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Bubble dynamics and spontaneous compactification

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As a laboratory for studying dimensional reduction in inhomogeneous spacetimes, we consider several five-dimensional models in which a bubble of a spacetime with a small compact dimension is sewn to a spacetime with all five dimensions macroscopic. Obliging the five-dimensional metric to be well behaved at the domain boundary constrains the parameters of the composite spacetimes under study. Given a suitable potential, it is possible that the domain wall is created by the rapid variation of the size of the compact dimension. Then it is appropriate to reduce the five-dimensional theory to an effective four-dimensional theory in which the size of the compact dimension plays the role of a scalar field. It is the effective four-dimensional metric which must be well defined at the domain boundary. We discuss several models of this latter type including some in which there is no singular contribution to the effective energy-momentum tensor even at the domain boundary.

I. INTRODUCTION

The idea that spacetime may have more than four dimensions has attracted a great deal of attention, beginning with the work of Kaluza[1] and Klein.[2] Modern string theory compels one to take this idea seriously. Of course, only four macroscopic dimensions are observed and one would like to understand the mechanism by which the others become small.

One possibility is that compactification occurs in the course of cosmological evolution. That is to say, the early Universe may have had a large number of macroscopic dimensions, with all but four driven to very small size by the Einstein equations. Several authors[3-5] have constructed homogeneous anisotropic cosmologies of this sort.

One can imagine, however, that the early Universe was not at all homogeneous. Rather, there may have been a domain structure so that the Universe was comprised of many regions of qualitatively different character. The mathematics of such an inhomogeneous spacetime can be very complex but we hope to get a feel for the effects of inhomogeneities by studying some simple examples: we shall study the dynamics of a bubble of an anisotropic spacetime separated by a thin wall from a simple ambient spacetime. We shall try to find models in which the bubble evolves to have four macroscopic dimensions.

There are two attitudes one can take when matching two higher-dimensional spacetimes across a thin wall. One can imagine that the thin wall represents a region in which some external matter field is varying rapidly. In the mathematical limit where the wall is infinitesimally thin, observers on either side of the wall must agree on all components of the metric which are tangent to the wall. We shall adopt this attitude in Secs. II and III, when we match the five-dimensional Schwarzschild–de Sitter cosmology to five-dimensional de Sitter space. This simple model may be a laboratory for understanding aspects of the more complicated scenario recently proposed by Linde and Zelnikov in the context of chaotic inflation.[6] In their scenario the Universe consists of domains of six-dimensional de Sitter space separated by walls from regions with the topology $M^4\times S^2$. In chaotic inflation the walls may not be thin, but we feel that the thin-wall limit is instructive nonetheless. The reason is that any pathologies in the thin-wall limit will hint that something interesting is happening in the vicinity of the physical (thick) wall. In the sample model we study, we find that matching the tangential components of the five-dimensional metric along the (four-dimensional) wall suffices to determine the equation of motion for the wall. This in turn determines the energy-momentum tensor of the wall. We cannot solve the equation of the wall except for in the asymptotic past, but we find that if the mass of the Schwarzschild–de Sitter cosmology is above a certain value, then the wall is necessarily tachyonic in the far past. One might expect the Linde and Zelnikov model to be even more complex and surprising.

The second attitude one may adopt is to suppose that the thin wall is a region where the sizes of the extra dimensions are varying rapidly. In this approach, one reduces the higher-dimensional gravity theory to an effective four-dimensional theory in which the extra dimensions appear as "matter" fields. The extra dimensions may vary rapidly in a small region of spacetime if, say, the effective "matter" fields are governed by a potential where two local minima are separated by a steep barrier. It is not difficult to create such a potential by, for example, introducing additional matter fields,[7,8] exploiting the Casimir energy of the extra dimensions[9-11] (for an interesting discussion of how Casimir effects and additional matter fields can be used to achieve a stable compactification of ten-dimensional supergravity see Ref. 12), or by introducing bosonic strings which wind around the extra dimensions.[13] In Sec. IV we apply this approach to a model consisting of a bubble of four-dimensional de Sitter space (plus one small spectator dimension) in an ambient five-dimensional de Sitter space. Such a cosmological compactification model is suggested by the work of Moss.[14] Our discussion consists mainly of combining the standard techniques for the reduction to an effective four-dimensional theory[9] with a review of Maeda's[15] formalism for connecting four-dimensional Robertson-Walker cosmologies. It is difficult to solve the resulting equations, but it is easy to see that the wall separating the four- and five-dimensional de Sitter spaces comprises a state of matter for which the pressure is the negative of the energy density, with both varying in time. Finally, we consider the reduction of the five-dimensional radiation-dominated Robertson-Walker cosmology. The reduction suggests some interesting spacetimes containing two regions sewn across a boundary which does not have a singular energy-momentum density. The first spacetime we discuss comprises a four-dimensional radiation-dominated universe (plus one small spectator dimension) sitting inside a five-dimensional radiation-dominated universe. The second contains an effective four-dimensional radiation-filled universe matched to an effective four-dimensional dust-filled spacetime.

II. EMPTY FIVE-DIMENSIONAL SPACETIME WITH COSMOLOGICAL CONSTANT

The Schwarzschild–de Sitter spacetime is the most general solution to the matter-free Einstein equations with cosmological constant which possesses the annealing property of spherical symmetry. In five dimensions its line element is

$$ds^2 = -A\,dY^2 + A^{-1}dR^2 + R^2 d\Omega_{(3)}^2 , \tag{2.1}$$

where

$$A = \left[1 - \frac{2M}{R^2} - X^2R^2\right]$$

and

$$d\Omega_{(3)}^2 = d\Psi^2 + \sin^2\Psi(d\Theta^2 + \sin^2 d\Phi^2) .$$

This line element looks very much like its four-dimensional counterpart. The differences are that the angular piece $d\Omega_{(3)}^2$ is the line element of the unit three-sphere, and R^2, not R, appears in the mass term of the metric function A. (Berezin and Kuzmin[16] discuss Schwarzschild–de Sitter space in arbitrary dimensions but they use a different normalization for the mass parameter.) In any dimension there is a transitional value of the product MX^2 which determines a radical change in the character of Schwarzschild–de Sitter space. Below the transitional value the spacetime has two horizons located at the two positive values of R for which the metric function A vanishes. The two horizons coincide at the transitional value, which in five dimensions is $8MX^2=1$. Above the transitional value there are no horizons; R and Y are globally well-defined timelike and spacelike coordinates, respectively, which along with the angular coordinates cover the entire Schwarzschild–de Sitter spacetime. Because the metric functions do not depend on Y, one may "compactify the Y direction," that is, restrict Y to the range $0 \le Y \le 2\pi R_5$ and identify the end points. Above transition, where Y is spacelike, this compactification does not lead to closed timelike curves in the Schwarzschild–de Sitter space.

Henceforth, when we work with the Schwarzschild–de Sitter spacetime, we shall always be working with $MX^2 > 1$, that is, above transition. The spacetime begins with a bang—a spacelike singularity at the origin of "time" $R=0$. For this reason we think of the Schwarzschild–de Sitter spacetime as a big-bang cosmology. Indeed, it is not difficult to find a cosmic time T for which the line element (2.1) assumes the form of a four-dimensional $k=1$ Robertson-Walker cosmology along with a dynamical extra dimension (such cosmologies have been studied by Davidson *et al.* in Ref. 17). Define $dT=(-A)^{-1/2}dR$ and find

$$ds^2 = \left[-1 + \frac{2M}{R^2(T)} + X^2R^2(T)\right]dY^2 - dT^2 + R^2(T)d\Omega_{(3)}^2 \tag{2.2}$$

with

$$R^2(T) = \frac{1}{2X^2}\{\sqrt{8MX^2-1}[\sinh(2XT)]+1\} .$$

Note that we have chosen conventions such that the Universe begins at a (negative) time given by $\sqrt{8MX^2-1}[\sinh(2XT)] = -1$.

At fixed time, the Schwarzschild–de Sitter cosmology is not isotropic at any point. As a manifestation of this it does exhibit cosmological compactification. A time profile of the metric function A is displayed in Fig. 1 from which it is seen that the compact dimension shrinks rapidly during the Universe's infancy. Ultimately the compact dimension will grow again so if one wishes to consider the possibility that the early Universe was Schwarzschild–de Sitter space, one will have to invoke some sort of phase transition at the time the compact dimension is small.

We are interested in the viability of the cosmological compactification mechanism in a universe with a complicated domain structure. In particular, we are interested in the constraints (if any) imposed by the requirement that spacetime be well behaved at the domain boundaries. As a toy model, we shall work out in the next section the matching of a bubble of Schwarzschild–de Sitter space to an exterior region of de Sitter space.

The de Sitter space is the special case of Schwarzshild–de Sitter space with $M=0$. It may be described as the surface

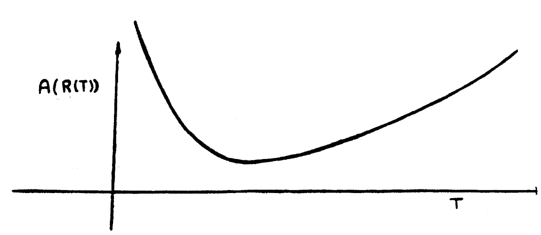

FIG. 1. Plot of the metric function $A(R(T)) = [-1+2M/R^2(T)+X^2R^2(T)]$ versus the time T.

$$g_{MN}u^M u^N = \frac{1}{\chi^2} \quad (M,N=0,\ldots,5) \tag{2.3}$$

with

$$g_{MN} = \mathrm{diag}(-1,1,1,1,1,1)$$

and

$$ds^2 = g_{MN}du^M du^N \ .$$

The line element of de Sitter space is not the $M=0$ limit of Eq. (2.2) because in deriving this equation we used $8MX^2>1$. However, it is possible to write the de Sitter line element in the form of a four-dimensional $k=-1$ Robertson-Walker cosmology with a dynamical extra dimension:[17]

$$ds^2 = \frac{1}{\chi^2}\cosh^2(\chi t)dy^2 - dt^2 + r^2(t)d\omega_{(3)}^2 \ , \tag{2.4}$$

where

$$r^2(t) = \frac{1}{\chi^2}\sinh^2(\chi t)$$

and

$$d\omega_{(3)}^2 = d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2) \ .$$

In these coordinates the surface swept out by the (y,t) coordinates at fixed angles is a subspace of the surface described by Eq. (2.3). This subspace is given by $-(u^0)^2+(u^1)^2+(u^2)^2=1/\chi^2$. So y is automatically an angular coordinate and lies in the range $0\le y\le 2\pi$. There are coordinate systems for de Sitter space, for which a dimension may be compactified by declaring an appropriate coordinate to lie in a fixed range (as we did for the Schwarzschild–de Sitter space). We choose, though, to work with the full de Sitter space with line element given by Eq. (2.4).

In the interest of simplicity, we shall try to match the Schwarzschild–de Sitter and de Sitter spacetimes across a boundary with the highest possible degree of symmetry. At first, it seems this may be achieved by specifying $Y=Y(T)$ to describe the dynamics of the boundary from the Schwarzschild–de Sitter point of view and by determining $y=y(t)$ to describe the de Sitter space dynamics. However, Schwarzschild–de Sitter and de Sitter observers will not agree on the topology of such a boundary, so such a case must be excluded. Even if the de Sitter space is expressed in coordinates so that it looks like a $k=+1$ Robertson-Walker cosmology with dynamical extra dimension, a boundary specified by $Y=Y(T)$ is still unacceptable. The reason is that the surface swept out by the (Y,T) coordinates has the topology of a cylinder. When projected onto this surface, the domain boundary appears as a line. A snapshot of the spacetime consisting of the Schwarzschild–de Sitter space sewn to an external de Sitter space is illustrated in Fig. 2. This composite spacetime ceases to be a manifold at the domain boundary. In particular, a particle entering the Schwarzschild–de Sitter space from the (shaded) de Sitter space would seem to have the option of turning left or right at the boundary. What will it do? Spacetimes such as are illustrated in Fig. 2 are inadmissable. Therefore, we shall specify the domain boundary dynamics by giving the equations $\Psi=\Psi(T)$ and $\psi=\psi(t)$. A snapshot of the resulting spacetime is illustrated in Fig. 3. The boundary has the topology $R\times S^2\times S^1$ and symmetry group $O(3)\times U(1)$. We shall find that, because of this relatively low degree of symmetry, obliging the metric to be well defined at the boundary suffices to determine the boundary dynamics. In the next section we shall work out the metric matching in detail. Along the way we shall find two surprising results: the scale of the Schwarzschild–de Sitter compact dimension is not a free parameter but rather is given by a relation $R_5=R_5(X,M,\chi)$ and that, in order that the domain boundary be timelike in the asymptotic past, $8MX^2$ cannot be too large.

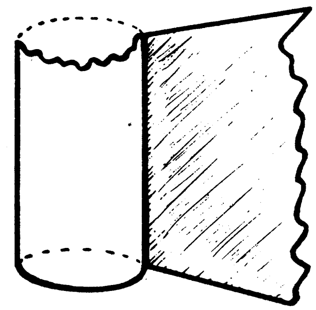

FIG. 2. Snapshot of Schwarzschild–de Sitter space sewn to de Sitter space (shaded) across a domain boundary with dynamics given by $Y=Y(T)$. This spacetime ceases to be manifold at the domain boundary.

III. THE DETAILED MATCHING

We would like to match the Schwarzschild–de Sitter line element

$$ds^2 = \left[-1+\frac{2M}{R^2(T)}+X^2R^2(T)\right]dY^2 - dT^2 + R^2(T)d\Omega_{(3)}^2 \ , \tag{3.1}$$

where

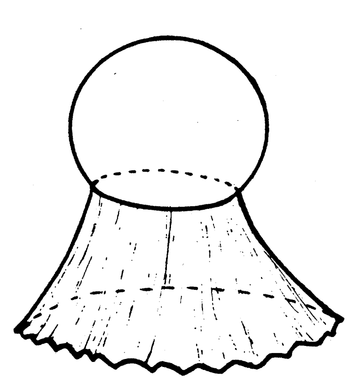

FIG. 3. Snapshot of Schwarzschild–de Sitter space sewn to de Sitter space (shaded) across a domain boundary with dynamics given by $\Psi=\Psi(T)$, $\psi=\psi(t)$. The polar angles Θ and θ along with the compact coordinates Y and y have been suppressed.

$$R^2(T)=\frac{1}{2X^2}\{\sqrt{8MX^2-1}[\sin(2XT)]+1\}$$

to the de Sitter line element

$$ds^2=\frac{1}{\chi^2}\cosh^2(\chi t)dy^2-dt^2+r^2(t)d\omega_{(3)}^2 , \qquad (3.2)$$

where

$$r^2(t)=\frac{1}{\chi^2}\sinh^2(\chi t)$$

across a boundary with topology $R\times S^2\times S^1$ and with dynamics specified by $\Psi=\Psi(T),\psi=\psi(t)$. Our strategy involves four steps.

(1) Demanding that the circumference of the compactified dimension is well defined at the boundary for all time gives the relation $T=T(t)$ and (perhaps surprisingly) $R_5=R_5(X,M,\chi)$.

(2) Requiring that the radius of the two-sphere is well defined gives a relation $\Psi=\Psi(\psi,t)$.

(3) Equating the domain boundary line element $ds^2=-dT^2+R^2d\Psi^2=-dt^2+r^2d\psi^2$ gives the domain-wall dynamics. We can solve the resulting equations explicitly in the asymptotic past.

(4) Obliging that $ds^2<0$ in the asymptotic past leads to $1<\sqrt{8MX^2}\le 2$.

(1) *Matching the compact dimension.* The condition that the size of the compact dimension be well defined at the domain boundary is

$$\left[-1+\frac{2M}{R^2(T)}+X^2R^2(T)\right]R_5^2=\frac{1}{\chi^2}\cosh^2(\chi t) , \qquad (3.3)$$

which has the solution

$$R^2(t)=\frac{1}{2X^2R_5^2}\left\{\left[\frac{1}{\chi^2}\cosh^2(\chi t)+R_5^2\right]\pm\left[\left[\frac{1}{\chi^2}\cosh^2(\chi t)+R_5^2\right]^2-8X^2MR_5^4\right]^{1/2}\right\} . \qquad (3.4)$$

For either choice of the sign in (3.4) the size of the compact dimension will be well defined at the domain boundary. Once we have determined which is the correct sign, Eqs. (3.4) and (3.1) give an implicit relation for $T=T(t)$. In the case of the negative branch of Eq. (3.4), $R^2(t)$ increases monotonically from zero at $t=-\infty$ to some maximum at $t=0$, then decreases monotonically back to zero at $t=+\infty$. For the positive branch, R^2 is infinite at $t=\pm\infty$ and achieves a minimum at $t=0$. Now, R is a timelike coordinate and must increase monotonically with t. This means that the negative branch is appropriate in the distant past while the positive branch is proper in the asymptotic future. The only way R can be a continuous function of t is if the "branch switch" occurs at $t=0$ and if the square root in Eq. (3.4) vanishes at this time. Thus, R_5 cannot be chosen freely, but must satisfy

$$\left[\frac{1}{\chi^2}+R_5^2\right]^2-8MX^2R_5^4=0 . \qquad (3.5)$$

The solution for this quadratic equation in R_5^2 assumes the convenient form

$$(\chi R_5)^2=\frac{1}{\sqrt{8MX^2}-1} . \qquad (3.6)$$

In order to match the domain-wall line element in step (3) we shall need to know how T varies with t. After a bit of algebra we find

$$\frac{dT}{dt}=\frac{1}{2X\chi R_5^2}\frac{1}{\{[2X^2R^2(t)-1]^2+(8MX^2-1)\}^{1/2}}\sinh 2\chi t\times\left[1\pm\frac{\frac{1}{\chi^2}\cosh^2\chi t+R_5^2}{[(1/\chi^2\cosh^2\chi t+R_5^2)^2-8MX^2R_5^4]^{1/2}}\right] , \qquad (3.7)$$

where $R^2(t)$ is given by Eq. (3.4). The minus sign is appropriate for $t<0$, the plus sign for $t>0$. Note that while T is a continuous function of t, dT/dt has a jump discontinuity at $t=0$. Equation (3.7) simplifies greatly in the asymptotic past:

$$\frac{dT}{dt}\mapsto 2\frac{\chi}{X}(\chi R_5)^2\sqrt{8MX^2}e^{2\chi t} . \qquad (3.8)$$

That dT/dt tends to zero reflects the fact that $R=0$ corresponds to a finite value of T but a (negative) infinite value for t.

(2) *Matching the two-sphere radius.* Because of the O(3) symmetry of the domain boundary we may identify the angles Θ with θ and Φ with ϕ. Then the condition that the coefficient of the two-sphere metric $d\Theta^2+\sin^2\Theta\, d\Phi^2$ be well defined at the domain boundary is

$$R^2\sin^2\Psi=r^2\sinh^2\psi . \qquad (3.9)$$

We know $R^2(t)$ [Eq. (3.4)] and $r^2(t)$ [Eq. (2.4)] so Eq. (3.9) determines the relation $\Psi=\Psi(\psi,t)$, up to sign. In order to match the domain-wall line element in step (3) we shall need to know how Ψ varies with ψ and t. The most convenient form for this variation is

$$\pm\frac{d\psi}{dt}=\frac{\cos\Psi\, d\Psi+\sin\Psi(d\sqrt{F(t)})/dt}{\sqrt{1+\sin^2\Psi F(t)}} , \qquad (3.10)$$

where

$$F(t)\equiv\frac{\cosh^2\chi t+(\chi R_5)^2\pm\{[\cosh^2\chi t+(\chi R_5)^2]^2-8MX^2(\chi R_5)^4\}^{1/2}}{2(XR_5)^2\sinh^2\chi t} .$$

The choice of sign in Eq. (3.10) has no impact on the argument we shall give in step (3) because only $(d\psi)^2$ appears in the domain-boundary line element. In the asymptotic past the complicated time functions in Eq. (3.10) have the form

$$F(t)\mapsto 32M\chi^2(\chi R_5)^2 e^{4\chi t} \tag{3.11}$$

and

$$\frac{d}{dt}\sqrt{F(t)}\mapsto 2\sqrt{32M}\,\chi^2(\chi R_5)e^{2\chi t}\ . \tag{3.12}$$

(3) *Matching the domain-boundary line element.* The domain boundary trajectory is described by the line element

$$ds^2=-dT^2+R^2 d\Psi^2=-dt^2+r^2 d\psi^2\ . \tag{3.13}$$

By substituting Eqs. (3.10) and (3.7) into this line element one obtains a quadratic equation

$$A(\Psi,t)\left[\frac{d\Psi}{dt}\right]^2+B(\Psi,t)\left[\frac{d\Psi}{dt}\right]+C(\Psi,t)=0 \tag{3.14}$$

which in principle gives the dynamics of the domain boundary. In practice the equation is difficult to analyze except in the asymptotic past where the coefficient functions are

$$\begin{aligned}A(\Psi,t)&\mapsto\frac{1}{4\chi^2}e^{-2\chi t}\cos^2\Psi\ ,\\ B(\Psi,t)&\mapsto\sqrt{8M}\,\sin 2\Psi(\chi R_5)\ ,\\ C(\Psi,t)&\mapsto -1\ .\end{aligned} \tag{3.15}$$

It is easy to see that $\cos\Psi\mapsto e^{\chi t}$ in the asymptotic past is not consistent with Eq. (3.14). Therefore,

$$\begin{aligned}\frac{d\Psi}{dt}&=\frac{-B(\Psi,t)\pm\sqrt{B^2(\Psi,t)-4A(\Psi,t)C(\Psi,t)}}{2A(\Psi,t)}\\ &\mapsto\pm\frac{1}{A^{1/2}(\Psi,t)}=\pm\frac{2\chi e^{\chi t}}{\cos\Psi}\end{aligned} \tag{3.16}$$

in the distant past. The asymptotic solution is

$$\sin\Psi\mapsto 2(\pm e^{\chi t}+\text{const})\ . \tag{3.17}$$

(4) *A timelike domain boundary in the asymptotic past.* Inserting Eqs. (3.4), (3.8), and (3.16) into

$$\left[\frac{ds}{dt}\right]^2=-\left[\frac{dT}{dt}\right]^2+R^2(t)\left[\frac{d\Psi}{dt}\right]^2 \tag{3.18}$$

yields, in the distant past,

$$\left[\frac{ds}{dt}\right]^2\mapsto -32M\chi^2(\chi R_5)^2 e^{4\chi t}\left[(\chi R_5)^2-\frac{1}{\cos^2\Psi_0}\right]\ , \tag{3.19}$$

where Ψ_0, an integration constant, is the asymptotic value for Ψ. In order that the domain boundary be timelike in the asymptotic past we require $(\chi R_5)^2-1/\cos^2\Psi_0>0$. Using Eq. (3.6) we find

$$\sqrt{8MX^2}-1<\cos^2\Psi_0\ . \tag{3.20}$$

So the critical product $8MX^2$, which was defined to be greater than unity, must surely be less than 2. Its range is squeezed even more tightly if $\cos^2\Psi_0$ does not assume its maximal value of unity.

A few caveats are in order regarding the constraints, (3.6) and (3.20), we have derived. We have been considering the classical evolution of a Schwarzschild–de Sitter space sewn to de Sitter space. In order that this spacetime be well defined for all time, we discovered Eq. (3.6) as a necessary condition. This, in turn, was used in deriving Eq. (3.20). If, however, one imagines that the Schwarzschild–de Sitter or de Sitter spaces undergo some quantum phase transition before $t=0$, then these constraints may be evaded. Some restrictions will obtain, for example, for as long as the Universe may be treated classically the square root in Eq. (3.4) must be real, but a detailed treatment is inherently quantum mechanical. Equation (3.20) was derived as a necessary condition in order that the domain boundary be timelike. But we cannot solve the equations of motion, (3.14), for all time, so we do not know if Eq. (3.20) is a sufficient condition. We also note that even if the constraints (3.6) and (3.20) are satisfied, it seems likely that the function $s(t)\equiv\int(ds/dt)dt$, though continuous, has a kink at $t=0$.

Finally we observe that the Einstein equations imply that the energy-momentum tensor has a contribution concentrated at the domain boundary. To see this, one splits the five-dimensional spacetime near the boundary into a one-parameter (η) family of four-dimensional slices such that $\eta=0$ is the domain-boundary. It is convenient (and always possible) to choose η so that the vector ξ^μ, which points in the direction of increasing η, is of unit length and normal to the slices it pierces. The five-dimensional Einstein equations at the point $x^\mu=(\eta,x^i)$ may be expressed in terms of four-dimensional quantities intrinsic to the slice coordinated by the x^i, along with the extrinsic curvature tensor

$$K_{ij}\equiv\xi_{i;j} \tag{3.21}$$

which describes how these slices are embedded in five-dimensional spacetime. The extrinsic curvature is ill defined as the domain boundary; Schwarzschild–de Sitter and de Sitter observers do not agree on its value there. And the Einstein tensor G_{ij} contains a normal derivative of the extrinsic curvature. Therefore, the energy-momentum tensor must have a δ-function contribution located at the domain boundary. This contribution is determined by integrating the five-dimensional Einstein equation across the domain boundary[18] (in four dimensions[19]):

$$[K_{ij}]-\tfrac{4}{3}g_{ij}[\mathrm{Tr}K]=-8\pi GS_{ij}\ , \tag{3.22}$$

where

$$S_{ij}\equiv\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}T_{ij}\,d\eta$$

and square brackets around a quantity denote the difference between the quantity viewed by de Sitter and Schwarzschild–de Sitter observers. Note that, because of the $O(3)\times U(1)$ symmetry of the boundary, S_{ij} can only depend on the boundary's proper time τ. Since the trajectory of the domain boundary determines its energy content via Eq. (3.22) one might wonder if there are any (or all) trajectories for which $S_{\tau\tau}$ is positive. The answer is, for some, but not all trajectories $S_{\tau\tau}$ must be positive in the asymptotic past. The reason is that if $(\Psi(\tau),\psi(\tau))$ is an admissible domain-boundary trajectory, so is $(\pi-\Psi(\tau),-\psi(\tau))$: after all, the metric functions involve only $\sin^2\Psi$ and $\sinh^2\psi$. Under the transformation $(\Psi,\psi)\mapsto(\pi-\Psi,-\psi)$ all the components of the extrinsic curvature (from both Schwarzschild–de Sitter and de Sitter points of view) change sign. Therefore, one and only one of the allowable trajectories (Ψ,ψ) and $(\pi-\Psi,-\psi)$ has a positive $S_{\tau\tau}$ in the asymptotic past. Pictorially, if Fig. 3 represents a snapshot of a trajectory with negative rest energy, then Fig. 4 shows a snapshot of a trajectory with positive rest energy. We have limited our remarks to the asymptotic past because we cannot rule out the possibility that $S_{\tau\tau}$ passes through zero during the domain-boundary history. In the Appendix we display the extrinsic curvature components explicitly and confirm that they change sign under $(\Psi,\psi)\mapsto(\pi-\Psi,-\psi)$.

IV. THE FIFTH DIMENSION AS A MATTER FIELD

Up until now, we have been considering the sewing of two higher-dimensional spacetimes across a domain boundary representing the rapid variation of some (unspecified) external matter field. However, it is possible that the extra dimensions themselves vary rapidly in a small region of spacetime, leading to a domain boundary. According to the (Einstein) equations of motion, this can occur if, for example, the extra dimensions are associated to a potential which is sharply peaked in a small region of spacetime. Such a potential can be generated by adding

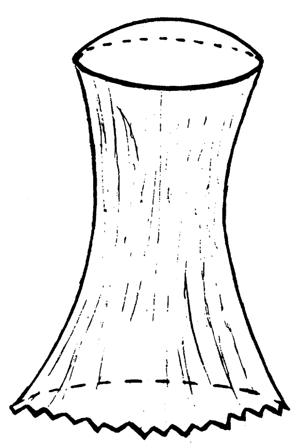

FIG. 4. The effect on Fig. 3 of the transformation $(\Psi,\psi)\mapsto(\pi-\Psi,-\psi)$. The surface stress S_{ij} corresponding to the trajectories frozen in Figs. 3 and 4 are equal in magnitude and opposite in sign.

matter fields to the higher-dimensional spacetime[7,8] introducing one-loop Casimir effects[9–11] or by considering bosonic strings which wind around the extra dimensions.[13] If any of these options is exploited, and if the domain boundary is idealized as infinitesimally thin, then the sizes of the compact dimensions jump at the boundary. So it is not the case that all the tangential components of the higher-dimensional metric are continuous at the domain boundary; matching spacetimes in the presence of a potential for the extra dimensions is qualitatively different from the matching previously discussed for which there was no such potential. In order to match spacetimes with a potential for the extra dimensions, one reduces the higher-dimensional theory to an effective four-dimensional theory in which the extra dimensions play the role of scalar matter fields. Then the situation is as we have previously discussed, only one matches the tangential components of an effective four-dimensional metric at a domain boundary which is effective a three-dimensional surface. In general, none of the components of the original five-dimensional metric are continuous at this surface.

The details of the reduction to an effective four-dimensional theory depend on the structure of the extra dimensions. The spacetimes we shall study will be five dimensional so we shall present the reduction of a five dimensional to an effective four-dimensional theory (for the reduction of higher-dimensional theories in which the extra dimensions have spherical symmetry see Ref. 9). We shall apply the reduction first to the case of five-dimensional de Sitter space enclosing a bubble of four-dimensional de Sitter space with constant fifth dimension. This example is suggested by the work of Moss.[14] We shall determine the equation of motion for the effective domain boundary in terms of the surface energy density and pressure of the boundary.

The reduction procedure suggests some interesting composite spacetimes which do not require a singular contribution to the energy-momentum tensor localized at the boundary separating two domains. One such spacetime consists of a five-dimensional Robertson-Walker radiation-dominated universe sewn to a four-dimensional Robertson-Walker radiation-dominated universe (plus constant extra dimension). As a second and final example, we sketch how effective four-dimensional radiation-filled and dust-filled spacetimes can be matched without a singular energy-momentum density at their boundary.

Now we turn to the reduction of five-dimensional gravity to an effective four-dimensional theory. The five-dimensional line element is

$$ds^2=g_{MN}dx^M dx^N \tag{4.1}$$

written in coordinates $x^M=(x^\mu,y)$. We shall be concerned with spacetimes which satisfy the Kaluza-Klein ansatz, that is to say, for which the coordinate y describes a compact dimension, and for which g_{MN} is independent of y. We shall also assume that the Kaluza-Klein gauge fields vanish: $g_{\mu y}=0$. Then the five-dimensional line element may be written in the form

$$ds^2=C^{-1}\gamma_{\mu\nu}dx^\mu dx^\nu+C^2dy^2\ . \tag{4.2}$$

We shall compactify the fifth dimension so that y lies in the range $0 \le y \le 2\pi R_5$. After choosing units such that the five-dimensional Newton's constant $G_5 \equiv R_5/4$ and defining the effective scalar field $\phi \equiv \sqrt{3/2}\ln C$, the five-dimensional Einstein equations become

$$ {}^{(4)}R_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}{}^{(4)}R = -\{\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\gamma_{\mu\nu}[(\partial_\sigma\phi\gamma^{\sigma\tau}\partial_\tau\phi) + \Lambda e^{-\sqrt{2/3}\phi}]\} , \quad (4.3) $$

$$ \Delta\phi = -\sqrt{2/3}\Lambda e^{-\sqrt{2/3}\phi} , \quad (4.4) $$

where Λ is the five-dimensional cosmological constant and where the Ricci tensor ${}^{(4)}R_{\mu\nu}$ and covariant derivatives are calculated with the effective four-dimensional metric $\gamma_{\mu\nu}$. With the parametrization defined by Eq. (4.2) the five-dimensional theory appears as a four-dimensional theory with a massless scalar field sitting in an exponentially falling potential.

Once additional matter fields or Casimir effects are introduced, potentials with interesting structure can be obtained. The potential illustrated in Fig. 5, for example, includes the one-loop Casimir energy of the graviton and additional fermionic matter fields.[9–11]

Given a potential such as is illustrated in Fig. 5 there is a metastable (i.e., classically stable) solution of the Einstein equations with ϕ frozen at the extremal value $\phi_{\min}$ and $\gamma_{\mu\nu}$ the metric of an effective de Sitter space with cosmological constant $V(\phi_{\min})$. The five-dimensional spacetime, then, is a product of a four-dimensional de Sitter space with a circle of constant circumference. Eventually ϕ tunnels through the potential barrier and begins rolling down the tail of the potential. Once $\phi \gg \phi_{\min}$ this potential is dominated by the cosmological-constant contribution $\Lambda e^{-\sqrt{2/3}\phi}$— the Casimir contributions can be ignored. During this rolling, the spacetime is a five-dimensional de Sitter space with cosmological constant Λ.

In an early universe with a complicated domain structure, it is plausible that some regions have ϕ frozen at its extremal value. Such regions may be enveloped in a five-dimensional de Sitter space. Moreover, it is possible to nucleate a region of four-dimensional de Sitter space (with an extra circle) within a larger domain of five-dimensional de Sitter space.[14] Therefore, it seems fair to enquire as to how these spacetimes may be matched across a domain wall across which the size of the extra dimension varies rapidly.

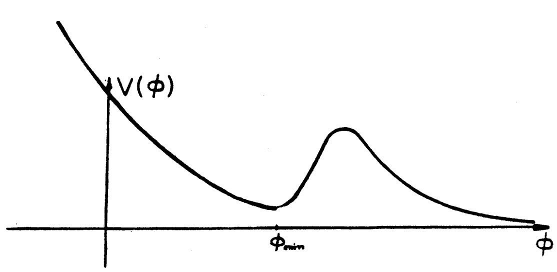

FIG. 5. The potential for the effective matter field ϕ obtained by considering the one-loop Casimir energy of graviton and Fermi fields, along with a modest cosmological-constant term.

The line element of the four-dimensional de Sitter space with an extra circle is

$$ ds^2 = -dT^2 + \frac{\cosh^2 XT}{X^2}(d\Psi^2 + \sin^2\Psi\, d\Omega^2) + dY^2 \quad (4.5) $$

with $d\Omega^2 = d\Theta^2 + \sin^2\Theta\, d\Phi^2$ and $0 \le Y \le 2\pi R_5$. For the five-dimensional de Sitter space the line element has the form

$$ ds^2 = -dt^2 + \frac{\sinh^2\chi t}{\chi^2}(d\psi^2 + \sinh^2\psi\, d\omega^2) + \frac{\cosh^2\chi t}{\chi^2}dy^2 \quad (4.6) $$

with $d\omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The effective four-dimensional line elements are

$$ d\sigma^2 = -dT^2 + \frac{\cosh^2 XT}{X^2}(d\Psi^2 + \sin^2\Psi\, d\Omega^2) \quad (4.7) $$

for the four-dimensional de Sitter space with circle, and

$$ d\sigma^2 = \frac{\cosh\chi t}{\chi}\left[-dt^2 + \frac{\sinh^2\chi t}{\chi^2}(d\psi^2 + \sinh^2\psi\, d\omega^2)\right] \quad (4.8) $$

for the five-dimensional de Sitter space. We shall match these effective four-dimensional line elements across a domain boundary with spherical symmetry. Thus we shall set $\Theta = \theta$ and $\Phi = \phi$. The line element (4.7) is that of a four-dimensional Robertson-Walker cosmology. By redefining the time t the line element (4.8) can also be put in such a form. Both Eqs. (4.7) and (4.8) can be summarized by

$$ d\sigma_\pm^2 = -dt_\pm^2 + a_\pm^2(t_\pm)[d\psi_\pm^2 + f_\pm^2(\psi_\pm)d\Omega^2] . \quad (4.9) $$

The plus subscripts refer to the four-dimensional de Sitter space with circle, which reduces to a $k=1$ four-dimensional Robertson-Walker cosmology. The minus subscripts refer to the five-dimensional de Sitter space which mimics a $k=-1$ cosmology.

The joining of two Robertson-Walker cosmologies across a domain-boundary has been examined by Maeda.[15] The composite spacetime may be described in terms of six functions—the energy densities and pressures of the two regions being joined:

$$ \rho_\pm \equiv 3\left[\left(\frac{\dot a}{a}\right)_\pm^2 \pm \frac{1}{a_\pm^2}\right] , \quad (4.10) $$

$$ p_\pm \equiv -2\left(\frac{\ddot a}{a}\right)_\pm - \left(\frac{\dot a}{a}\right)_\pm^2 \mp \frac{1}{a_\pm^1} \quad (4.11) $$

(an overdot denotes differentiation with respect to the proper time τ) along with the energy density (ς) and pressure (ϖ) of the domain boundary. All six parameters are related by the effective four-dimensional (covariant) conservation of energy, and the parameters ς and ϖ are further related by the equation of state for a thin domain boundary, which we now discuss.

In general, when two spacetimes are matched across a thin wall, there is a contribution to the energy-momentum tensor concentrated at this wall. In order to describe this contribution it is convenient to choose coordinates in a neighborhood of the wall $x^\mu=(\eta,x^i)$ such that $\eta=0$ defines the wall (now idealized as infinitesimally thin) and ξ^μ, the unit vector which points toward increasing η, is normal to the wall. The effective four-dimensional energy-momentum tensor may then be written

$$T_{\mu\nu}=S_{\mu\nu}(x^i)\delta(\eta)+\text{regular terms} . \tag{4.12}$$

It is not difficult to check that four-dimensional conservation of energy ensures that $S_{\eta\eta}=S_{\eta i}=0$. The most general form for $S_{\mu\nu}$ which respects these orthogonality conditions, and which is rotationally invariant is

$$S_{\mu\nu}=\varsigma(\tau)U_\mu U_\nu+\varpi(\tau)(h_{\mu\nu}+U_\mu U_\nu) , \tag{4.13}$$

where U^μ is the four-velocity of the domain-boundary and $h_{\mu\nu}\equiv\gamma_{\mu\nu}-\xi_\mu\xi_\nu$. In the effective four-dimensional picture the source of the energy-momentum reflected in $S_{\mu\nu}$ is the rapid variation of an effective scalar field ϕ with energy-momentum tensor

$$T^\phi_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\gamma_{\mu\nu}[\tfrac{1}{2}\partial_\alpha\phi\gamma^{\alpha\beta}\partial_\beta\phi+V(\phi)] . \tag{4.14}$$

In the thin-wall limit, derivatives normal to the wall dominate all others so that $\partial_\mu\phi\propto\xi_\mu$. It follows that $S_{\mu\nu}$ can have only contributions proportional to $\xi_\mu\xi_\nu$ and $\gamma_{\mu\nu}$ which forces $\varpi=-\varsigma$ and

$$S_{\mu\nu}=-\varsigma(\tau)h_{\mu\nu}(\eta=0,x^i) . \tag{4.15}$$

The domain boundary can exchange energy with the spacetimes it separates, so it need not be the case that ς is a constant. Indeed, the proper-time variation of ς is determined by the conservation of energy:[15]

$$\frac{d}{d\tau}(\varsigma R^2)+\varpi\frac{d}{d\tau}(R^2)=\left[(\rho+p)R^2a\gamma\frac{d\psi}{d\tau}\right] , \tag{4.16}$$

where square brackets about a quantity denotes the quantity measured by a five-dimensional de Sitter observer at the boundary subtracted from the quantity measured by a four-dimensional (with extra circle) de Sitter observer there; $R\equiv af$ is the radius of the boundary's two-sphere, on which both observers agree and $\gamma_\pm\equiv(dt/d\tau)_\pm=[1+a^2(d\psi/d\tau)^2]^{1/2}_\pm$ is the usual relativistic expansion factor. Equation (4.16) can be simplified by substituting $\varpi=-\varsigma$ and the four-dimensional de Sitter relation $\rho_+=-p_+$ (the extra circle does not alter this relation):

$$\varsigma=-(\rho_-+p_-)a\gamma\frac{d\psi}{d\tau} \tag{4.17}$$

with all quantities evaluated at the domain boundary. The equation of motion for the domain boundary is[15]

$$\left[\frac{a}{\gamma}\left[\frac{d^2\psi}{d\tau^2}+2\gamma\frac{\dot a}{a}\frac{d\psi}{d\tau}\right]\right]=-\varsigma . \tag{4.18}$$

As before, a bracketed quantity is a difference measured between four- and five-dimensional de Sitter observers. Given the initial data $R(\tau=0)$ and $\dot R(\tau=0)$, Eq. (4.18) determines the initial value $\varsigma(\tau=0)$. Equations (4.17) and (4.18) together then determine the trajectory of the domain boundary along with $\varsigma(\tau)$.

Finally, we shall apply the technique of reducing a five-dimensional spacetime to a four-dimensional one to the case of a five-dimensional radiation-dominated Robertson-Walker cosmology with compact dimension. This will suggest a scenario where an inhomogeneous anisotropic spacetime can be sewn to a radiation-dominated spacetime without a concentration of energy at the domain-boundary. We shall also discuss a spacetime in which two five-dimensional radiation-dominated cosmologies with rather different scales for the compactified dimension can be attached without a concentration of energy density. From an effective four-dimensional viewpoint, these spacetimes would be described quite differently.

The five-dimensional radiation-dominated cosmology is a solution to the Einstein equations with energy-momentum tensor $T^N_M=\text{diag}(-\rho,\rho/4,\rho/4,\rho/4,\rho/4)$. The line element may be written

$$ds^2=C^{-1}\gamma_{\mu\nu}dx^\mu dx^\nu+C^2dy^2 \tag{4.19}$$

with $0\le y\le 2\pi$. The effective four-dimensional line element is

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu=-dt^2+t(dr^2+r^2d\Omega^2) . \tag{4.20}$$

The relationship between the effective cosmic time t and the metric function C of Eq. (4.19) is $C=kt^{1/3}$ (k can be any positive number and the five-dimensional energy density ρ is independent of k). The effective four-dimensional scalar field is defined by $\phi\equiv\sqrt{3/2}\ln C$. The effective four-dimensional Einstein equations have the form

$$ {}^{(4)}R_{\mu\nu}-\tfrac{1}{2}\gamma_{\mu\nu}{}^{(4)}R =-\{\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}\gamma_{\mu\nu}[\partial_\sigma\phi\gamma^{\sigma\tau}\partial_\tau\phi+V(\phi)]\}+T^{\text{eff}}_{\mu\nu} \tag{4.21}$$

and the equation of motion for the scalar field is

$$\Delta\phi=\frac{dV}{d\phi}-\sqrt{3/2}\gamma^{\mu\nu}T^{\text{eff}}_{\mu\nu} ; \tag{4.22}$$

where $T^{\text{eff}}_{\mu\nu}$ is an effective four-dimensional source term. In obtaining Eq. (4.22) we used the fact that the five-dimensional energy-momentum tensor describing radiation is traceless. The four-dimensional effective source $T^{\text{eff}}_{\mu\nu}$ is not, in general, traceless. Equations (4.21) and (4.22) do not uniquely determine the potential and effective source terms. (In the cases we have previously studied, with $T^N_M=\Lambda\delta^N_M$, it is always possible to set the effective source equal to zero.) One consistent choice is $V(\phi)=\lambda e^{-2\sqrt{3}\phi}$ and $T^{\nu(\text{eff})}_\mu=\text{diag}(-\rho_r,p_r,p_r,p_r)$ with the effective radiation energy density and pressure defined via

$$\rho=\rho_r+V(\phi) , \tag{4.23}$$

$$\rho=4[p_r-V(\phi)] . \tag{4.24}$$

From the four-dimensional viewpoint, the energy density ρ is interpreted as a sum of an effective four-dimensional radiation energy density and a Casimir energy, $V(\phi)$. Likewise the pressure $p=\rho/4$ is seen as a four-dimensional effective radiation pressure along with a "Casimir pressure"—$V(\phi)$. For more details, in particular the justification for interpreting ρ_r and p_r as effectively arising from radiation, see Ref. 20.

In Fig. 6 we have reproduced Fig. 5, only we have fine-tuned the five-dimensional cosmological constant so that the potential vanishes at its minimum. The potential is dominated by the Casimir term when ϕ is sufficiently small. Therefore, if the spacetime is such that ϕ is sufficiently small, then its line element is well described by Eq. (4.19) and there is an effective radiation source. By choosing the parameters of the potential properly, it is possible to arrange for the size of the compact dimension to be macroscopic even when ϕ is "sufficiently small" that the Casimir term dominates the potential. In time, the scalar field rolls down the potential and settles at the value ϕ_{min}. At this point the line element (4.19) becomes, after a rescaling of coordinates,

$$ds^2=-dT^2+T(dR^2+R^2d\Omega^2)+dY^2 \tag{4.25}$$

with $0\leq Y\leq 2\pi R_5$; in the present case $R_5=\exp(\sqrt{2/3}\phi_{\text{min}})$ but we want to be able to discuss the line element (4.25) for general R_5. The line element (4.25) describes a solution of Einstein's equations with energy-momentum tensor $T_M^N=\text{diag}(-\rho,\rho/3,\rho/3,\rho/3,0)$ with ρ independent of R_5. The spacetime is a four-dimensional radiation-dominated cosmology with an extra constant circle.

For any R_5, the effective four-dimensional metrics arising from Eqs. (4.19) and (4.25) are identical, and give Eq. (4.20). This means that the five-dimensional radiation-dominated cosmology can be attached to the four-dimensional radiation-dominated cosmology (with circle) across a small region of spacetime across which the fifth dimension varies rapidly, with no singular concentration of energy density in this region. The trick is to find a potential which enforces the rapid variation of the fifth dimension.

As a final example we consider the potential illustrated in Fig. 7. It was obtained by Rubin and Roth[11] who considered both repulsive (from fermions) and attractive

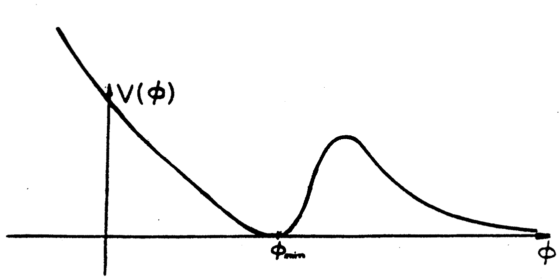

FIG. 6. The potential for the effective matter field ϕ obtained by considering the one-loop Casimir energy of graviton and Fermi fields and by choosing the cosmological constant Λ so that the potential vanishes at its minimum.

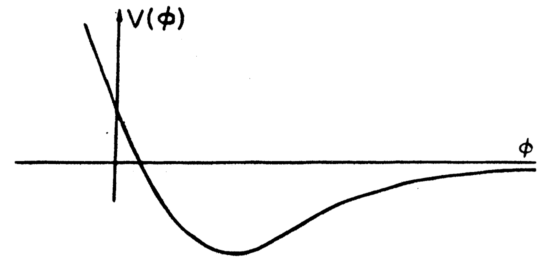

FIG. 7. The potential for the effective matter field ϕ obtained by considering the one-loop Casimir energy of graviton and Fermi fields but with no cosmological-constant term.

(from gravitons) Casimir effects, but who did not include a five-dimensional cosmological constant. The asymptotic behavior of the potential function is $V(\phi)\mapsto\lambda_-e^{-2\sqrt{6}\phi}\equiv V_-(\phi)$ for large negative ϕ and $V(\phi)\mapsto-\lambda_+e^{-2\sqrt{6}\phi}\equiv V_+(\phi)$ for large positive ϕ. This behavior at large ϕ is noteworthy because a second consistent choice for the potential and source term of Eqs. (4.21) and (4.22) is $V(\phi)\mapsto V_+(\phi)=-\lambda_+e^{-2\sqrt{6}\phi}$ and $T_\mu^{\nu(\text{eff})}=\text{diag}(\rho_d,0,0,0)$ with the effective dust energy defined by

$$\rho=\rho_d+V_+(\phi)\ . \tag{4.26}$$

From the four-dimensional viewpoint, the energy density ρ is interpreted as a sum of an effective four-dimensional dust energy and a Casimir energy $V_+(\phi)$. The pressure arises exclusively from Casimir effects. For more details see Ref. 21. If a region of spacetime is such that $\phi\gg 0$ then its line element is well described by Eq. (4.19) and there is an effective dust source. Indeed the spacetime with $\phi\ll 0$ (and effective radiation source) and the spacetime with $\phi\gg 0$ (and effective dust source) have the same effective four-dimensional metric. From the effective four-dimensional point of view, spacetimes of two very different characters may be connected across a region across which ϕ varies from very small to very large. The size of this region depends on the details of the potential $V(\phi)$. No matter how small it is though, there will not be a singular contribution to the energy-momentum tensor localized in this region. All our remarks about effective four-dimensional spacetimes apply equally well to genuine four-dimensional spacetimes with appropriate external matter fields.

V. SUMMARY

As a laboratory for studying the dimensional reduction in inhomogeneous spacetimes, we have considered several five-dimensional models in which a bubble of a spacetime with a small compact dimension is matched to a spacetime with all five dimensions macroscopic. The matching is across a domain boundary which is a small region of spacetime across which some field is varying rapidly. This field could represent external matter (Secs. II and III) or the size of the compact dimension (Sec. IV). In the former case, all the tangential components of the five-dimensional metric must be well defined on the domain boundary. For the model we analyzed, this condition was enough to determine the dynamics of the domain boundary, and also the energy-momentum content of the boundary. We also found that obliging the metric to be well defined for all time constrained the parameters describing the composite spacetime. In our opinion, these features are generic to models involving the sewing together of spacetimes, at least one of which is inhomogeneous.

The compact dimension could vary rapidly across a small region of spacetime if a suitable potential is introduced for the size of this dimension. Such a potential can arise if one includes additional matter fields or Casimir effects. When a compact dimension varies rapidly it is appropriate to reduce the five-dimensional theory to an effective four-dimensional theory in which the size of the compact dimension plays the role of a scalar matter field. It is the effective four-dimensional metric which must be well defined at the (effectively three-dimensional) domain-boundary. In general, the domain boundary represents a delta-function concentration of effective energy-momentum density (in the limit of infinitesimal boundary), but it is possible to find scenarios for which this is not the case.

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APPENDIX: THE EXTRINSIC CURVATURE COMPONENTS

We present the components of the extrinsic curvature tensor of the domain boundary from the Schwarzschild–de Sitter and de Sitter points of view, and observe that they change sign under the transformation $(\Psi,\psi)\mapsto(\pi-\Psi,-\psi)$. The extrinsic curvature is defined as

$$K_{ij}\equiv\xi_{i;j} ,$$

where ξ^μ is the unit normal to the domain boundary pointing (by convention) from Schwarzschild–de Sitter to de Sitter space. We choose coordinates so that ξ^μ points to increasing values of both Ψ and ψ. In the following an overdot denotes differentiation with respect to the proper time τ.

From the Schwarzschild–de Sitter viewpoint,

$$K_{\tau\tau}=\frac{1}{\dot T}R(T)\ddot\Psi+\frac{\sqrt{8MX^2-1}\dot\Psi}{R(T)} ,$$

$$K_{\Theta\Theta}=\frac{R(T)\sqrt{8MX^2-1}\sinh 2XT\sin^2\Psi\dot\Psi}{2X}+R(T)\sin\Psi\cos\psi\dot T ,$$

$$K_{\Phi\Phi}=\cos^2\Theta K_{\Theta\Theta} ,$$

$$K_{YY}=R(T)\frac{d}{dT}[-1+2M/R^2(T)+X^2R^2(T)]\dot\Psi$$

with

$$R^2(T)=\frac{1}{2X^2}(\sqrt{8MX^2-1}\sinh 2XT+1) .$$

All components change sign under $\Psi\mapsto\pi-\Psi$.

From the de Sitter viewpoint,

$$K_{\tau\tau}=\frac{|\sinh\chi t|}{\sinh\chi t}\left[\frac{\sinh\chi t\ddot\psi}{\chi\dot t}+2\cosh(\chi t)\dot\psi\right] ,$$

$$K_{\theta\theta}=\frac{|\sinh\chi t|\sinh\psi}{\chi^2}\times(\sinh\chi t\cosh\chi t\sinh\psi\dot\psi+\chi\cosh\psi\dot t) ,$$

$$K_{\phi\phi}=\cos^2\theta K_{\theta\theta} ,$$

$$K_{yy}=\frac{|\sinh\chi t|}{\chi^2}\sinh\chi t\cosh\chi t\dot\psi .$$

All components change sign under $\psi\mapsto-\psi$.

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