde g\ .\end{aligned} \tag{9}$$

$\mathcal{A}$ and $\mathcal{F}$ denote forms purely in x. Now, the anomaly for a finite gauge transformation g is given by[16]

$$\int_0^1\int_{S^{2n}}\Omega^1_{2n}(\mathcal{A},\mathcal{F},v)=\int_{D^{2n+1}}\Omega^1_{2n}(\mathcal{A},\mathcal{F},v)\ , \tag{10}$$

where we integrated the gauge transformation from 1 to g (i.e., $t=0$ to $t=1$). Note that, in general,[16]

$$\begin{aligned}\Omega^0_{2n+1}(A^{\tilde g},F^{\tilde g})&=\Omega^0_{2n+1}(\mathcal{A}+v,\mathcal{F})\\&=\Omega^0_{2n+1}(\mathcal{A},\mathcal{F})+\Omega^1_{2n}(\mathcal{A},\mathcal{F},v)+\cdots\ .\end{aligned}$$

However, we have only one parameter extension of $g(x)$ to $\tilde g(x,t)$ and thus in the present case we have

$$\Omega^0_{2n+1}(A^{\tilde g},F^{\tilde g})=\Omega^1_{2n}(\mathcal{A},\mathcal{F},v)\ .$$

Thus, the anomaly is given by

$$\int_{D^{2n+1}}\Omega^0_{2n+1}(A^{\tilde g},F^{\tilde g})\ . \tag{11}$$

By adding vanishing terms, we have

$$\begin{aligned}&\int_{D^{2n+1}}\{[\omega^0_{2n+1}(A^{\tilde g},F^{\tilde g})-\omega^0_{2n+1}(A,F)]-[\omega^0_{2l-1}(A^{\tilde g},F^{\tilde g})-\omega^0_{2l-1}(A,F)]X_{2n+2-2l}(F)\}\\&\qquad=\int_{D^{2n+1}}[\gamma_{2n+1}(\tilde g,A,F)-\gamma_{2l-1}(\tilde g,A,F)X_{2n+2-2l}(F)]\ ,\end{aligned} \tag{12}$$

where we have defined, for an arbitrary m,

$$\gamma_{2m+1}(\tilde g,A,F)=\omega^0_{2m+1}(A^{\tilde g},F^{\tilde g})-\omega^0_{2m+1}(A,F)\ . \tag{13}$$

Note that this form γ_{2m+1} can always be written as[16]

$$\begin{aligned}\gamma_{2m+1}(\tilde g,A,F)=&\left[\frac{i}{2\pi}\right]^m\frac{m!}{(2m+1)!}\mathrm{Tr}(\tilde g^{-1}d\tilde g)^{2m+1}\\&+d\alpha(\tilde g,A,F)\ ,\end{aligned} \tag{14}$$

where the exterior derivative acts on the disk D^{2n+1}. That is, the dependence on the field A and F is contained only in the exact form α. The asterisk denotes the normalization factor.

In order to calculate the global anomaly caused by a nontrivial gauge transformation h of H [i.e., h is a nontrivial element of $\Pi_{2n}(H)$], we limit our $\tilde g$ for those which reduce to the nontrivial gauge transformation h of H at the boundary of the disk, S^{2n}. Then the anomaly calculated above corresponds to the global anomaly caused by

h of H. Since $\tilde{g}$ is now a map, $(D^{2n+1}, S^{2n}) \to (G,H)$, the extension $\tilde{g}$ is now classified by the relative homotopy group Π_{2n+1} (G,H). Using the homotopy diagram,[8]

$$\begin{array}{ccc} (D^{2n+1}, S^{2n}) & \xrightarrow{\tilde{g}} & (G,H) \\ \downarrow & & \downarrow \\ (S^{2n+1}, *) & \xrightarrow{f} & (G/H, *) \end{array} \tag{15}$$

we can find an element f, which is a map $S^{2n+1} \to G/H$, corresponding to $\tilde{g}$. Then, the integration over the disk for $\tilde{g}$ is reduced to the integration over the sphere for f (Ref. 8):

$$\Gamma = \int_{S^{2n+1}} [\gamma_{2n+1}(f, A, F) - \gamma_{2l-1}(f, A, F) X_{2n+2-2l}(F)] , \tag{16}$$

since the integrand vanishes on the boundary S^{2n} of the disk D^{2n+1}. Because

$$\int_{S^{2n+1}} d\alpha(f, A, F)_{2l-2} X_{2n+2-2l}(F) = \int_{S^{2n+1}} \{ d[\alpha(f, A, F)_{2l-2} X_{2n+2-2l}(F)] + \alpha(f, A, F)_{2l-2} dX_{2n+2-2l}(F) \} = 0 ,$$

we can ignore the dependence on A and F in $\gamma(f, A, F)$. The second term in Eq. (16) indicates that the global anomaly depends on the field strength F. Note that the index formula for the global anomalies also shows this fact.[11] However, in the present case the local Poincaré lemma

$$X_{2n+2-2l}(F) = d\omega_{2n+1-l}(A, F)$$

holds globally for $l \geq 1$, because the de Rham cohomologies on the sphere are nonvanishing only at the dimension of the sphere. Thus, it can be ignored, because then the second term can be rewritten as

$$\int_{S^{2n+1}} \gamma_{2l-1} d\omega^0_{2n+1-2l} = -\int_{S^{2n+1}} d(\gamma_{2l-1}\omega_{2n+1-2l}) + \int_{S^{2n+1}} d\gamma_{2l-1}\omega^0_{2n+1-2l} = 0 , \tag{17}$$

since $d\gamma_{2l-1} = 0$ and the sphere has no boundary. Therefore, we obtain the general formula for the global anomaly.

Proposition. The global gauge anomaly of H is given by

$$\Gamma = \int_{S^{2n+1}} \gamma_{2n+1}(f) = \left[\frac{i}{2\pi} \right]^n \frac{n!}{(2n+1)!} \times \int_{S^{2n+1}} \mathrm{Tr}(f^{-1} df)^{2n+1} , \tag{18}$$

where f is the element of $\Pi_{2n+1}(G/H)$. The group G contains H as a subgroup and its homotopy group $\Pi_{2n}(G)$ vanishes. The representation of G must reduce to the representation of H under consideration and possibly singlets.

This is the same form as for the complete cancellation of the local anomaly. Since this integration is homotopy invariant, the global anomaly only depends on the nontrivial elements in $\Pi_{2n+1}(G/H)$. Since it is homomorphic, no global anomaly exists for a homotopy group $\Pi_{2n+1}(G/H)$ being finite.[8]

So far we have ignored the dependence on the representation of H. Actually, in order to obtain the global anomaly for a representation ω of H, we must find a representation $\tilde{\omega}$ of G such that $\tilde{\omega}$ reduces to ω plus singlets under reduction of G to H. Thus, the application of our formula depends upon whether we can find such $\tilde{\omega}$ and G. We can always do so in the case of any representations of $H = \mathrm{SU}(N)$, $\mathrm{Sp}(2N)$, and G_2, and tensor representations (except self-dual ones) of $\mathrm{SO}(N)$.[5–9] Other groups we so far have to do case by case.

Consider the case where both $\Pi_{2n+1}(G)$ and $\Pi_{2n+1}(G/H)$ contain only one $\mathbb{Z}$. Then Eq. (19) takes the integral multiple of the form[4–9]

$$\Gamma_0 = 2\pi i \frac{1}{a} Q^G_{n+1}(\tilde{\omega}) , \tag{19}$$

where a is an integer of the map $\alpha \to a\beta$ with α $[\beta]$ the generator of $\mathbb{Z}$ for $\Pi_{2n+1}(G)$ $[\Pi_{2n+1}(G/H)]$ (Ref. 8). The quantity $Q^G_{n+1}(\tilde{\omega})$ denotes the Dynkin index (closely related to the Casimir invariant) of G for the representation $\tilde{\omega}$.[17]

It may be very difficult to obtain the constraint on the Dynkin index, since we do not have $\mathrm{Tr}X^{n+1} = 0$ for H in general, although the Green-Schwarz mechanism certainly picks up particular representations of a group. Thus, the analysis of the global gauge anomalies will become case by case, contrary to the case of the complete cancellation of the local anomalies where we have obtained representation-independent results. Only for self-contragredient representations in $D \equiv 0$ (mod 4), which automatically satisfy $\mathrm{Tr}X^{n+1} = 0$, can we prove the following general results:[9] No global anomalies exist for real reps in $D \equiv 4$ (mod 8), while for pseudoreal representations no global anomalies exist in $D \equiv 0$ (mod 8).

As examples of the application of our formula Eq. (19), we discuss SU(3) theories in $D = 6$ where $\Pi_6(\mathrm{SU}(3)) = \mathbb{Z}_6$. We choose $G = \mathrm{SU}(4)$. Then, $\Pi_6(\mathrm{SU}(4)) = 0$, $\Pi_7(\mathrm{SU}(4)) = \mathbb{Z}$, and $\Pi_7(\mathrm{SU}(4)/\mathrm{SU}(3)) = \mathbb{Z}$ and thus the condition for Eq. (19) is satisfied. In addition we know $a = 3! = 6$ and thus the largest possible global anomaly is of type $\mathbb{Z}_6$ (Ref. 4). However, in the case of the complete cancellation of the local anomaly, Eq. (1), we have shown in Refs. 4, 7, and 9 that no global anomalies exist for SU(3) in any even dimensions. Here, we show that with the use of the Green-Schwarz mechanism, there may exist a $\mathbb{Z}_6$ global anomaly in some cases.

First of all, we must find solutions with the Green-Schwarz mechanism of canceling the local anomalies. Fortunately, we found *regular* solutions in $D = 6$ with $H = \mathrm{SU}(3)$ in Ref. 18, which have neither the local Yang-Mills anomalies nor the local gravitational anomalies. The two solutions we take are (a) $\omega = \lambda_2 + 5\lambda_1$ and (b) $\omega = \text{adjoint} + m\lambda_1$ (arbitrary m), where λ_2 (3^*) is a two-index antisymmetric tensor representation and λ_1 (3) is the basic vector representation of SU(3).

For the solution (a), we use $G = \mathrm{SU}(4)$ with

$$\tilde{\omega} = \lambda_2 + 4\lambda_1 ,$$

where λ_2 is the **6** and λ_1 is the **4** of SU(4). For this choice of $\tilde{\omega}$, we have

$$Q_4^{SU(4)}(\tilde{\omega}) = -4 + 4\times 1 = 0$$

since $Q_4^{SU(4)}(\lambda_2) = 4 - 2^3$ and $Q_4^{SU(4)}(\lambda_1) = 1$ (Ref. 19). Thus, although we use the Green-Schwarz mechanism of canceling the local anomalies,

$$\mathrm{Tr}^{(\omega)}F^4 = \tfrac{1}{12}(\mathrm{Tr}^{(\omega)}F^2)^2 \neq 0 ,$$

the global anomaly does not exist for this solution (a).

For the second solution (b), we use again $G = SU(4)$ with

$$\tilde{\omega} = \text{adjoint} - \lambda_2 - \{2\lambda_1\} + (m+1)\lambda_1 ,$$

where $\{2\lambda_1\}$ is the two-index symmetric tensor. For this choice, we have

$$Q_4^{SU(4)}(\tilde{\omega}) = (16 - 3^3) - (4 - 2^3) - (4 + 2^3) + (m+1)$$
$$= m - 18 \equiv m \pmod 6 .$$

Consequently, we have the global anomaly of

$$\mathbb{Z}_2 \text{ for } m \equiv 3 \pmod 6 ,$$
$$\mathbb{Z}_3 \text{ for } m \equiv 2,4 \pmod 6 ,$$
$$\mathbb{Z}_6 \text{ for } m \equiv 1,5 \pmod 6 ,$$

while for $m \equiv 0 \pmod 6$ we have no global anomaly. Thus, we can have the global anomaly of the largest kind, $\mathbb{Z}_6$, for the Green-Schwarz mechanism

$$\mathrm{Tr}^{(\omega)}F^4 = \frac{1}{2}\frac{m+18}{(m+6)^2}(\mathrm{Tr}^{(\omega)}F^2)^2 \neq 0 .$$

As can be seen from these two examples, the global YM gauge anomalies with the Green-Schwarz mechanism may yield global anomalies of the possible worst kind, i.e., $\mathbb{Z}_a$, where a is completely fixed by the topology of the choice of the group G. Thus, we have to make sure that no global anomalies exist for theories with the Green-Schwarz mechanism of canceling the local anomalies.

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