Further study of global gauge anomalies of simple groups

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We generalize results of our previous studies for global gauge anomalies of simple Lie groups in even-dimensional space $D=2n$. Assuming the absence of local anomalies, we first show that any real (or orthogonal) representation has no global anomaly in $D\equiv 2,4,6 \pmod 8$, and that the pseudoreal (or symplectic) one is free of global anomalies in $D\equiv 0,2,6 \pmod 8$. Second, we prove that the SU(3) group has no global anomalies in any even dimension and neither does SU(4) in $D\equiv 4 \pmod 8$.

I. INTRODUCTION AND SUMMARY OF MAIN RESULTS

It was noted by Witten[1] in 1982 that any self-consistent gauge theory containing Weyl fermions must be free of global (or nonperturbative) as well as local (or perturbative) anomalies. In four-dimensional space-time, the conditions for the absence of these anomalies are now well known. Let ω designate the representation content of Weyl fermions for a compact gauge group H, and let $Q_p(\omega)$ be the pth Dynkin index[2] of ω. Any theory is free of local anomaly, provided that we have $Q_3(\omega)=0$. Moreover, it then possesses no global anomaly with the exceptions of Sp(2N) and SU(2) groups.[3] For these groups, the absence of the global anomaly is to have $Q_2(\omega)=0 \pmod 2$ as will be shortly explained. These facts would be useful for any phenomenological grand-unified-theory (GUT) model construction.

It may be of some interest to study these anomalies in higher dimensions in view of the emergence of superstring theories as the ultimate unified theory. It is quite remarkable that recent superstring theories in ten dimensions possess neither local nor global Yang-Mills anomalies at all. (However, the absence of global Yang-Mills anomalies is the result of a rather trivial fact: the vanishing of the relevant homotopy groups.) Because of the topological origin of anomalies, the presence of anomalies is more serious than the lack of renormalizability. The breakdown of gauge invariance almost certainly cannot be cured by a short-distance cutoff. Thus, anomalies can serve a severe constraint for any theories dealing with long- or short-distance behavior as pointed out by 't Hooft. After seeing that the uniqueness of string theories has gone and many possibilities have opened up even in string theories, one may speculate that there may exist alternative viable theories. In order to look for clues for such theories, many authors[4] including us have searched local anomaly-free configurations (anomaly-free in *both* gravitation *and* Yang-Mills sector) with more relaxed conditions and found many possibilities. Because the absence of local anomalies does not guarantee the absence of global anomalies, separate studies on global anomalies are needed.

In a series of papers,[5–9] we have analyzed the global (nonperturbative) anomaly of compact connected simple Lie gauge group H in *even*-dimensional space with

$$D=2n \ , \tag{1.1}$$

where the list of compact connected simple Lie groups is as follows:[10] SO(N) ($N\geq 3$, $N\neq 4$), Spin(N) ($N\geq 3$, $N\neq 4$), Sp(2N) ($N\geq 1$), SU(N) ($N\geq 2$), G_2, F_4, E_6, E_7, and E_8. There exist isomorphisms among lower-rank groups: Spin(3)=SU(2)=Sp(2)=S^3, Spin(5)=Sp(4), Spin(6)=SU(4). All the groups are simply connected except SO(N) ($N\geq 3$). The group Spin(N) doubly covers SO(N) for $N\geq 2$. The groups, SO(2), Spin(2), SO(4), and Spin(4) are not simple, i.e., Spin(2)=SO(2)=U(1)=S^1 and Spin(4)=SU(2)×SU(2). Because of the isomorphisms for homotopy groups,

$$\Pi_k(\mathrm{Spin}(N))=\Pi_k(\mathrm{SO}(N)) \quad (k\geq 2,\ N\geq 3) \ ,$$

we hereafter use the notation SO(N) for both Spin(N) and SO(N). The U(1) symmetry does not contribute to the global anomaly, since $\Pi_k(\mathrm{U}(1))=0$ for $k\geq 2$. We use the convention of Sp(2N) with N being the rank, while the notation of Sp(N) with N being the rank is used in mathematics literature.

Let X be the generic element of the Lie algebra $\mathcal{H}$ of H in a generic representation ω. Hereafter, we assume the absence of the local (or perturbative) anomaly,[11]

$$\mathrm{Tr}^{(\omega)}X^{n+1}=0 \ , \tag{1.2}$$

unless it is stated otherwise. Although this condition is stronger than the Green-Schwarz ansatz,[12] it may be noted that Eq. (1.2) is automatically satisfied for cases of n being even, i.e., for $D=0 \pmod 4$ when ω is self-contragredient (either real or pseudoreal) representation. (See Appendix A for the meaning of the adjectives self-contragredient, real, and pseudoreal.) Especially, if H is any one of the following groups:

$$\mathrm{SU}(2),\ \mathrm{Sp}(2N), \mathrm{SO}(2N+1),\ \mathrm{SO}(4N),\ \mathrm{G}_2,\ \mathrm{F}_4,\ \mathrm{E}_7,\ \mathrm{E}_8\ , \tag{1.3}$$

then the local anomaly-free condition, Eq. (1.2), is automatically satisfied for any $D \equiv 0 \pmod 4$ dimension, since all representations of groups listed above are known to be self-contragredient.[13] Let $A(\omega)$ be the fundamental global anomaly coefficient of the representation ω of the group H, so that any global anomaly is some integer power of $A(\omega)$. Then, utilizing the general formula of Bismut and Freed,[14] we have proved the following proposition in Ref. 5.

Proposition 1. (i) The global anomaly coefficient $A(\omega)$ is of the type $\mathbb{Z}_2$, i.e., it can assume only two values, $+1$ or -1, provided that the local anomaly-free condition, Eq. (1.2), is valid. (ii) Any self-contragredient representation ω of any group H has no global anomaly in $D \equiv 2 \pmod 4$ under the same condition, Eq. (1.2).

In the next section, we will first prove a further generalization of the statement (ii) to self-contragredient representations in dimensions $D \equiv 0 \pmod 4$ without assuming Eq. (1.2).

Proposition 2. (i) Let ω be a real representation of H. Then, ω has neither local nor global anomaly in $D \equiv 4 \pmod 8$. (ii) Let ω be a pseudoreal representation of H. Then, ω has no global and local anomalies in $D \equiv 0 \pmod 8$.

Note the representations can only be either complex or self-contragredient. Thus, propositions 1 and 2 cover many representations in not only the groups in Eq. (1.3), but also SU(N), SO($4N+2$), and E_6. Now, possible global anomalies for self-contragredient representations may occur only in either $D \equiv 0 \pmod 8$ for real representations $D \equiv 4 \pmod 8$ for pseudoreal representations. For some groups in Eq. (1.3), we can have stronger results. A complete result exists for the case of $H = \mathrm{Sp}(2N)$ (N is the rank of H). In Ref. 8, we have proved the following.

(a) Any representation ω of Sp($2N$) has neither global nor local anomaly in $D \equiv 0 \pmod 8$. Any locally anomaly-free representation ω of Sp($2N$) has also no global anomaly in $D \equiv 2$ or $6 \pmod 8$.

(b) The fundamental global anomaly coefficient $A(\omega)$ of Sp($2N$) in $D \equiv 4 \pmod 8$ is given by

$$A(\omega) = \exp[i\pi Q_2(\omega)]\ , \tag{1.4}$$

where $Q_2(\omega)$ is the second Dynkin index of Sp($2N$), normalized to $Q_2(\square) = 1$ for $2N$-dimensional basic representation $\square$.

Note that Sp($2N$) has both real and pseudoreal representations. (See Appendix A.) Thus, for real representations, the result (a) is nontrivial. The result (b) implies that for a pseudoreal representation of Sp($2N$), there may be a global anomaly, depending upon the value of the second-order index Q_2. When we note the isomorphism, SU(2) = Sp(2), then the case of $H = \mathrm{SU}(2)$ is also covered as a special case of this statement. In particular, the doublet representation of SU(2) is pseudoreal with $Q_2 = 1$ and thus has the global anomaly of type $\mathbb{Z}_2$ in $D \equiv 4 \pmod 8$. This result includes Witten's first example of global gauge anomaly.[1]

The cases of $H = \mathrm{SO}(N)$ ($N \geq 7$) are partially complete with the following results.

(c) Any tensor representation ω with the possible exception of self-dual cases for $N =$ even has no global and local anomalies in $D \equiv 4 \pmod 8$. The absence of global anomalies holds in $D \equiv 2$ or $6 \pmod 8$, provided that we have no local anomalies.

(d) The fundamental global anomaly coefficient $A(\omega)$ of non-self-dual tensor representations ω of $H = \mathrm{SO}(N)$ ($N \geq 6$) in $D \equiv 0 \pmod 8$ is given by

$$A(\omega) = \exp[i\pi Q_2(\omega)]\ , \tag{1.5}$$

where $Q_2(\omega)$ is the second Dynkin index of SO(N) with normalization $Q_2(\square) = 1$ for the N-dimensional basic (or vector) representation $\square$. [See Appendix A for the notion of self-dual tensor representation of SO(N).]

Note that the bound on N in (d) has been lowered by the use of the recent result.[15] The cases of spinor and self-dual tensor representations of $H = \mathrm{SO}(N)$ are not covered in the statement above. However, if we use proposition 2, we can say something more for these cases as we will see in Sec. II.

Next, let us consider the more difficult case of $H = \mathrm{SU}(N)$, where its representation ω need not be self-contragredient. First of all, we note that SU(N) with $N \geq n+1$ has no global anomaly in $D = 2n$, since the homotopy group $\Pi_{2n}(\mathrm{SU}(N))$ vanishes for this case in view of the Bott periodicity theorem.[16] Hence, we need consider only cases of $N \leq n$, and we have shown elsewhere[7,9] that the fundamental global anomaly coefficient $A(\omega)$ for $H = \mathrm{SU}(n-k)$ with $0 \leq k \leq n-2$ is given by

$$A(\omega) = \exp\left[2\pi i \frac{W(n+1,k+1)}{n!} Q_{n+1}(\tilde{\omega})\right] \tag{1.6}$$

assuming the absence of local anomalies, Eq. (1.2). Here, $W(n+1,k+1)$ is the James number[17] of the complex Stiefel manifold $\mathrm{SU}(n+1)/\mathrm{SU}(n-k)$, and $Q_{n+1}(\tilde{\omega})$ is the $(n+1)$th Dynkin index[2] of SU($n+1$) group where the formal representation $\tilde{\omega}$ must be chosen in such a way that it reduces to a direct sum of ω and singlets of $H = \mathrm{SU}(n-k)$ under the restriction of SU($n+1$) to H. We note that the formula, Eq. (1.6), reduces to that given by Elitzer and Nair[18] for the special case of $k = 0$, since then $W(n+1,1) = 1$.

The usefulness of the formula (1.6) is that the knowledge of the homotopy group of SU($n-k$) is not needed. We have more knowledge on the James numbers than the homotopy group itself for general n and k. [The James number denotes a purely topological number which gives the information on how the generator of the infinite order of the homotopy group of $\mathrm{SU}(n+1)/\mathrm{SU}(n-k)$ is mapped into the generator of infinite order of the homotopy group of the sphere S^{2n+1}.] Consequently, we can get a general statement on the global anomaly for general n and k, not just for particular values. However, the complication is twofold. First and foremost, the computation of the group-theoretical Dynkin index $Q_{n+1}(\tilde{\omega})$ is complicated. Second, the James numbers $W(n+1,k+1)$ are not completely known for arbitrary n and k (Refs. 19–22).

Nevertheless, we could prove in our earlier paper[5] that $H=\mathrm{SU}(n)$ in $D=2n$ has no global anomaly in $D\equiv 2$ (mod 4), while $H=\mathrm{SU}(n-1)$ in $D=2n$ has no global anomaly in any even dimension, provided the absence of local anomaly. The case of SU(2) has been solved from Eq. (1.6) in agreement with the case of $H=\mathrm{Sp}(2N)$ with $N=1$ discussed earlier. We have also shown elsewhere[8] that $H=\mathrm{SU}(3)$ has no global anomaly in $D\equiv 2$ (mod 4). Using the recent calculation of $W(n,n-3)$ by Walker,[21] we will show in Sec. III that SU(3) has no global anomaly in $D\equiv 0$ (mod 4). Therefore, we conclude that SU(3) has no global anomaly at all for all even D, provided that it has no local anomaly. Moreover, we will prove that SU(4) has no global anomaly in $D\equiv 4$ (mod 8).

For other cases of SU(N), we have failed to obtain general results similar to the ones given above. However, by explicitly calculating the Dynkin indices as we have done in Ref. 5, we can conclude (but we will not discuss in this paper) that $\mathrm{SU}(n-2)$ $(n\geq 4)$ in $D=2n$ up to $D=26$ does not have global anomalies, provided the absence of local anomaly. Then, by the use of the James number congruence relation given in Ref. 7, we can say that $\mathrm{SU}(n-3)$ $(n\geq 5)$ in $D=2n$ up to $D=26$ does not possess global anomaly, provided the absence of local anomaly. These results are very indicative of the general results, but we have not been able to obtain the clues for proving them.

In Appendix A we discuss self-contragredient, real, and pseudoreal representations. In Appendix B we will present some congruence relations among Dynkin indices, which are needed in Sec. III.

II. SELF-CONTRAGREDIENT REPRESENTATIONS

First, we briefly recapitulate here the method of computing the fundamental global anomaly coefficient $A(\omega)$ of H. Unless it is stated otherwise, we always assume the validity of the locally anomaly-free condition Eq. (1.2). Our method is based upon a generalization of methods originally due to Witten[23] as well as Elitzer and Nair.[18] For a given ω and H, we suppose that we can find a simple and simply connected compact group G and its formal representation $\tilde{\omega}$, satisfying the following two conditions: (i) G has H as the subgroup and its homotopy group, $\Pi_{2n}(G)$, vanishes; (ii) when we restrict G to H then $\tilde{\omega}$ reduces to a direct sum of ω and singlet representations of H. We call these two conditions *the representation condition*. As we often emphasized earlier, the formal representation $\tilde{\omega}$ of G as well as ω of H for some cases may contain negative multiplicity coefficients corresponding to negative helicity when we decompose it in terms of irreducible components. Then utilizing the exact homotopic sequence

$$\Pi_{2n+1}(G)\xrightarrow{p_*}\Pi_{2n+1}(G/H)\xrightarrow{\Delta_*}\Pi_{2n}(H)\xrightarrow{i_*}\Pi_{2n}(G)=0\ , \quad (2.1)$$

we have obtained[5-9] the generalization of the formula Eq. (1.6) as well as some theorems. Here, we only need the following particular proposition.

Proposition 3. For the case when both representation conditions and the locally anomaly-free condition Eq. (1.2) are satisfied, the representation ω of H is free of global anomaly under one of the following two conditions: (i) $\Pi_{2n+1}(G)$ or $\Pi_{2n+1}(G/H)$ is a finite group (including the trivial case); (ii) the Lie algebra $\mathcal{G}$ of G does not possess the fundamental $(n+1)$th-order Casimir invariant.

We assume in this section that ω is a self-contragredient representation of H. Then, it is either a real (i.e., orthogonal) or pseudoreal (i.e., symplectic) representation. The question of deciding whether a given irreducible ω of H is orthogonal or symplectic has been answered by many authors.[24-26] (See Appendix A for more information on self-contragredient representations.) In order to apply our proposition 3, we have to find a pair $(\tilde{\omega},G)$, satisfying the representation conditions. To be definite, we assume that ω is a nontrivial representation of H with its dimension

$$d=d(\omega)\ . \quad (2.2)$$

A. Global anomaly for pseudoreal representation

Consider first the case of ω being pseudoreal (i.e., symplectic). Since any nontrivial representation of H is faithful under our assumption of H, we may identify H with its representation under the present consideration. Then, the group H can be regarded as a subgroup of a d-dimensional symplectic group $G_0=\mathrm{Sp}(d)$ in view of Eqs. (2.2) and (A5). Note that d must be of necessity an even integer. Moreover, let $\square$ be the d-dimensional basic (i.e., vector) representation of Sp(d). Then the choice $\tilde{\omega}_0=\square$ clearly reduces to ω itself, when we restrict G_0 to H. However, in order to satisfy $\Pi_{2n}(G)=0$, we require in general the following slightly generalized choice for G. Let $\tilde{N}$ be an arbitrary positive integer satisfying

$$2\tilde{N}\geq \mathrm{Max}(d,n)\ . \quad (2.3)$$

Now, we identify $G=\mathrm{Sp}(2\tilde{N})$, and choose $\tilde{\omega}=\tilde{\square}$ to be the $2\tilde{N}$-dimensional basic (i.e., vector) representation of $\mathrm{Sp}(2\tilde{N})$. If we restrict $G=\mathrm{Sp}(2\tilde{N})$ to $G_0=\mathrm{Sp}(d)$ in the canonical way, then $\tilde{\omega}=\tilde{\square}$ reduces to a direct sum of $\tilde{\omega}_0=\square$ and singlet representations of $G_0=\mathrm{Sp}(d)$. Therefore, reducing G_0 further to H, we see that $\tilde{\omega}=\tilde{\square}$ of G reduces to a direct sum of ω and singlets of H. Next, in order to satisfy $\Pi_{2n}(G)=0$, we note the Bott periodicity theorem[16]

$$\Pi_j(\mathrm{Sp}(2N))=\begin{cases}\mathbb{Z} & \text{if } j=3,7 \pmod 8\ ,\\ \mathbb{Z}_2 & \text{if } j=4,5 \pmod 8\ ,\\ 0 & \text{if } j=0,1,2,6 \pmod 8\end{cases} \quad (2.4)$$

for the stable region $4N\geq j-1$. Therefore, if $\tilde{N}$ satisfies Eq. (2.3), then the remaining condition $\Pi_{2n}(G)=0$ for the representation is automatically satisfied for $D=0,2,6$ (mod 8). Moreover, if $D=0$ (mod 8), then we see also $\Pi_{2n+1}(G)=0$ so that the first part of our proposition 3 implies the absence of the global anomaly for this case. Here, we need not assume Eq. (1.2), since it is also automatically satisfied for $D\equiv 0$ (mod 8). Also, the same result follows from the second part of proposition 3. When we have $D\equiv 2$ or 6 (mod 8), Eq. (2.4) tells us that $\Pi_{2n}(G)=\mathbb{Z}$ is infinite so that the same reasoning is not applicable. However, the cases for $D\equiv 2$ or 6 (mod 8) can be derived from the second part of proposition 1.

B. Global anomaly for real representations

Next we consider the case of ω being a real (i.e., orthogonal) representation of H. Then, we may regard H as a subgroup of the orthogonal group SO(d) [or of Spin(d) if we wish] in view of Eqs. (2.2) and (A3). In order to satisfy the condition $\Pi_{2n}(G)=0$, let $\tilde{N}$ be any integer satisfying

$$\tilde{N}\geq \mathrm{Max}(d,2n+2) \tag{2.5}$$

and note the Bott periodicity theorem[16]

$$\Pi_j(\mathrm{SO}(N))=\begin{cases}\mathbb{Z} & \text{if } j=3,7 \pmod 8\ ,\\ \mathbb{Z}_2 & \text{if } j=0,1 \pmod 8\ ,\\ 0 & \text{if } j=2,4,5,6 \pmod 8\end{cases} \tag{2.6}$$

for the stable region $N\geq j+2$. Now, we choose $G=\mathrm{SO}(\tilde{N})$ with $\tilde{\omega}=\square$ being the $\tilde{N}$-dimensional basic (i.e., vector) representation of SO($\tilde{N}$). Then, reducing $G=\mathrm{SO}(\tilde{N})$ first to $G_0=\mathrm{SO}(d)$ and then to its subgroup H, we see that $\tilde{\omega}$ reduces to a direct sum of ω and singlet representations of H under the reduction of G to H. Further for dimensions $D\equiv 2,4,6$ (mod 8), Eq. (2.6) gives $\Pi_{2n}(G)=0$, so that the pair $(\tilde{\omega},G)$ satisfies the desired representation conditions. Moreover, for $D\equiv 4$ (mod 8), we have $\Pi_{2n+1}(G)=0$ so that the first part of proposition 3 implies the absence of global anomaly for ω. Also, this follows from the second part of proposition 3, if we choose $\tilde{N}$ to be odd. For $D\equiv 2,6$ (mod 8), we appeal to the second half of proposition 1.

C. Property of the $Q_2(\omega)$

Using the results given in Appendix A, we first remark the following. For the case of $H=\mathrm{Sp}(2k)$ in $D\equiv 4$ (mod 8), Eq. (1.4) implies

$$A(\omega)=\exp[i\pi Q_2(\omega)]$$

for the global anomaly, while proposition 2 together with the result (v) of (I) in Appendix A requires that $A(\omega)=1$ for $m_1+m_3+m_5+\cdots=\text{even}$. Therefore, consistency demands that we must have $Q_2(\omega)=\text{even}$ when $m_1+m_3+m_5+\cdots=\text{even}$ for $H=\mathrm{Sp}(2k)$. In terms of the Young tableau, the condition is equivalent to even numbers of boxes contained in the Young tableau of the representation under consideration. Actually, we can prove the following more general statement.

Proposition 4. Any irreducible representation ω of SU($2N$) and Sp($2N$) as well as any non-self-dual tensor representation of SO($2N$) satisfies $Q_2(\omega)=\text{even}$, provided that the Young tableau corresponding to the irreducible representation ω contains an even number of boxes. Here ω need not be self-contragredient. We can prove this proposition by an induction which is analogous to that used elsewhere for a proof of congruence relation,[3] but we will not go into detail in this paper.

III. SU(N) CASES

We discuss the case of $H=\mathrm{SU}(N)$ ($N\geq 3$), since the SU(2) group is isomorphic to Sp(2). If the representation ω is self-contragredient, then we can utilize results of the previous section to obtain some information on the global anomaly coefficient $A(\omega)$. However, for general non-self-contragredient representation, we must base our calculation on the formula (1.6), assuming the locally anomaly-free condition (1.2). As we have already stated, the complications are due to our inability of computing the Dynkin index $Q_{n+1}(\tilde{\omega})$ of the SU($n+1$) group as well as our insufficient knowledge of the James number $W(n+1,k+1)$ of the Stiefel manifold SU($n+1$)/SU($n-k$). The known James numbers (and other homotopy group results) have been tabulated in a useful form by Lundell.[22] Using the result for $k=0$ and 1 as well as $k=n-2$, we have analyzed global anomaly coefficients $A(\omega)$ of groups, $H=\mathrm{SU}(n)$, SU($n-1$), and SU(2) in $D=2n$ in some detail elsewhere.[5-9]

A. SU(3) in $D\equiv 2$ (mod 4)

Recently, Walker[20,21] has completed his computation of $W(n+1,n-2)$ for an arbitrary n. His results are

$$W(n+1,n-2)=\begin{cases}\dfrac{n!}{\mathrm{denom}B_{n-1}} & \text{if } n=\text{odd}\ ,\\[2ex] \dfrac{n!}{\mathrm{denom}B_{n-2}(\mathrm{denom}B_{n-2},n-2)} & \text{if } n=\text{even}\ .\end{cases} \tag{3.1}$$

Here, (x,y) stands for the largest common divisor of two integers x and y, while $\mathrm{denom}B_n$ implies the denominator of the nth Bernouli number B_n. Since the calculation of $A(\omega)$ for $n=\text{even}$ is rather complicated, we first sketch briefly our earlier calculation for the simpler case of $n=\text{odd}$ in Eq. (3.1). Formula (1.6) together with Eq. (3.1) gives

$$A(\omega)=\exp\left[\frac{2\pi i}{\mathrm{denom}B_{n-1}}Q_{n+1}(\tilde{\omega})\right]\quad(n=\text{odd})\ . \tag{3.2}$$

We show $A(\omega)=1$, provided that we use the local-anomaly-free condition (1.2),

$$\mathrm{Tr}^{(\omega)}X^{n+1}=0 \tag{3.3}$$

for any generic element X in the representation ω of the Lie algebra $\mathcal{A}_2$ of the Lie group $H=\mathrm{SU}(3)$. In order to avoid possible confusion, we write, for example, the lth Dynkin index of a representation ω of H explicitly as $Q_l(\omega,H)$.

Now, let $G=\mathrm{SU}(N)$ for arbitrary N. Then, we can first show the validity of

$$Q_{n+1}(\tilde{\omega},G)-Q_2(\tilde{\omega},G)=k\,\mathrm{denom}B_{n-1} \tag{3.4}$$

for any representation $\tilde{\omega}$ of G and any integer n satisfying $1<n\leq N-1$ (see Appendix B). Here k is an integer. Next, let X be a suitably chosen Cartan subalgebra element of $\mathcal{H}$ such that it can assume only integer weights in a representation ω of $\mathcal{H}$. Then, if n is odd, we will prove in Appendix B that

$$\mathrm{Tr}^{(\omega)}X^{n+1}-\mathrm{Tr}^{(\omega)}X^2=2k'\mathrm{denom}B_{n-1} \tag{3.5}$$

for another integer k', as well as

$$\mathrm{Tr}^{(\omega)}X^2=2Q_2(\omega,H)\ . \tag{3.6}$$

Moreover, the branching index sum rule[2] implies

$$Q_2(\tilde{\omega},G)=Q_2(\omega,H)\ , \tag{3.7}$$

if the representation $\tilde{\omega}$ of G reduces to a direct sum of ω and singlet representations of H under restriction of G to H. Now, assuming the validity of the locally anomaly-free condition Eq. (3.3), all equations, Eqs. (3.4)–(3.7) lead to

$$Q_{n+1}(\tilde{\omega},G)=k''\mathrm{denom}B_{n-1} \tag{3.8}$$

for some integer k''. Then, Eq. (3.2) gives the desired result $A(\omega)=1$ for $D\equiv 2$ (mod 4) (odd n) with $G=\mathrm{SU}(n+1)$ and $H=\mathrm{SU}(3)$.

B. SU(3) in $D\equiv 0$ (mod 4)

We will turn our attention to the more difficult case for n being even. Then the anomaly coefficient $A(\omega)$ of $H=\mathrm{SU}(3)$ is given by

$$A(\omega)=\exp\left[2\pi i\frac{1}{\mathrm{denom}B_{n-2}(n-2,\mathrm{denom}B_{n-2})}Q_{n+1}(\tilde{\omega},G)\right] \tag{3.9}$$

for $G=\mathrm{SU}(n+1)$ in $D=2n\equiv 0$ (mod 4). First, we recall the fact that $\mathrm{denom}B_n$ is a product of all prime numbers p such that $p-1$ divides n. Therefore, if n = even, then

$$\mathrm{denom}B_{n-2}(n-2,\mathrm{denom}B_{n-2})=4b$$

for some odd integer b as we will see in Appendix B. Hence, Eq. (3.9) gives

$$A(\omega)=\exp\left[\pi i\frac{1}{2b}Q_{n+1}(\tilde{\omega},G)\right]\ .$$

However, as we have already noted in proposition 1, $A(\omega)$ can assume only two possible values, $+1$ or -1. Therefore, $Q_{n+1}(\tilde{\omega},G)$ must be necessarily an integer multiple of $2b$. Especially, we can set $b=1$, since b is odd. Hence, we can rewrite our formula as

$$A(\omega)=\exp[i\pi\tfrac{1}{2}Q_{n+1}(\tilde{\omega},G)]\ . \tag{3.10}$$

In Appendix B, we will prove the congruence relation

$$Q_{n+1}(\tilde{\omega},G)-Q_3(\tilde{\omega},G)=4k \tag{3.11}$$

for some integer k for any even integer $n\geq 3$. Moreover, the branching index sum rule[2] implies

$$Q_3(\tilde{\omega},G)=Q_3(\omega,H) \tag{3.12}$$

since $\tilde{\omega}$ reduces to a direct sum of ω and singlet representations of H under restriction of $G=\mathrm{SU}(n+1)$ to $H=\mathrm{SU}(3)$. Therefore, Eqs. (3.10)–(3.12) provide the formula

$$A(\omega)=\exp[i\pi\tfrac{1}{2}Q_3(\omega,H)] \tag{3.13}$$

for n = even, directly in terms of the third-order Dynkin index $Q_3(\omega,H)$ of $H=\mathrm{SU}(3)$. Note that any reference to $\tilde{\omega}$ of $\mathrm{SU}(n+1)$ has disappeared completely in our formula (3.13). Now, we evaluate $Q_3(\omega,H)$ under the assumption of the locally anomaly-free condition

$$\mathrm{Tr}^{(\omega)}X^{n+1}=0\ . \tag{3.14}$$

If ω is self-contragredient, then Eq. (3.14) is automatically satisfied and $Q_3(\omega,H)=0$. Thus, we conclude that $A(\omega)=1$ for a self-contragredient representation. For general cases, it is difficult to utilize the locally anomaly-free condition Eq. (3.14) effectively. However, as we will show in Appendix B, Eq. (3.14) implies that

$$Q_3(\omega,H)\equiv 0\quad(\mathrm{mod}\ 4)\quad\text{for } n=\text{even}\ .$$

Thus, we conclude that $A(\omega)=1$ for $D\equiv 0$ (mod 4). Consequently, we have found that *any locally anomaly-free representation of H = SU(3) has no global anomaly in any even dimension D.*

For $H=\mathrm{SU}(4)$, the James number for n = even has been also computed by Walker[21] and the result is

$$W(n+1,n-3)=\frac{2n!}{\mathrm{denom}B_{n-2}(n,4)(n-2,\mathrm{denom}B_{n-2})}\ . \tag{3.15}$$

If $D\equiv 4$ (mod 8), then we can show similarly to the previous case that the global anomaly coefficient is given by

$$A(\omega)=\exp[i\pi\tfrac{1}{2}Q_3(\omega,H)] \tag{3.16}$$

which has the same form as Eq. (3.13). However, since $H=\mathrm{SU}(3)$ has been shown to have no global anomaly in $D\equiv 4$ (mod 8), this fact implies the same result also for SU(4) by the following reasoning. We write the reduction of the representation ω of SU(4) into SU(3) as ω_0. Then, the branching index sum rule implies that

$$Q_3(\omega,\mathrm{SU}(4))=Q_3(\omega_0,\mathrm{SU}(3))\ . \tag{3.17}$$

Therefore, the comparison of Eqs. (3.14) and (3.17) gives

$$A(\omega)|_{H=\mathrm{SU}(4)}=A(\omega_0)|_{H=\mathrm{SU}(3)}\ , \tag{3.18}$$

for $D\equiv 4$ (mod 8). Moreover, we have

$$\mathrm{Tr}^{(\omega_0)}X^{n+1}|_{\mathrm{SU}(3)}=0\ . \tag{3.19}$$

However, since we just proved that SU(3) does not have global anomaly, we conclude that SU(4) does not have any global anomaly. We remark that the relation (3.18) for $D\equiv 4$ (mod 8) has been also derived in a different way elsewhere,[7] using a result obtained by Crabb and Knapp.[27]

The analysis for $H=\mathrm{SU}(4)$ in other even dimensions has not been successful. We may need a better congruence relation than what we have done so far.

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APPENDIX A: SELF-CONTRAGREDIENT, REAL OR PSEUDOREAL REPRESENTATIONS

In this appendix, we explain the adjectives self-contragredient, real, and pseudoreal which attach to representations of Lie groups or Lie algebras.[24-26]

We call a representation ω *self-contragredient* if there exists an invariant real bilinear form on the vector space V for this representation. That is, for x and y in V and the bilinear form (x,y), we have

$$(gx,gy)=(x,y) \quad \text{for } g\in H \ , \tag{A1}$$

where we do not distinguish the representation of $g\in H$ and the element g itself in H. If the representation space V is irreducible over the complex number field, then we can prove that this bilinear form is nondegenerate and unique up to a constant factor, and it is either symmetric, $(x,y)=(y,x)$, or skew symmetric, $(x,y)=-(y,x)$, but not both.[26] Hereafter, we consider the irreducible representation space. If it is symmetric, we can find basis vectors $\{e_j\}$ such that

$$(e_i,e_j)=\delta_{ij} \ . \tag{A2}$$

Then Eq. (A1) can be written as

$$\begin{aligned}(ge_i,ge_j)&=g_{ik}g_{jl}(e_k,e_l)\\ &=g_{ik}g_{jk}=(G\cdot G^t)_{ij}=\delta_{ij} \ ,\end{aligned} \tag{A3}$$

where $(G)_{ij}=g_{ij}$. That is, G is an orthogonal matrix. We constructed an orthogonal matrix from a representation of H, which has a symmetric bilinear form. Note that this orthogonal matrix corresponds to a fundamental representation of an orthogonal group with its dimension equal to the dimension of the representation of H. If it is skew symmetric, we can find basis vectors $\{e_j\}$ such that

$$(e_i,e_j)=(J)_{ij} \ , \tag{A4}$$

where $J=\text{diag}(J_1,J_1,\ldots,J_1)$ and

$$J_1=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} .$$

Then, Eq. (A1) can be written as

$$(ge_i,ge_j)=g_{ik}g_{jl}J_{kl}=(G\cdot J\cdot G^t)_{ij}=(J)_{ij} \ . \tag{A5}$$

That is, G is a symplectic matrix, satisfying $GJG^t=J$.

Thus, irreducible self-contragredient representations divide into classes: *orthogonal* representations if the bilinear form is symmetric and *symplectic* ones if the bilinear form is skew symmetric.

In Lie algebra $\mathcal{H}$ of H, the condition for being self-contragredient, Eq. (A1), becomes

$$(Tx,y)+(x,Ty)=0 \quad \text{for } T\in\mathcal{H} \ . \tag{A6}$$

For a symmetric bilinear form, we have in matrix form

$$T+T^t=0 \ , \tag{A7}$$

while for a skew-symmetric bilinear form we have

$$TJ+JT^t=0 \ . \tag{A8}$$

The condition for a representation being self-contragredient is that there exists an element of the Weyl group of $\mathcal{H}$ which transforms the highest weight Λ into $-\Lambda$. This gives the method of deciding which groups can have only self-contragredient representations. The result is Eq. (1.3).

Now, we explain the adjectives real and pseudoreal. For a unitary representation of H (which we physicists use) which acts on $\mathbb{C}^n$, consider a representation which satisfies

$$g^*=S^{-1}\cdot g\cdot S \quad \text{for } g\in H \tag{A9}$$

for some nonsingular matrix S where an asterisk denotes complex conjugation. (This shows the reason for the adjectives real or pseudoreal.) In terms of its Lie algebra, the condition becomes

$$T=-S^{-1}\cdot T^t\cdot S \tag{A10}$$

for a Hermitian matrix T, since $g^*=\exp(-iT^*)=\exp(-iT^t)$. Therefore, we have

$$\text{Tr}T^{\text{odd}}=0 \ . \tag{A11}$$

The matrix S is either symmetric or skew symmetric for an irreducible representation since

$$T=-S^{-1}\cdot T^t\cdot S=-S^{-1}\cdot(-S^t)\cdot T\cdot(S^{-1})^t\cdot S$$

which leads to

$$[T,(S^{-1}\cdot S^t)]=0 \ .$$

Thus, $S=\lambda S^t=\lambda^2 S$. That is, $\lambda=\pm 1$. We call the representation *real* if S is symmetric, i.e., $S^t=S$, while we call the representation *pseudoreal* if S is skew symmetric, $S^t=-S$. Furthermore, the matrix S is unique up to a constant if the representation is irreducible.

We show the equivalence of the definition of "orthogonal" and "real" and the equivalence of "symplectic" and "pseudoreal." We can define a bilinear form using the matrix S as follows:

$$(x,y)_S\equiv\sum x_jS_{jk}y_k \tag{A12}$$

for $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$. Then, we have, first of all,

$$\begin{aligned}(x,Ty)_S&=x_jS_{jk}T_{km}y_m\\ &=x_jS_{jk}(-S^{-1}\cdot T^t\cdot S)_{km}y_m=-(Tx,y)_S \ ,\end{aligned}$$

which is Eq. (A6). Furthermore, it is easy to see that for a symmetric S, we have $(x,y)_S=(y,x)_S$, while for a skew symmetric S, we have $(x,y)_S=-(y,x)_S$.

We can also show the following properties.[26] (i) The tensor product of two orthogonal or symplectic representations is orthogonal; (ii) the tensor product of an orthogonal and a symplectic representations is symplectic.

The question of deciding whether a given irreducible representation ω of H is orthogonal or symplectic has been answered by many authors.[24–26] Here, we state the results by Bose and Patera.[25] Let $\mathcal{H}$ be a simple Lie algebra of rank r and let Λ be the highest weight of the irreducible representation ω under consideration. If $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ are the fundamental weight system of $\mathcal{H}$, then we can express Λ as

$$\Lambda = m_1\Lambda_1 + m_2\Lambda_2 + \cdots + m_r\Lambda_r \tag{A13}$$

in terms of non-negative integers $m_1, m_2, \ldots, m_r$. We adopt also their lexicographical ordering of the simple root system. Then their results are as follows.

(I) Orthogonal (i.e., real) representations

(i) Any representation of SO($8k$), SO($8k+1$), SO($8k-1$), G_2, F_4, and E_8, where k is a positive integer.

(ii) Any representation of SO($8k+3$) and SO($8k+5$) such that m_r=even. This excludes spinor representations.

(iii) Any self-contragredient representation of SO($8k+2$) and SO($8k+6$). This condition is equivalent to the condition $m_{r-1}=m_r$. Again this excludes spinor representations.

(iv) Any representation of SO($8k+4$) such that $m_{r-1}+m_r$=even.

(v) Any representation of Sp($2k$) such that $m_1+m_3+m_5+\cdots$=even.

(vi) Any representation of E_7 satisfying $m_4+m_6+m_7$=even.

(vii) Any self-contragredient representation of SU($2k+1$), SU($4k+4$), and E_6.

(viii) Any self-contragredient representation of SU($4k+2$) such that m_{2k+1}=even.

(II) Symplectic (i.e., pseudoreal) representations

(i) Any spinor representation of SO($8k+3$) and SO($8k+5$). The condition is equivalent to m_r=odd.

(ii) Any spinor representation of SO($8k+4$), i.e., $m_{r-1}+m_r$=odd.

(iii) Any representation of E_7 satisfying $m_4+m_6+m_7$=odd.

(iv) Any representation of Sp($2k$) satisfying $m_1+m_3+m_5+\cdots$=odd.

(v) Any self-contragredient representation of SU($4k+2$) such that m_{2k+1}=odd.

Note that for SO($2r+1$) the condition m_r=odd implies the representation is a spinor. For SO($2r$), the case of m_r+m_{r-1}=odd gives spinor representations, while the case of m_r+m_{r-1}=even$\neq 0$ defines self-dual tensor representations.

APPENDIX B: CONGRUENCE RELATIONS FOR DYNKIN INDICES

We prove congruence relations for Dynkin indices, which are used in this paper. Let ω be a representation of the SU(N) group and let $Q_n(\omega)$ be the nth Dynkin index[2] of SU(N) ($2 \le n \le N$).

Relation 1. For any two integers n and m which satisfy the relation $1 \le m \le n \le N-1$, we have

$$Q_{n+1}(\omega) - Q_{m+1}(\omega) = k \ \text{denom}B_{n-m} \tag{B1}$$

with an integer k.

Here, denomB_n is the denominator of the nth Bernoulli number. More precisely, it is a product of all prime numbers p such that $p-1$ divides n: denom$B_0=1$, denom$B_1=2$, denom$B_2=6$, denom$B_4=30$, denom$B_6=42$, etc. Here, we define denom$B_{\text{odd}}=2$, although B_m with m=odd (>1) vanish. (For practical use in this paper, we use only B_m with m=even.) In particular, the Bernoulli number satisfies the relation

$$\text{denom}B_{n-m} \equiv 2 \pmod 4 . \tag{B2}$$

Equation (B1) has been already proved in Ref. 8 and used in Eq. (3.4) for $m=1$.

For the result of Sec, III we require some improvement over Eq. (B1) as we stated. We have the following.

Relation 2. For $n-m=2^l\times$(odd integer) for some positive integer $l \ge 1$, we have

$$Q_{n+1}(\omega) - Q_{m+1}(\omega) = k' \times 2^{\min(m,l+2)-1}\text{denom}B_{n-m} \tag{B3}$$

for another integer k', where Min($m,l+2$) stands for the minimum of two positive integers m and $l+2$.

Now, we proceed to the proof of Eq. (B3). Let x be any integer. If n and m are two positive integers which satisfy the assumption, then we first show the validity of

$$x^n - x^m = k'' \times 2^{\text{Min}(m,l+2)-1}\text{denom}B_{n-m} . \tag{B4}$$

Let p be a prime number such that $(p-1)$ divides $(n-m)$. If x is not divisible by p, then, for $n > m \ge 1$,

$$x^n - x^m \equiv 0 \pmod p , \tag{B5}$$

using the Fermat theorem $x^{n-m}-1 \equiv 0 \pmod p$. This relation is trivially valid also, if x is an integral multiple of p. Then, because of the definition of denomB_{n-m}, Eq. (B5) will prove Eq. (B1) as we have done in Ref. 8. If x is even, then we have, for $m \ge 1$,

$$x^n - x^m \equiv 0 \pmod{2^m} . \tag{B6}$$

For the case where x is odd, we will show shortly that

$$x^n - x^m \equiv 0 \pmod{2^{l+2}} , \tag{B7}$$

for $n-m=2^l\times$(odd integer) with $l \ge 1$. Combining both facts and noting Eq. (B2), we establish the desired result Eq. (B4). Therefore, we need only prove Eq. (B7).

The equivalent statement to Eq. (B7) is

$$y^{2^l} - 1 \equiv 0 \pmod{2^{l+2}} \tag{B8}$$

for any odd integer y, where $y=x^b$=odd for odd x with $n-m=2^l\times b$ for an odd integer b with $l \ge 1$. We prove Eq. (B8) by induction. By writing an odd y as $4m\pm 1$, we have

$$y^2 = 1 \pm 8m + 16m^2 \equiv 1 \pmod{2^3} ,$$

which corresponds to the case of $l=1$. Now, let us assume Eq. (B8) is valid for $l=p$. Then, we have, for some integer q,

$$y^{2(p+1)}=(1+2^{(p+2)}q)^2 =1+2^{(p+3)}q+2^{2(p+2)}q^2\equiv 1 \pmod{2^{(p+3)}} ,$$

which corresponds to Eq. (B8) with $l=p+1$. Note that Eq. (B8) may fail for $l=0$.

The next task is to show that Eq. (B3) follows from Eq. (B4). For this, we must restrict the values of n and m such that $1\le m<n\le N-1$, since $Q_{n+1}(\omega)$ for SU(N) is not well defined for $n\ge N$ as we have emphasized elsewhere.[2] Let $\Lambda_1,\Lambda_2,\ldots,\Lambda_{N-1}$ be the $N-1$ fundamental weight system of the Lie algebra $\mathcal{A}_{N-1}$ of the SU(N) group. Then, we know that

$$Q_{n+1}(\Lambda_f)-Q_{m+1}(\Lambda_f) =-\sum_{l=0}^{f-1}(-)^{f-l}[(f-l)^n-(f-l)^m]\frac{N!}{l!(N-l)!} , \tag{B9}$$

for $1\le f\le N-1$. Identifying $x=f-l$, then Eqs. (B4) and (B9) imply the validity of Eq. (B3) when ω is one of the $N-1$ fundamental representations of the SU(N). The general case can be then obtained by induction as in Ref. 3 by using index sum rules of the direct product of two representations, or more directly from the general formula for $Q_n(\omega)$ found in Ref. 2.

Now, Eq. (3.11) is a special case of Eq. (B3) for $m=2$ with $l\ge 1$ when we note Eq. (B2), while Eq. (3.4) results from Eq. (B3) with $m=1$. As we see from our proof, we did not fully utilize special form Eq. (B9) for $Q_{n+1}(\Lambda_f)-Q_{m+1}(\Lambda_f)$. Thus, it is plausible that there may be room for improvement. Also, we remark at this point that somewhat analogous congruence relations have been studied by Braden[28] from a different point of view. Moreover, proposition 4 of Sec. II can be proven by induction in a similar way.

The next relation is used in Sec. III B.

Relation 3. For any representation ω of SU(N) ($N\ge 3$) which satisfies $\mathrm{Tr}^{(\omega)}X^{n+1}=0$ with even $n\ge 4$, we have

$$Q_3(\omega,\mathrm{SU}(N))\equiv 0 \pmod 4 . \tag{B10}$$

Let X be an element of a Cartan subalgebra of the Lie algebra $\mathcal{A}_{N-1}$ of the SU(N) group such that its weight M in any representation ω is always an integer. As we will show shortly, we can always find such X. We calculate

$$\mathrm{Tr}^{(\omega)}X^{n+1}-\mathrm{Tr}^{(\omega)}X^{m+1}=\sum_{M\in\omega}(M^{n+1}-M^{m+1}) . \tag{B11}$$

Therefore, if n and m are positive integers satisfying Eq. (B3) with $n>m\ge 1$ and $l\ge 1$, then, Eq. (B4) for $x=|M|$ gives

$$\mathrm{Tr}^{(\omega)}X^{n+1}-\mathrm{Tr}^{(\omega)}X^{m+1}=k_0\times 2^{\mathrm{Min}(m+1,l+2)-1} \times \mathrm{denom}B_{n-m} \tag{B12}$$

for some integer k_0, when we replace n and m by $(n+1)$ and $(m+1)$, respectively. Equation (3.5) is a special case of Eq. (B12) for $m=1$ and odd n. Actually, if we can choose X in such a way that both M and $-M$ ($M\neq 0$) are simultaneously weights for all M, then k_0 in Eq. (B11) is an even integer.

Now, we shall give examples that we can find X satisfying these conditions. Let I_+, I_-, and I_3 be the su(2) sub-Lie algebra of our Lie algebra $\mathcal{A}_{N-1}$ ($N\ge 2$). Then, the choice $X=2I_3$ is one example where Eq. (3.6) holds, since $\mathrm{Tr}^{(\omega)}X^2=Q_2(\omega,\mathrm{SU}(N))\mathrm{Tr}^{(\square)}X^2$. However, for our present problem, we require a slightly more complicated choice. The reason is that $\mathrm{Tr}X^3=0$ for any $X\in$ su(2) algebra. Write the Lie algebra $\mathcal{A}_{N-1}$ of the SU(N) ($N\ge 3$) group in the standard Cartesian form of

$$[B^\mu_\nu,B^\alpha_\beta]=\delta^\mu_\beta B^\alpha_\nu-\delta^\alpha_\nu B^\mu_\beta,\quad \sum_{\mu=1}^{N}B^\mu_\mu=0 , \tag{B13}$$

for $\mu,\nu,\alpha,\beta=1,2,\ldots,N$. Then, choosing

$$X=(B^1_1-B^2_2)-(B^2_2-B^3_3) , \tag{B14}$$

we can readily verify that X assumes only integer eigenvalues, since X belongs to the Cartan subalgebra of SU(N) [actually of SU(3)]. Note that for $H_\alpha\equiv B^\alpha_\alpha-B^{\alpha+1}_{\alpha+1}$ ($\alpha=1,2,\ldots,N-1$), we have $[H_\alpha,H_\beta]=0$ and thus the set $\{H_\alpha\}$ is the Cartan subalgebra of SU(N).

After these preparations, we now choose $m=2$ and n even in Eq. (B12) and utilize $\mathrm{Tr}X^{n+1}=0$ ($n\ge 4$) for SU(N) ($N\ge 3$). Then, we find

$$\mathrm{Tr}^{(\omega)}X^3=-4k_0\mathrm{denom}B_{n-2} \tag{B15}$$

for any given integer $n\ge 4$, since then we have $l\ge 1$. On the other hand, we know[2]

$$\mathrm{Tr}^{(\omega)}X^3=Q_3(\omega,\mathrm{SU}(N))\mathrm{Tr}^{(\square)}X^3 , \tag{B16}$$

where $\square$ is the basic representation of SU(N). With our choice of X as in Eq. (B14), we calculate easily

$$\mathrm{Tr}^{(\square)}X^3=-6 , \tag{B17}$$

since the diagonal elements of $X(\square)$ are $(1,-2,1,0,\ldots,0)$. Then, Eqs. (B15)–(B17) give

$$Q_3(\omega,\mathrm{SU}(N))=\frac{4k_0}{6}\mathrm{denom}B_{n-2} .$$

Moreover, since $n-2$ is an even integer by assumption, $\mathrm{denom}B_{n-2}$ is an integer multiple of six. Therefore, we proved Eq. (B10). Especially, with the choice of SU(3), Eqs. (3.13) and (B10) establish the fact that SU(3) has no global anomaly in $D\equiv 0 \pmod 4$. Combining with the previous result for $D\equiv 2 \pmod 4$, we see that the SU(3) group has always no global anomaly in any even dimension as long as the local anomaly vanishes.

We can similarly calculate the case of $H=\mathrm{SU}(4)$ in $D\equiv 4 \pmod 8$ to find the statement given in Sec. III. We emphasize the fact that our congruence relations are most likely not the optimal ones and we may be able to improve them.

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