Covariant and gauge-independent perfect-fluid Robertson-Walker perturbations

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In the preceding paper, covariant and gauge-invariant quantities were defined that characterize density inhomogeneities in an almost-uniform model universe in a transparent way. In this paper second-order propagation equations are derived for these quantities in the case of a general "perfect fluid," and their properties examined. We do not use a harmonic decomposition in our definitions, but when such a decomposition is applied, our results are compatible with those obtained by Bardeen in his harmonically based gauge-invariant analysis. Our second-order equation enables a unified and transparent derivation of a series of results in the literature, without any ambiguity from choice of any particular gauge.

I. INTRODUCTION

There is a long history of study of perturbations of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe models, and their use to examine galaxy formation (e.g., Refs. 1–5). In general the discussions are plagued by the problem that the choice of variables to represent the inhomogeneities depends on the gauge chosen. Bardeen's major paper[5] gives a set of gauge-invariant quantities to describe the perturbations and propagation equations for these quantities; but their definition and geometric meaning are complex, and lose direct geometric meaning because a Fourier decomposition is applied at the outset of the analysis.

In the preceding article Ellis and Bruni[6] (EB) gave a set of covariantly defined gauge-invariant quantities with a simple geometrical and physical meaning, that code the information we need to discuss density inhomogeneities in an almost-FLRW model, and examined their dynamics in the case of pressure-free flows. Here we extend the dynamic discussion to the case of a general perfect fluid with nonzero pressure, the physics of which is expressed by a suitable equation of state (the extension to the imperfect-fluid cases is discussed in a separate paper[7]). The formalism used here is essentially equivalent to Bardeen's in its content,[7] but is more transparent because of the direct interpretation of the variables used.

As in the work of EB, the basic philosophy is not to perturb a FLRW model, but rather to solve the exact equations in the full four-dimensional space-time when the variables take values close to those they take in a FLRW universe; the approximation takes place by neglecting higher-order terms in the exact equations. Thus we do not distinguish between real and background variables, but do all calculations in the real space-time and linearize the resulting equations.

II. COVARIANTLY DEFINED GAUGE-INVARIANT VARIABLES

The exact covariant fluid equations for a completely general fluid flow in a curved space-time are presented in Refs. 8, 3, 9, and 10. The four-velocity vector tangent to the flow lines (the world lines of a typical observer in the universe) is u^a ($u^a u_a = -1$). The projection tensor into the tangent three-spaces orthogonal to u^a is $h_{ab} \equiv g_{ab} + u_a u_b$. The time derivative of any tensor $T^{ab}{}_{cd}$ along the fluid flow lines is $\dot{T}^{ab}{}_{cd} \equiv T^{ab}{}_{cd;e} u^e$. The first covariant derivative of the four-velocity vector is

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} - \dot{u}_a u_b, \tag{1}$$

where $\Theta \equiv u^a{}_{;a}$ is the expansion, $\omega_{ab} = \omega_{[ab]}$ is the vorticity tensor ($\omega_{ab}u^b = 0$), and $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab}u^b = 0$, $\sigma^a{}_a = 0$). A representative length scale S along the flow lines is defined by

$$\dot{S}/S = \tfrac{1}{3}\Theta. \tag{2}$$

The vorticity and shear magnitudes are defined by $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$, $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$.

Because we are here considering the case of a perfect fluid, the matter stress tensor will take the form

$$T_{ab} = \mu u_a u_b + p h_{ab} \tag{3}$$

where the pressure p and energy density μ are related by a suitable equation of state; for example, if s is the entropy then we can express it in the form

$$p = p(\mu, s). \tag{4}$$

However, as always in thermodynamics many other representations are possible, and other forms may be more suitable for some applications (see Sec. IV B).

A. Gauge-invariant quantities

In a FLRW universe model the shear, vorticity, acceleration, and Weyl tensor vanish,[8,9] and the energy density μ, pressure p, and expansion Θ are functions of the cosmic time t only. Covariantly defined quantities representing the spatial variation of the zero-order variables μ, p, and Θ, are their orthogonal spatial gradients:

$$X_a \equiv \kappa h_a{}^b \mu_{,b}\ ,\quad Y_a \equiv \kappa h_a{}^b p_{,b}\ ,\quad Z_a \equiv h_a{}^b \Theta_{,b} \tag{5}$$

where κ is the gravitational constant. Each of these quantities is gauge invariant, as they all vanish in FLRW universes (EB). Two other gauge-invariant quantities will be important, namely, the divergence of the acceleration, and its spatial gradient:

$$A \equiv \dot{u}^c{}_{;c}\ ,\quad A_a \equiv h_a{}^b A_{,b}. \tag{6}$$

B. The key variables

Two simple gauge-invariant quantities give us the information we need to discuss the time evolution of density fluctuations. The basic quantity we start with is the orthogonal projection of the energy-density gradient, i.e., the vector $X_a \equiv h_a{}^b \kappa\mu_{,b}$. This can be determined (a) from virial theorem estimates and (b) by observing gradients in the numbers of observed sources and estimating the mass-to-light ratio [Kristian and Sachs,[11] Eq. (39)]. However, this does not directly correspond to the quantities usually calculated; but two closely related quantities do.

The first is the comoving fractional density gradient

$$\mathcal{D}_a \equiv S\frac{X_a}{\kappa\mu}, \tag{7}$$

which is gauge invariant and dimensionless, and represents the spatial density variation over a fixed comoving scale (EB). Note that S, and so $\mathcal{D}_a$, is defined only up to a constant by Eq. (2); this allows it to represent the density variation between any neighboring world lines. The vector $\mathcal{D}_a$ can be separated into a direction e_a and magnitude $\mathcal{D}$ where

$$\mathcal{D}_a = \mathcal{D}e_a,\quad e_a e^a = 1,\quad e_a u^a = 0 \quad\Rightarrow\quad \mathcal{D} = (\mathcal{D}^a\mathcal{D}_a)^{1/2}. \tag{8}$$

The magnitude $\mathcal{D}$ is the gauge-invariant variable that most closely corresponds to the intention of the usual $\delta\mu/\mu$. The crucial difference from the usual definition is that $\mathcal{D}$ represents a (real) spatial fluctuation, rather than a (fictitious) time fluctuation (EB).

The second quantity is the vector

$$\Phi_a \equiv \kappa\mu S^2 \mathcal{D}_a \ =\ S^3 X_a \tag{9}$$

corresponding to the Bardeen variable Φ_H up to a constant, except that this is a vector; and we have not required a harmonic analysis to make this definition. We shall later see its propagation equations correspond to those for the Bardeen variable.

III. DYNAMIC EQUATIONS

We can determine exact propagation equations along the fluid flow lines for the quantities defined in the previous section, and then linearize these to the almost-FLRW case. The basic linearized equations are given by Hawking[3] [see his Eqs. (13)–(19)]; we add to them (EB) the linearized propagation equations for the gauge-invariant spatial gradients defined above.

A. Basic equations

The basic equations for the linearized case are the energy and momentum-conservation equations for perfect fluids,

$$\dot{\mu} + (\mu + p)\Theta = 0, \tag{10}$$

$$\kappa(\mu + p)\dot{u}_a + Y_a = 0; \tag{11}$$

the linearized Raychaudhuri equation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - A + \tfrac{1}{2}\kappa(\mu + 3p) - \Lambda = 0, \tag{12}$$

where A is defined by (6); and the propagation equations for the gauge-invariant variables in (5) and (7). We will focus on the gauge-invariant variable $\mathcal{D}_a$; then the equations we need are

$$h_a{}^c(\mathcal{D}_c)^{\cdot} = \frac{p}{\mu}\Theta\mathcal{D}_a - \left(\frac{p}{\mu} + 1\right)\mathcal{Z}_a; \tag{13}$$

$$h_c{}^a(\mathcal{Z}_a)^{\cdot} = -\tfrac{2}{3}\Theta\mathcal{Z}_c - \tfrac{1}{2}\kappa\mu\mathcal{D}_c + S(\tfrac{1}{2}\mathcal{K}\dot{u}_c + A_c) \tag{14}$$

(see EB), where $\mathcal{Z}_a \equiv SZ_a$ and

$$\mathcal{K} = 2(-\tfrac{1}{3}\Theta^2 + \kappa\mu + \Lambda),\quad \dot{\mathcal{K}} = -\tfrac{2}{3}\Theta(\mathcal{K} + 2A). \tag{15}$$

When $\omega = 0$, $\mathcal{K}$ is the Ricci scalar ${}^{(3)}R$ of the three-dimensional spaces orthogonal to the fluid flow (there are no such three-spaces when $\omega \neq 0$). In evaluating this equation, we may ignore the term A in (15) because the difference it makes in determining $\mathcal{K}$ is first order and so the difference resulting in determining $\mathcal{Z}_a$ via (14) is second order; that is, for the purposes of evaluating (14) we may take

$$\mathcal{K} = 6k/S^2,\quad \dot{k} = 0 \tag{16}$$

(k corresponds to the curvature constant for the background FLRW universe model).

To evaluate the last two terms in (14), we introduce ${}^{(3)}\nabla_a$, the covariant derivative operator orthogonal to u^a (obtained by totally projecting the four-dimensional covariant derivative operator; see, e.g., Refs. 9 and 10); when $\omega = 0$ this is the covariant derivative in the surfaces Σ orthogonal to the fundamental flow lines. Now, from the definition (6) of A_a and (11) we see that, to first order,

$$A_a = -\frac{{}^{(3)}\nabla_a {}^{(3)}\nabla_b {}^{(3)}\nabla^b p}{(\mu+p)}. \tag{17}$$

But

$$\begin{aligned}{}^{(3)}\nabla_b\, {}^{(3)}\nabla^b {}^{(3)}\nabla_a p &= {}^{(3)}\nabla_b\, {}^{(3)}\nabla_a {}^{(3)}\nabla^b p \\ &= \frac{1}{\kappa} {}^{(3)}\nabla_b\, {}^{(3)}\nabla_a Y^b \end{aligned} \tag{18}$$

and using the Ricci identitiy for the ${}^{(3)}\nabla_a$'s and the zero-order relation ${}^{(3)}R^b{}_a = \frac{1}{3} h_a{}^{b(3)}R$ for the three-dimensional Ricci scalar ${}^{(3)}R$ we obtain

$${}^{(3)}\nabla_a {}^{(3)}\nabla_b Y^b = {}^{(3)}\nabla_b {}^{(3)}\nabla_a Y^b - \tfrac{1}{3} Y_a {}^{(3)}R. \tag{19}$$

Thus on using ${}^{(3)}R = \mathcal{K} = 6k/S^2$ we find

$$\begin{aligned} \tfrac{1}{2}\mathcal{K}\dot{u}_a &= -\frac{1}{\kappa(\mu+p)}\frac{3k}{S^2} Y_a, \\ A_a &= \frac{1}{\kappa(\mu+p)}\left(\frac{2k}{S^2}Y_a - {}^{(3)}\nabla^2 Y_a\right), \end{aligned} \tag{20}$$

on using the notation ${}^{(3)}\nabla^2 \mathcal{D}_a \equiv {}^{(3)}\nabla_b {}^{(3)}\nabla^b \mathcal{D}_a$. In performing this calculation, note that there will *not* be three-spaces orthogonal to the fluid flow if $\omega \neq 0$, but still we can calculate the three-dimensional orthogonal derivatives as usual (by using the projection tensor $h_a{}^b$); the difference from when $\omega = 0$ will be that the quantity we calculate as a curvature tensor, using the usual definition from commutation of second derivatives, will not have all the usual curvature tensor symmetries. Nevertheless the zero-order equations, representing the curvature of the three-spaces orthogonal to the fluid flow in the background model, will agree with the linearized equations up to the required accuracy.

Now, from (4),

$$\frac{1}{\kappa}Y_a = \left(\frac{\partial p}{\partial \mu}\right) {}^{(3)}\nabla_a \mu + \left(\frac{\partial p}{\partial s}\right) {}^{(3)}\nabla_a s. \tag{21}$$

We assume we can ignore the second term (pressure variations caused by spatial entropy variations) relative to the first (pressure variations caused by energy density variations), and spatial variations in the scale function S (which would at most cause second-order variations in the propagation equations). Then (ignoring terms due to the spatial variation of $dp/d\mu$, which will again cause second-order variations) we find

$$S(\tfrac{1}{2}\mathcal{K}\dot{u}_a + A_a) = -\frac{1}{1+p/\mu}\left(\frac{dp}{d\mu}\right)\left(\frac{k}{S^2}\mathcal{D}_a + {}^{(3)}\nabla^2 \mathcal{D}_a\right). \tag{22}$$

This is the result that we need in our future use of (14).

1. The curvature gradient

The orthogonal spatial gradient of $\mathcal{K}$ [see (15)] is

$$\mathcal{K}_a \equiv h_a{}^b \mathcal{K}_{,b} \;\Rightarrow\; S\mathcal{K}_a = -\tfrac{4}{3}\Theta \mathcal{Z}_a + 2\kappa\mu\mathcal{D}_a. \tag{23}$$

This can be used to substitute for $\mathcal{Z}_a$ in (13) and the other propagation equations; for example, we find

$$\begin{aligned} h_a{}^b(SX_b)^{\cdot} = &-(SX_a)[\Theta + \tfrac{3}{2}\kappa(\mu+p)\Theta^{-1}] \\ &+ \tfrac{3}{4}\Theta^{-1}\kappa(\mu+p)\frac{S^3\mathcal{K}_a}{S^2} \end{aligned} \tag{24}$$

which (remembering the Hubble parameter H is given by $H = \frac{1}{3}\Theta$) is essentially Eq. (24) of Ref. 12.

From the definition (23) and the above equations we obtain a propagation equation for this gradient:

$$\begin{aligned}(S^3\mathcal{K}_a)^{\cdot} = \frac{2}{3}\frac{S^2}{1+p/\mu}\Bigg\{ &\frac{6k}{S^2}\left[-\mathcal{Z}_a\left(1+\frac{p}{\mu}\right) + \Theta\frac{dp}{d\mu}\mathcal{D}_a\right] \\ &-2\Theta\frac{dp}{d\mu}\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right)\Bigg\} \end{aligned} \tag{25}$$

where again one can substitute for $\mathcal{Z}_a$ from (23) if desired.

2. The Bardeen variable

Finally, the rate of variation of the Bardeen variable (9) follows directly from the equations above; it is given by

$$\dot{\Phi}_{\perp a} + \tfrac{1}{3}\Theta\Phi_a + \tfrac{3}{2}(w+1)\frac{\kappa\mu}{\Theta}\Phi_a = \tfrac{3}{4}(1+w)\frac{\kappa\mu}{\Theta}(S^3\mathcal{K}_a), \tag{26}$$

where we have written it in terms of the spatial curvature variation (23) and we use the subscript $\perp$ to denote projection orthogonal to u^a.

B. Second-order equations

The equations for propagation can now be used to obtain second-order equations for $\mathcal{D}_a$ and Φ_a. For easy comparison, we follow Bardeen[5] by defining

$$w = p/\mu, \quad c_s^2 = dp/d\mu$$
$$\Rightarrow \left(\frac{p}{\mu}\right)^{\cdot} \equiv \dot{w} = -(1+w)(c_s^2 - w)\Theta. \tag{27}$$

1. The fractional comoving gradient

Now differentiation of (13), projection orthogonal to u^a, and linearization gives a second-order equation for $\mathcal{D}_a$ [we use (12), (14), (27), and (22) in the process]. As before we use the subscript $\perp$ to denote projection orthogonal to u^a; i.e., we write $h_a{}^c(\mathcal{D}_c)^{\cdot\cdot} \equiv \ddot{\mathcal{D}}_{\perp a}$, $h_a{}^c(\mathcal{D}_c)^{\cdot} \equiv \dot{\mathcal{D}}_{\perp a}$. We find

$$\ddot{\mathcal{D}}_{\perp a} + (\tfrac{2}{3} - 2w + c_s^2)\Theta\dot{\mathcal{D}}_{\perp a} - \left((\tfrac{1}{2} + 4w - \tfrac{3}{2}w^2 - 3c_s^2)\kappa\mu + (c_s^2 - w)\frac{12k}{S^2} + (5w - 3c_s^2)\Lambda\right)\mathcal{D}_a + c_s^2\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0. \tag{28}$$

This equation is the basic result of this paper; the rest of the discussion examines its properties and special cases. It is a second-order equation determining the evolution of the gauge-invariant density variation variable $\mathcal{D}_a$ along the fluid flow lines, equivalent to the central equation of Bardeen's paper[5] (see Ref. 7). It has the form of a wave equation with extra terms due to the expansion of the Universe, gravity, the spatial curvature, and the cosmological constant. We group the last two terms together because when we make a harmonic decomposition corresponding to that made by Bardeen (Sec. III B 3), these terms together give the harmonic eigenvalues n^2.

This form of the equations allows for a variation of $w = p/\mu$ with time. However, if w =const, then from (27) $c_s^2 = w$, and the equation simplifies to

$$\ddot{\mathcal{D}}_{\perp a} + (\tfrac{2}{3} - w)\Theta\dot{\mathcal{D}}_{\perp a} - \left(\frac{(1-w)(1+3w)}{2}\kappa\mu + 2w\Lambda\right)\mathcal{D}_a + w\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0. \tag{29}$$

The matter source term vanishes if $w = 1$ (the case of "stiff matter" $\Leftrightarrow p = \mu$) or $w = -\frac{1}{3}$ (the case $p = -\mu/3$, corresponding to matter with no active gravitational mass). Between these two limits ("ordinary matter"), the matter term is positive and tends to cause the density gradient to increase ("gravitational aggregation"); outside these limits, the term is negative and tends to cause the density gradient to decrease ("gravitational smoothing"). A positive Λ term tends to cause gravitational aggregation if w is positive (but smoothing if w is negative). Also the sign of the damping term (giving the adiabatic decay of inhomogeneities) is positive if $\frac{2}{3} > w$ (that is, $2\mu > 3p$) but negative otherwise (they adiabatically grow rather than decay in this case). The equation reduces correctly to the corresponding dust equation [(58) in EB] in the case $w = 0$.

While this form is expressed in terms of $\kappa\mu$, it is convenient for many applications to substitute for μ from (15) and (16). We do so and drop Λ (which can be represented by setting $w = -1$) to obtain

$$\ddot{\mathcal{D}}_{\perp a} + (\tfrac{2}{3} - w)\Theta\dot{\mathcal{D}}_{\perp a} - \frac{(1-w)(1+3w)}{2}\left(\tfrac{1}{3}\Theta^2 + \frac{3k}{S^2}\right)\mathcal{D}_a + w\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0, \tag{30}$$

a form convenient for most applications.

2. The Bardeen variable

We can directly find the second-order equation for Φ_a [see (9)] from the equations above, obtaining

$$\ddot{\Phi}_{\perp a} + \dot{\Phi}_{\perp a}\Theta(\tfrac{4}{3} + c_s^2) + \left((c_s^2 - w)\kappa\mu - \frac{2k}{S^2}(1 + 3c_s^2) + \Lambda(1 + c_s^2)\right)\Phi_a + c_s^2\left(\frac{2k}{S^2}\Phi_a - {}^{(3)}\nabla^2\Phi_a\right) = 0, \tag{31}$$

which simplifies in the case $c_s^2 = w$, $\Lambda = 0$ to the form

$$\ddot{\Phi}_{\perp a} + \dot{\Phi}_{\perp a}\Theta(\tfrac{4}{3} + w) - \frac{2k}{S^2}(1 + 3w)\Phi_a + w\left(\frac{2k}{S^2}\Phi_a - {}^{(3)}\nabla^2\Phi_a\right) = 0. \tag{32}$$

3. Harmonic decomposition

It is standard[1–3,5] to decompose the variables harmonically, thus effectively separating out the time and space variations; this conveniently represents the idea of a comoving wavelength for the matter inhomogeneities. In our case we do so by writing $\mathcal{D}_a$ in terms of harmonic vectors $Q_a^{(n)}$ from which the background expansion has been factored out.

We start with the defining equations

$$(Q^{(n)})_{;c}u^c = 0, \quad {}^{(3)}\nabla^2 Q^{(n)} = -\frac{n^2}{S^2}Q^{(n)}, \tag{33}$$

corresponding to Bardeen's scalar Helmholtz equation (2.7), but expressed covariantly following Hawking.[3] From these quantities we define the vector harmonics [cf. Ref. 5, Eqs. (2.8), (2.10); we do not divide by the wave number, however, so our equations are valid even if $n = 0$]

$$Q_a^{(n)} \equiv S\ {}^{(3)}\nabla_a Q^{(n)} \Rightarrow \quad Q_a^{(n)}u^a = 0, \quad (Q_a^{(n)})_{;c}u^c \simeq 0,$$

$${}^{(3)}\nabla^2 Q_a^{(n)} = -\frac{n^2 - 2k}{S^2}Q_a^{(n)} \tag{34}$$

(the factor S ensuring these vector harmonics are approximately covariantly constant along the fluid flow lines in the almost-FLRW case). Then we can write μ in terms of these harmonics as

$$\mu = \sum_n \mu^{(n)} Q^{(n)}, \quad {}^{(3)}\nabla \mu^{(n)} \simeq 0, \tag{35}$$

$\mu^{(n)}$ being the nth harmonic component of μ (approximately constant in the directions orthogonal to u^a; they cannot be *exactly* constant in all these directions if $\omega \neq 0$, for if they were they would define surfaces orthogonal to the fluid flow lines). As usual the harmonics are orthogonal to each other in a suitable measure (the details depending on whether $k = +1$, 0, or -1), so the coefficients $\mu^{(n)}$ can be determined by suitable weighted integrals of μ. However, we have to worry about the convergence of these integrals; this may require consideration of finite boxes in the Universe, or subtraction of a time-varying function from μ before doing the harmonic analysis. In the latter case it may be preferable to use an alternative representation:

$$\mu = \mu_0 \left(1 + \sum_n \delta^{(n)} Q^{(n)} \right), \quad {}^{(3)}\nabla \mu_0 \simeq 0, \quad {}^{(3)}\nabla \delta^{(n)} \simeq 0, \tag{36}$$

where μ_0 is a solution of the zeroth-order equations and $\delta^{(n)}$ is the nth fractional harmonic component of μ (again, these functions are approximately constant in the directions orthogonal to u^a). Now suitable choice of the background solution (e.g., such that the Traschen integral constraints[13] are satisfied) will make all the harmonic coefficients small and ensure these integrals (specifically, that for $n = 0$) converge. In this case there is a gauge arbitrariness in defining the harmonics, that will affect the higher-order terms but not the linear calculations of this paper (because we do not use the absolute values of these coefficients, but rather their spatial gradients).

In either case it then follows from the definition of $\mathcal{D}_a$ that

$$\mathcal{D}_a = \sum_n \mathcal{D}^{(n)} Q_a^{(n)}, \quad {}^{(3)}\nabla_b \mathcal{D}^{(n)} \simeq 0, \tag{37}$$

where $\mathcal{D}^{(n)}$ is the harmonic component of $\mathcal{D}_a$ corresponding to the comoving wave number n, containing the time variation of that component; to first order, $\mathcal{D}^{(n)} \equiv (\mu^{(n)}/\mu) \equiv \delta^{(n)}$. Putting this decomposition in the linearized equations (28), (29), or (30), the harmonics decouple. Thus for example we obtain from (30) the nth harmonic equation

$$\ddot{\mathcal{D}}^{(n)} + (\tfrac{2}{3} - w)\Theta \dot{\mathcal{D}}^{(n)} - \left[\frac{(1-w)(1+3w)}{2} \left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2} \right) - w\frac{n^2}{S^2} \right] \mathcal{D}^{(n)} = 0 \tag{38}$$

(valid for each $n \geq 0$), showing how the growth of the inhomogeneity depends on the comoving wavelength. Clearly we can similarly harmonically analyze the other equations, e.g., the second-order equation (32) for the Bardeen variable.

C. Implications

To determine the solutions explicitly, we have to substitute for μ, Θ, and S from the zero-order equations.

1. Speed of sound

We can examine solutions in the case where the divergence term is the dominant term, by examining the case where Θ, $\kappa\mu$, k/S^2 and, Λ can be neglected. We see then directly from (28) that c_s introduced above *is* the speed of sound (and that imaginary values of c_s, that is, negative values of $dp/d\mu$, lead to exponential growth or decay rather than oscillations).

2. Jeans instability

The Jeans criterion is that gravitational collapse will tend to occur if the combination of the matter term and the divergence term in (28) or (29) is positive; that is, if

$$\tfrac{1}{2}(1-w)(1+3w)\kappa\mu\mathcal{D}_a > \; w\left(\tfrac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2 \mathcal{D}_a\right) \tag{39}$$

when $c_s^2 = w$ (we include the curvature term also, because it comes from the divergence term A_a). Using the harmonic decomposition, this can be expressed in terms of an equivalent scale: from (38), gravitational collapse tends to occur for a mode $\mathcal{D}^{(n)}$ if

$$\tfrac{1}{2}(1-w)(1+3w)\kappa\mu > \; w\tfrac{n^2}{S^2}, \tag{40}$$

that is

$$\left[(1-w)\left(\frac{1}{w} + 3\right) \frac{\kappa\mu}{2} \right]^{1/2} > \frac{n}{S}. \tag{41}$$

In terms of wavelengths, the Jeans length is defined by

$$\lambda_J \equiv \frac{2\pi S}{n_J} = c_s \left(\frac{\pi}{G\mu} \frac{1}{(1-w)(1+3w)} \right)^{1/2} \tag{42}$$

where we have expressed the result in terms of the usual gravitational constant G. Thus gravitational collapse will occur for small n (wavelengths longer than λ_J), but not for sufficiently large n (wavelengths less than λ_J), for the pressure gradients are then large enough to resist the collapse and lead to oscillations instead (cf. Jackson,[14] but his answer appears to be in error; we here present a corrected version of his result).

3. Long-wavelength solutions

Suppose we can ignore A and so A_a; then provided $\dot{S} \neq 0$ we can multiply (12) by $S\dot{S}$ and integrate: we find the Friedmann equation

$$3(\dot{S})^2 - (\kappa\mu + \Lambda)S^2 = -3k, \quad \dot{k} = 0 \tag{43}$$

applies along each world line. Thus in this case *there is a separate FLRW evolution along each world line*;[15,12] these evolutions will differ only in their energies and starting times (as in the case of dust, cf. EB, Ref. 3).

Note the difference from (15), (16) here: in general we are able to use (16) to determine $\mathcal{K}$ as far as the propagation equation for $\mathcal{D}_a$ is concerned; but this ignores first-order corrections to this equation, which we must take into account if we use it to determine $\Theta(t)$ or $\mu(t)$; the separate world lines evolve separately in general, and (16) does not describe their evolution accurately. However, in the case considered now, we *can* use (16) for these purposes, that equation being the same as (43) under these conditions, and giving the independent evolution of $S(t)$ along each world line. The evolution of the spatial variation of density will be then governed by the equations above, where now we drop the divergence terms, that is from (6) and (20),

$$A = 0 \ \Rightarrow \ A_a = 0 \ \Leftrightarrow \ c_s^2\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0; \tag{44}$$

then the second-order propagation equations become ordinary differential equations along the fluid flow lines, easily solved for particular equations of state (e.g., see Sec. IV below).

There is a first integral in this case if additionally

$$\Lambda = 0, \ \ k = 0, \tag{45}$$

which has been used extensively in analyzing perturbations during inflation (cf. Refs. 16 and 17). It is obtained in the following way: define Φ_a by (9). Under the restrictions (44) and (45) from (26) and remembering that now by (15) and (16) $\frac{1}{3}\Theta^2 = \kappa\mu$,

$$\dot{\Phi}_{\perp a} + [\tfrac{1}{3} + \tfrac{1}{2}(w+1)]\Theta\Phi_a = \tfrac{1}{4}(1+w)\Theta(S^3\mathcal{K}_a) \tag{46}$$

while (25) shows

$$S^3\mathcal{K}_a = C_a, \ \ (C_a)^{\cdot} = 0. \tag{47}$$

Combining these results, we find

$$\Phi_a + \frac{2}{\Theta}\frac{\dot{\Phi}_a + \frac{1}{3}\Theta\Phi_a}{1+w} = \frac{C_a}{2}, \tag{48}$$

a first integral of the equations (cf. Bardeen *et al.*[16] and Lyth[15,12]).

If additionally $\{w = \text{const}\} \Leftrightarrow c_s^2 = w$, the second-order equation (32) reduces to

$$\ddot{\Phi}_{\perp a} + \dot{\Phi}_{\perp a}\Theta(\tfrac{4}{3} + w) = 0 \tag{49}$$

which can be directly integrated to give

$$\dot{\Phi}_{\perp a} = \frac{\phi_a}{S^{(4+3w)}}, \ \ \dot{\phi}_{\perp a} = 0 \tag{50}$$

which can be put in (48) to give Φ_a explicitly in terms of two constants. In the particular case where $\phi_a = 0$ (no decaying mode), then we have the integral

$$\Phi_a = \Phi_{0a} = \frac{C_a}{2\left(1 + \dfrac{2}{3(1+w)}\right)} \tag{51}$$

showing how this constant relates to the spatial variation in the curvature of the perturbed model. A generalization of this argument by Mukhanov and Vishniac applies if w is not constant. If now the equation of state varies dramatically over a short time interval during a phase transition, although w varies C_a stays constant at the transition [see (47) and Ref. 17], so Φ_a varies greatly; enabling us to follow the evolution of Φ_a through the different stages of an inflationary universe.[16,17] We obtain the isocurvature case (see Sec. III D below) when $C_a = 0$.

4. Scalar modes

In general, $\mathcal{D}_a$ and $\mathcal{Z}_a$ are not parallel. Even if they are parallel at one event p on a world line γ, the divergence term A_a in (14) will in general mean that they will not stay parallel; thus they will be essentially vector rather than scalar solutions. However, each harmonic mode is effectively a scalar solution (as it is an eigenmode). Also, when the divergence term may be ignored, that is, (44) is satisfied, then [as in the dust case (EB)] there is scalar solution, arising from initial data for which $\mathcal{D}_a$ and $\dot{\mathcal{D}}_a$ are parallel (so $\mathcal{D}_a$ and $\mathcal{Z}_a$ are parallel to each other all along the world line). For example (29) has a scalar mode obeying

$$\ddot{\mathcal{D}} + (\tfrac{2}{3} - w)\Theta\dot{\mathcal{D}} - \left(\frac{(1-w)(1+3w)}{2}\kappa\mu + 2w\Lambda\right)\mathcal{D} = 0, \tag{52}$$

where $\mathcal{D}$ is the magnitude defined in (8). We can always find such "scalar" solutions [take initial data at p on γ with $\dot{\mathcal{D}}_a$ parallel to $\mathcal{D}_a$, and (44) satisfied]; they will indicate the extreme behavior of the vector solutions. Thus we may use the scalar equations to investigate the maximum rates at which density inhomogeneities can grow. Note that the scalar equation (52) is just the harmonic equation (38) for $n = 0$. As we obtain the same equations in both cases, $\mathcal{D}^{(0)}$ varies precisely as $\mathcal{D}$ along the flow lines; that is the $n = 0$ harmonic equation *is* the scalar mode.

D. Isocurvature modes

From (25) we find that isocurvature inhomogeneities, that is, perturbations with $\mathcal{D}_a \neq 0$, such that

$$\mathcal{K}_a = 0 \ \ \Leftrightarrow \ \ \mathcal{Z}_a = \frac{3}{2}\frac{\kappa\mu}{\Theta}\mathcal{D}_a, \tag{53}$$

are possible only if

$$\frac{3k}{S^2}[-\tfrac{3}{2}(1+w)\kappa\mu + c_s^2\Theta^2]\mathcal{D}_a - c_s^2\Theta^2\left(\frac{2k}{S^2}\mathcal{D}_a - {}^3\nabla^2\mathcal{D}_a\right) = 0. \tag{54}$$

Thus as in the dust case, such solutions are possible if $k = 0$ and the divergence term vanishes, i.e., (44) is satisfied, whatever the value of Λ. When $c_s^2 = w$ and $\Lambda = 0$ this corresponds to the particular case $C_a = 0$ of the integral discussed above. Although we have not obtained a formal proof, it seems likely these are the only such solutions, that is, (54) can only remain true at all times for ordinary matter (more specifically, matter such that $w \neq \frac{2}{3}$) if $k = 0$ and (44) holds.

IV. PARTICULAR APPLICATIONS

A. Radiation

In the case of pure radiation, $\gamma = \frac{4}{3}$, $w = \frac{1}{3} = c_s^2$. Then we find, from (30),

$$\ddot{\mathcal{D}}_{\perp a} + \frac{1}{3}\Theta\dot{\mathcal{D}}_{\perp a} - \frac{2}{3}\left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2}\right)\mathcal{D}_a + \frac{1}{3}\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0 \quad (55)$$

and from (52) the scalar form

$$\ddot{\mathcal{D}} + \frac{1}{3}\Theta\dot{\mathcal{D}} - \frac{2}{3}\left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2}\right)\mathcal{D} = 0, \quad (56)$$

valid when we can ignore the divergence term (that is, in the low-frequency limit). When $k = 0$ then $S(t) \propto t^{1/2}$ and we obtain, in the long-wavelength limit,

$$\mathcal{D}_a = t\, d_{+a} + t^{-1/2} d_{-a}, \quad \dot{d}_{ia} = 0 \quad (57)$$

(where t is proper time along the flow lines). The corresponding standard result in the synchronous and comoving proper time gauges is different, being modes proportional to t and to $t^{1/2}$ (cf., e.g., Refs. 4 and 5); however, we obtain the same growth law as derived in the comoving time orthogonal gauge and equivalent gauges (cf., e.g., Refs. 18 and 5). As our variables are gauge-independent and covariantly defined, we believe they show the latter gauges represent the physics more accurately than any other. Note that we obtain no fictitious modes (proportional to t^{-1}) as happens, e.g., in Olson's paper, because we are using gauge-independent variables.

The Jeans-length criterion (41) is now

$$(2\kappa\mu)^{1/2} > n/S \quad \Leftrightarrow \quad \lambda < \lambda_J = \left(\frac{1}{4}\frac{\pi}{G\mu}\right)^{1/2} \quad (58)$$

as usual. Because our equations reduce effectively to the Bardeen equations, their further properties (e.g., solutions when $k \neq 0$) are essentially dealt with in his paper, so we will not discuss them further here.

B. A mixture of simple fluids

Consider a multicomponent fluid (matter plus radiation plus a cosmological constant). A noninteracting mixture of matter and radiation with the *same* four-velocity is like a perfect fluid; that is, (3) applies where the total energy density μ is now given by

$$\mu = \mu_1 + \mu_2 + \mu_3 \equiv M_1/S^3 + M_2/S^4 + M_3, \quad \dot{M}_i = 0 \quad (59)$$

and the total pressure p by

$$p = p_2 + p_3 \equiv \tfrac{1}{3}M_2/S^4 - M_3, \quad (60)$$

where M_1 represents the amount of matter present, M_2 the amount of radiation present, and M_3 a cosmological constant.

The relativistic γ-law equation of state

$$p = (\gamma - 1)\mu \quad (61)$$

can still be used in this case; it is related to w and c_s^2 [see (27)] by

$$w = \gamma - 1, \quad c_s^2 = (\gamma - 1) + \mu\frac{d\gamma}{d\mu}. \quad (62)$$

The quantity γ is a constant for a simple fluid; in the present case we have an effective $\gamma(S)$ of the form

$$\gamma = \frac{M_1/S^3 + \frac{4}{3}M_2/S^4}{M_1/S^3 + M_2/S^4 + M_3} \quad (63)$$

(Madsen and Ellis[19]). When Λ vanishes ($M_3 = 0$), γ smoothly decreases from $\frac{4}{3}$ to 1 as the universe expands, and there is a smooth transition from radiation-dominated to matter-dominated behavior.

We can use the equations in the rest of this paper in this situation, with μ representing the total energy density and p the total pressure; at most stages $\dot{\gamma}$ will be small and can be neglected, so we can use (29) rather than (28). The Jeans length will be given by (42), where w is given by (62) and (63). Because of the possible independent spatial variation of M_1 and M_2, the isocurvature behavior will be richer than in the simple fluid case, but (25) remains valid and we find as before that perturbations such that $\mathcal{K}_a = 0$ are consistent with the evolution equations if $k = 0$ and

$$c_s^2\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0;$$

and it seems likely that these are the only isocurvature solutions.

If the fluids interact significantly, we can no longer describe the situation by the simple equation of state (63); nevertheless, just as in the case of a dissipative "perfect fluid" [i.e., a fluid with stress-tensor (3) but nonzero bulk viscosity] we can still use the equations in this paper for the evolution of density gradients, provided we add suitable equations of state describing the interactions and dissipative processes occurring.

However, the situation for multicomponent fluids with different four-velocities is more complex; generalizations of the equations given here are needed for that case.

V. CONCLUSION

We have utilized covariant gauge-invariant quantities which directly characterize the spatial variation of the relevant physical variables in the observable Universe, to represent inhomogeneities in an almost-FLRW universe with a perfect-fluid matter content. We have obtained dynamic equations obeyed by these quantities; in particular, we give second-order propagation equations along the fluid flow lines (in terms of proper time along those flow lines) for $\mathcal{D}_a$, the comoving fractional density gradient, which is the covariant and gauge-invariant quantity that embodies most closely the intention of the usual (gauge-dependent) quantity $\delta\mu/\mu$; and for Φ_a, a vector corresponding to the Bardeeen variable Φ_H. Our basic definitions and equations are valid independent of any harmonic analysis, but they can be harmonically decomposed if desired.

We have considered their solutions in the case of pure radiation, and related them to standard approaches. We obtain equations equivalent to standard ones but in a more transparent way, because in the usual approach the *definition* of the density fluctuation $\delta\mu$ depends on the gauge chosen. In our case we need a specific gauge to write down the solutions to the equations, but the definitions of the fundamental quantities are gauge invariant. The key difference is that the standard approach compares two evolutions (the actual one, and a fictitious comparison one) along a world line, whereas our variables specifically reflect the spatial density variation in the fluid (they compare evolutions along neighboring world lines in the actual Universe).

Our analysis reproduces many standard results in a unified and transparent way: we deduce (1) the speed of sound in a barotropic relativistic fluid, (2) the Jeans criterion for gravitational instability, (3) the general linearized equations for perfect fluid inhomogeneities in an almost FLRW model; these equations are equivalent to the general Bardeen gauge-invariant equations (see Ref. 7); and (4) the long-wavelength limit of those equations, and corresponding integrals in particular cases; (5) restrictions on the nature of isocurvature inhomogeneities. Our equations cover the case of a mixture of perfect fluids when those perfect fluids all have the same four-velocity vector u^a.

In the case of pure radiation with $k = 0$, we obtain different growth rates (relative to proper time along the fluid flow lines) than standard analyses using the synchronous and comoving proper time gauges; our results agree with those obtained in the comoving time-orthogonal gauge. As our variables are covariantly defined and gauge invariant, we believe they describe the situation accurately, and indicate that if one does choose to deal with the usual variable $\delta\mu/\mu$, the latter gauge should be used.

Finally we note that we have not considered the constraint equations here, nor verified that their solutions are preserved along the fluid flow lines under the propagation equations. A full analysis of almost-FLRW universe models must of course examine these issues.

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[1]E. M. Lifshitz, J. Phys. (Moscow) **10**, 116 (1946).
[2]E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[3]S. W. Hawking, Astrophys. J. **145**, 544 (1966).
[4]D. W. Olson, Phys. Rev. D **14**, 327 (1976).
[5]J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[6]G. F. R. Ellis and M. Bruni, preceding paper, Phys. Rev. D **40**, 1804 (1989).
[7]J. Hwang and E. Vishniac, University of Texas report, 1989 (unpublished).
[8]J. Ehlers, Abh. Mainz Akad. Wiss. Lit. (Math. Nat. Kl.) **11**, 1 (1961).
[9]G. F. R. Ellis, in *General Relativity and Cosmology*, proceedings of the XLVII Enrico Fermi Summer School, edited by R. K. Sachs (Academic, New York, 1971).
[10]G. F. R. Ellis, in *Cargese Lectures in Physics*, edited by E. Schatzmann (Gordon and Breach, New York, 1973), Vol. 6, p. 1.
[11]J. Kristian and R. K. Sachs, Astrophys. J. **143**, 379 (1966).
[12]D. H. Lyth and M. Mukherjee, Phys. Rev. D **38**, 485 (1988).
[13]G. F. R. Ellis and M. Jaklitsch, Astrophys. J. (to be published).
[14]J. C. Jackson, Proc. R. Soc. London **A328**, 561 (1972).
[15]D. H. Lyth, Phys. Rev. D **31**, 1792 (1985).
[16]J. M. Bardeen, P. Steinhardt, and M. Turner, Phys. Rev. D **28**, 679 (1983).
[17]J. A. Frieman and M. S. Turner, Phys. Rev. D **30**, 265 (1984).
[18]K. Sakai, Prog. Theor. Phys. **41**, 1461 (1969).
[19]M. Madsen and G. F. R. Ellis, Mon. Not. R. Astron. Soc. **234**, 67 (1988).