Covariant and gauge-invariant approach to cosmological density fluctuations

G. F. R. Ellis
Physics Department, University of Texas, Austin, Texas 78712,
Applied Mathematics Department, University of Cape Town, Rondebosch 7700, Cape Town, South Africa,
*and Scuola Internazionale Superiore di Studi Avanzati, Strada Costiera 11, Miramare, 34014 Trieste, Italy**

M. Bruni
Scuola Internazionale Superiore di Studi Avanzati, Strada Costiera 11, Miramare, Trieste 34014 Trieste, Italy

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It has been known for a long time that the gauge problem plagues the study of density perturbations in cosmology. The quantity $\delta\mu/\mu$ (the fractional variation in density along a world line) usually determined in perturbation calculations is completely dependent on the gauge chosen. Even the fully covariant approach of Hawking (1966) is not immune. Bardeen's major paper (1980) determines a set of gauge-invariant quantities that are related to density perturbations, but are not those perturbations themselves. We give a simple alternative representation of density fluctuations. This representation is both fully covariant and gauge invariant; thus it sidesteps the usual problems. The basic quantity used to represent density inhomogeneities is the *comoving fractional gradient of the energy density orthogonal to the fluid flow*. Our description does not make the usual assumption that this gradient is small. Exact (fully nonlinear) propagation equations are derived for this quantity. They are then linearized to give propagation equations appropriate to the case of an almost-Robertson-Walker universe. Their solutions are obtained in a simple case which can be compared with the standard theory; we recover the usual growing and decaying modes. Thus the result is standard, but its derivation is completely transparent. We give an interpretation of the Bardeen variables in terms of our formalism.

I. INTRODUCTION

It has been known for a long time[1,2] that the gauge problem plagues the study of density perturbations in cosmology. To define perturbations, we have to choose a correspondence between a fictitious background space-time, and the physical, inhomogeneous Universe. A change in this correspondence, keeping the background space-time fixed, is a *gauge transformation*. The basic point is that[3] "a gauge transformation . . . changes the point in the background space-time corresponding to a point in the physical space-time. Thus even if a quantity is a scalar under coordinate transformations, the value of the *perturbation* in the quantity will *not* be invariant under gauge transformations if the quantity is non-zero and position dependent in the background."

Consequently,[3] "if the gauge condition imposed to simplify the metric leaves a residual gauge freedom, the perturbation equations will have spurious gauge mode solutions which can be completely annulled by a gauge transformation and have no physical reality."

Even the fully covariant approach of Hawking[4] is not immune.[5] The resulting problem is that the quantity $\delta\mu$ (the variation in density along a single world line) often calculated in perturbation calculations is completely dependent on the gauge chosen, and unless this gauge is fully specified the modes discovered for this quantity are spurious modes (due to residual gauge freedom); while if it is fully specified, its relation to what we really want to know (the spatial variation of density in the Universe) is convoluted and difficult to interpret.

Bardeen's major paper[3] determines a set of gauge-invariant quantities that are related to density perturbations, but are not those perturbations themselves (they include metric tensor Fourier components and other quantities in cunning combinations). His is a theory of some complexity, which can now be regarded as the standard theory of density perturbations.

The purpose of this paper is to present a simple alternative representation of density inhomogeneities in an almost-Robertson-Walker universe (the background space-time is exactly spatially homogeneous and isotropic). This representation is both fully covariant and gauge invariant; thus it sidesteps the usual problems. Section II discusses the gauge problem for density perturbations, and poses the problem of arbitrariness of $\delta\mu$. Section III summarizes the covariant approach to describing a fluid flow, and defines gauge-invariant variables that satisfactorily represent density inhomogeneities independent of any assumption of "smallness" of these inhomgeneities. Section IV gives exact (fully nonlinear) equations for these variables, and obtains from them the basic perturbation equations. Section V discusses those equations in various physical conditions, and gives their solutions in a simple case which can be compared with the standard theory. Appendixes discuss technical details of how to compare evolution of the Universe on neighboring world lines, and the interpretation of a naturally occurring quantity in the case of rotating-universe models.

When we consider the pressure-free case, we regain the standard results. However, our derivation is completely transparent, as our variables are not restricted to the description of "small" density fluctuations, and are all covariantly defined quantities which are gauge-invariant in the almost Friedmann-Lemaître-Robertson-Walker (FLRW) case. To make clear the relation to Bardeen's paper,[3] we give an interpretation of how his variables relate to our formalism.

II. THE GAUGE PROBLEM

It is very easy to be misled by the "obvious" way of investigating density perturbations. In this approach we consider an idealized universe model $\bar{S}$ (usually taken to be a FLRW universe); each quantity in this model will be denoted by an overbar, e.g., the energy density will be $\bar{\mu}$ and pressure $\bar{p}$. We perturb this model to obtain a "realistic" or "lumpy" universe S, where the physical quantities will be denoted by the same symbols as in $\bar{S}$ but without overbars (e.g., the energy density is μ and the pressure p). Then the perturbation of each quantity at a given space-time point q is the difference between these quantities at q; considering all points, the perturbation field is determined. For example, the metric perturbation is

$$\delta g_{ab} = g_{ab} - \bar{g}_{ab} \tag{1}$$

and the energy-density perturbation is

$$\delta\mu \equiv \mu - \bar{\mu}. \tag{2}$$

However, this approach obscures the real situation. It suggests that there is something very special about the way the original model $\bar{S}$ is related to the lumpy model, whereas in reality this is not so. Suppose we consider the lumpy universe model S, not knowing how the model $\bar{S}$ was used to make the construction; can we uniquely recover $\bar{S}$ from S? Without further restriction, the answer is no; for without a specific prescription for approximating the lumpy model by the smooth one, the quantities in the background model $\bar{S}$ are not uniquely determined from the lumpy model S [in Eq. (1), the only restriction relating the two models is that δg_{ab} is "small" in some suitable sense; it is far from obvious how one can extract $\bar{g}_{ab}$ from g_{ab} in a unique way]. In fact the definition of the background model in S is equivalent to defining a map Φ from $\bar{S}$ to S, mapping the density in $\bar{S}$ into a background density $\bar{\mu}$ in S [for notational convenience, we use the same symbol for quantities in $\bar{S}$ and their images in S, e.g., the image $\Phi(\bar{\mu})$ in S of $\bar{\mu}$ in $\bar{S}$ is simply denoted by $\bar{\mu}$]. The perturbations defined are completely dependent on how that map is chosen (Fig. 1). This is the gauge freedom in defining the perturbation.

The situation is usually expressed in terms of the coordinate choice in S, it being understood that the coordinates in S correspond to coordinates chosen in $\bar{S}$, so that a choice of coordinates determines a map from $\bar{S}$ into S; thus the gauge freedom is represented as a freedom of coordinate choice in S. It is clearer to specifically consider the map Φ from $\bar{S}$ into S, noting that we have coordinate freedom both in $\bar{S}$ and in S which we can usefully adapt to the chosen map Φ.

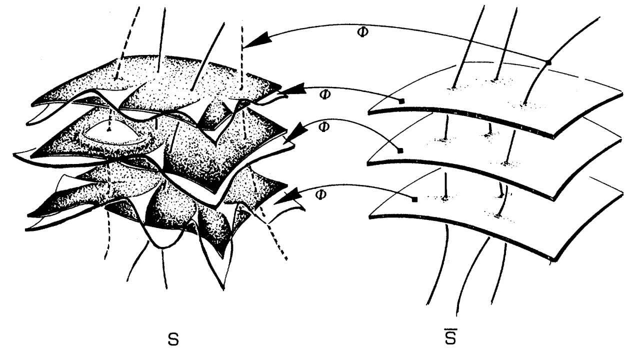

FIG. 1. The perturbed density $\delta\mu$ is defined by a mapping Φ of an idealized world model $\bar{S}$ into a more accurate world model S, for Φ maps surfaces $\{\bar{\mu} = \text{const}\}$ from $\bar{S}$ into S, where they can be compared with the actual surfaces $\{\mu = \text{const}\}$.

Thus the actual situation is that what we are given to study is the real (lumpy) universe S (this is all we can measure), and we define the perturbed quantities and their evolution by the way we specify a mapping Φ of the (fictitious) idealized space-time $\bar{S}$ into S. The determination of the best way to make this correspondence can be called the "fitting problem" for cosmology;[6,7] there are various ways to do this, so the answer is not unique. Once we completely specify the map Φ, there is no arbitrariness in $\delta\mu$; insofar as Φ is unspecified, that quantity is arbitrary.

A. Gauge specification

It is convenient to think of this map as having four aspects (Fig. 2).

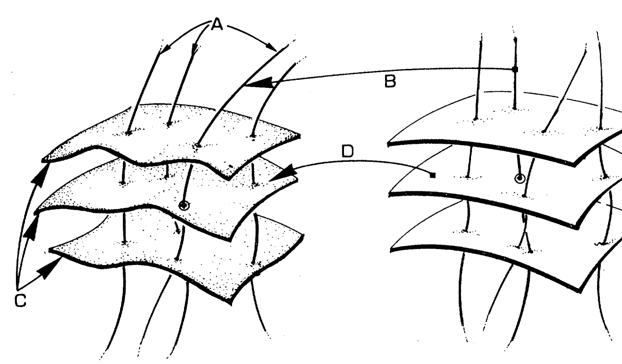

FIG. 2. The map Φ has four aspects: (A) choice of a family of time lines in each spacetime; (B) choice of a particular correspondence of time lines in the family in $\bar{S}$ to particular time lines in the family in S; (C) choice of a family of spacelike surfaces in each spacetime; (D) choice of a particular correspondence of surfaces from the family in $\bar{S}$ to surfaces in the family in S.

(A) We define a family of world lines $\bar{\gamma}$ in $\bar{S}$ and a corresponding family of world lines $\bar{\gamma}$ in S. This determines the world lines in each space-time along which we will compare the evolution of density fluctuations. There is an obvious choice in $\bar{S}$, namely, the fundamental flow lines; this will often be the best choice in S also, but others (e.g., normals to a chosen set of surfaces) may be convenient.

(B) We define a specific correspondence between individual world lines $\bar{\gamma}_i$ in $\bar{S}$ and individual world lines $\bar{\gamma}_i$ in S. This specifies which specific observer's observations we shall compare with which. In the case where $\bar{S}$ is an FLRW universe, this choice does not matter because of the spatial homogeneity of those models.

(C) We define a family of spacelike surfaces $\bar{\Sigma}$ in $\bar{S}$ and a corresponding family $\bar{\Sigma}$ in S; these are the "time surfaces" in each space-time. There is an obvious choice in $\bar{S}$: namely, the surfaces of homogeneity $\{\bar{t} = \text{const}\}$; this means the image of these surfaces in S (that is, the surfaces $\{\bar{t} = \text{const}\}$ in S) *are* the idealized surfaces of constant density $\{\bar{\mu} = \text{const}\}$ we use to define the density perturbations. There is a variety of choice for the surfaces $\bar{\Sigma}$ in S, as discussed in depth by Bardeen.[3]

(D) We define a correspondence between particular surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in $\bar{S}$ and particular surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in S, and so assign particular time values $\bar{t}$ to each event q in S. This is crucial: this specifies which specific point q in S corresponds to a point $\bar{q}$ in $\bar{S}$, and completes the specification of the map Φ. In particular, the time evolution of a density perturbation $\delta\mu$ is now defined, because this choice, by assigning particular values $\bar{\mu}$ to each surface $\bar{\Sigma}_i$ in S (the "unperturbed value" of the density) defines $\delta\mu$ via Eq. (2).

If we follow the normal convention, we understand (C) to *define* the coordinate surfaces $\{t = \text{const}\}$ in S (taking them as the same as the surfaces $\{\bar{t} = \text{const}\}$); and (D) to assign particular values to t at each event q in S by this map: $t_q = \bar{t}_q$. However, this choice is not forced on us. Note that in general neither t nor $\bar{t}$ will measure proper time along the world lines in S.

B. The arbitrariness of $\delta\mu$

The problem is that the definition of $\delta\mu$ depends both on the choice of the surfaces $\bar{\Sigma}$ in S and on the allocation of density values to these surfaces. We can for example choose $t = \bar{t}$ and then set the dependence of $\delta\mu$ on the spatial coordinates to zero through the gauge freedom (C), by choosing the surfaces $\bar{\Sigma}$ as surfaces of constant density μ in S; because these surfaces are regarded as surfaces of constant reference density, we will then have $\delta\mu$ constant on these surfaces (they will be spacelike if the universe S is sufficiently like a FLRW universe), and as they are also surfaces of constant $\bar{t}$, we will find $\delta\mu = \delta\mu(t)$. In many ways this is an obvious choice for the time surfaces (the constant-density surfaces are covariantly defined in S, and correspond precisely to the surfaces of homogeneity in the idealized model $\bar{S}$, which are also surfaces of constant density).

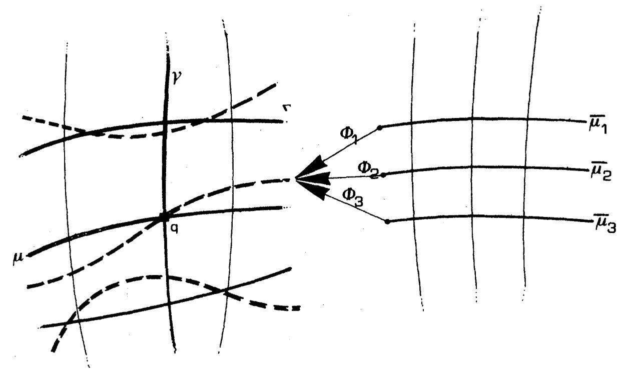

FIG. 3. By varying the assignation (D) of particular surfaces in $\bar{S}$ to surfaces in S, we can give the density perturbation $\delta\mu = \mu - \bar{\mu}$ at the event q in S (where the world line γ intersects the surface $\{\bar{\mu} = \text{const}\}$) any value we like.

Furthermore, given a choice of the family of surfaces $\bar{\Sigma}$ in S, we can still assign any value we like to $\delta\mu$ at a particular event through the gauge freedom (D), by changing the assignation of values $\bar{\mu}$ to the surfaces $\bar{\Sigma}$. Thus in particular, given any choice whatever of the time surfaces, we can set $\delta\mu$ to zero at an event q at $t = t_0$ on any world line γ, by choosing $\bar{\mu}_q = \mu_q$; this is a possible assignation of a value of the "ideal" density $\bar{\mu}$ to the event q where $t = t_0$ intersects γ (Fig. 3). How this propagates along the chosen time lines then depends on the gauge choice and the fluid equation of state. We can choose a gauge where $\delta\mu$ vanishes at every point of γ by assigning the mapping of densities to satisfy the condition $\mu(t) = \bar{\mu}(t)$ on γ. This choice is obtained in Bardeen's formalism[3] by choosing the arbitrary function $T(\tau)$ [his notation; see his Eq. (3.1)] to be given (in terms of his variables) by

$$T = -\frac{\delta}{3(1+w)(\dot{S}/S)}$$

on γ, where the right-hand side will only depend on the conformal time τ along any chosen world line γ. Then his Eq. (3.7) shows $\tilde{\delta} = 0$; i.e., the energy-density perturbation vanishes along γ in the new gauge.

If we combine these two choices, we will have chosen a gauge where $\delta\mu = 0$ identically; we map the FLRW model into the lumpy universes by mapping surfaces of constant density $\bar{\mu}$ into surfaces of constant density μ with the same numerical values (Fig. 4). We might call this the *zero density-perturbation gauge*. This possibility will not of course mean that there are no spatial variations of density; in this gauge, inhomogeneities will be represented by the fact that the proper time separating a surface of coordinate time $\bar{t}_1$ from a surface of coordinate time $\bar{t}_2$, measured along the normals to these surfaces, varies spatially (corresponding to the normals to these surfaces being nongeodesic).

The basic problem, then, is this arbitrariness in definition of $\delta\mu$, because $\delta\mu$ (a) is not gauge invariant: it can be assigned any value we like at any event by ap-

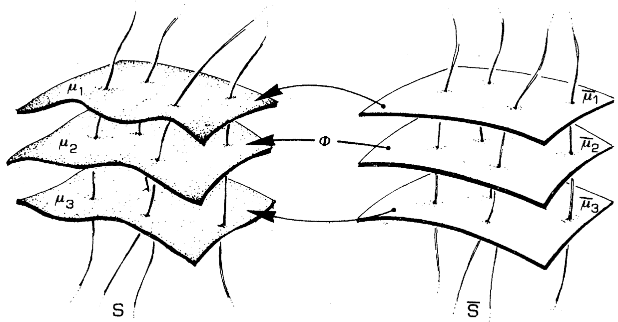

FIG. 4. By choosing Φ so that the surfaces $\{\bar{\mu} = \text{const}\}$ in S are the same as the surfaces $\{\mu = \text{const}\}$, and then choosing the correspondence (D) to assign the same numerical values to $\bar{\mu}$ on each surface as μ has on it, we obtain a *zero density-perturbation* gauge. Note that the proper time τ between any two of these surfaces in S will vary spatially, in general; the physical density variation is coded in this spatial variation of $dt/d\tau$.

propriate gauge choice; and (b) is not observable even in principle, unless the gauge is fully specified by an *observationally based* procedure (as otherwise $\bar{\mu}$ is not an observable quantity).

As a result, if we are to use $\delta\mu$ in a satisfactory way to describe density perturbations, we must either leave some gauge freedom, and keep full track of the consequences of *all* this freedom; or find a satisfactory, unique way of making the gauge choices (A)–(D) discussed above. The alternative is to look for gauge-invariant quantities that code the information we want.

C. Fixing a gauge

One way of approaching the problem is to choose a satisfactory specific gauge [specifying completely (A)–(D) above]. We mention four possibilities.[8] In each case we choose the corresponding world lines in S and $\bar{S}$ to be the fundamental flow lines. The issue then is the choice of time surfaces, and then a specific correspondence between these surfaces.

1. Proper-time gauges

One possibility is to define clearly equivalent proper times in the two models, and use this to completely specify both time functions and so fix the gauge. The obvious choice (cf. Olson[5]) is to choose proper time along the fluid flow lines from the big bang in both models. This is conceptually a clean solution to the problem, provided we can start at the big bang and follow the evolution of each model from then on.

The problem, as pointed out by Bardeen,[3] who refers to this as a *synchronous gauge*, is that the definition is nonlocal. If we observe the Universe *today*, this proposal means we cannot define $\delta\mu$ directly from these observations but have to do so by integrating the field equations all the way back to the big bang and then deducing from this integration what $\delta\mu$ is today. Apart from issues of practicality, this is clearly an unsatisfactory procedure.

2. Flow-orthogonal hypersurfaces

A second possibility is to choose the surfaces of constant time as surfaces orthogonal to the fluid flow. However, this choice (called *comoving hypersurfaces* by Bardeen) is only possible if the fluid vorticity is zero, so it is not a generic strategy. Furthermore it is not clear how to assign specific values of time or density uniquely to these surfaces (unless the acceleration is zero, proper time measured along one flow line from the big bang to a given surface will be different from that time measured along another world line).

3. Equivalent scalars

A third possibility is to identify equivalent scalars in S and $\bar{S}$, which define spacelike surfaces in $\bar{S}$. The obvious choices are the energy density μ (leading to the "zero density perturbations" discussed above, with $\bar{\mu}_q = \mu_q$) or the fluid expansion Θ (giving Bardeen' s *uniform-Hubble-constant hypersurfaces*, with $\bar{\Theta}_q = \Theta_q$). The problem is that then the information on spatial density fluctuations is coded in a way that is hard to unravel.

4. Spatial averaging

A fourth approach is to define the ideal density $\bar{\mu}$ in the lumpy model S as a suitable *average density* in S: $\bar{\mu} = \langle\mu\rangle$, where $\langle\ \rangle$ denotes some suitable spatial average (cf. Lyth and Mukherjee[9]). This is equivalent to specifying a fitting procedure of the fictitious model to the real Universe based on this averaging. This is indeed a reasonable thing to do,[6,7] and may be expected to lead to integral conditions such as the Traschen integral constraints,[10–12] as discussed by Ellis and Jaklitsch.[13]

This procedure may well give us the physical information we want. However, one will then have to take seriously the problems associated with averaging in general relativity, for example the degree to which averaging commutes with the Einstein field equations.[14–16] It also demands investigation of how this average depends on the choice of space sections over which the average is taken.

The results obtained for the evolution of $\delta\mu/\mu$ from the various gauge choices are different (see Bardeen's paper[3] for an extensive discussion; and see also Goode[17]). In each of the last three cases considered, we have to concern ourselves with the relation between coordinate time and proper time along the fluid flow lines. In the first three cases, clearly the definitions are such that they have the correct correspondence limit: if S *is* a FLRW model, they define as surfaces $\{\bar{\mu} = \text{const}\}$ the surfaces $\{\mu = \text{const}\}$ in those universes. However, the fourth approach is the most fundamental; it tackles the major issue: on what scale is the real Universe approximated by the FLRW model?[14] From the viewpoint of the present paper, the averaging implied is a sophisticated way of comparing evolution along neighboring world lines in the real fluid. In the next section we shall see there are simpler and more direct ways of making this comparison.

D. Gauge-invariant variables

The fundamental requirement for a gauge-invariant quantity is that it be invariant under the choice of the mapping Φ. The simplest case is a scalar $\bar{f}$ that is constant in the unperturbed space-time $\bar{S}$ ($\bar{f}$ = const), or any tensor $\bar{f}^{ab}{}_{cd}$ that vanishes in $\bar{S}$: $\bar{f}^{ab}{}_{cd} = 0$. The reason is that in each case the mapped quantity $\bar{f}$ in S will also be constant, so *the choice of correspondence* Φ *does not matter; they will all define the same perturbation* $\delta f = f - \bar{f}$. The only other possibility for gauge-invariant quantities is a tensor that is a constant linear combination of products of Kronecker deltas (Stewart and Walker,[18] Lemma 2.2).

What are the simple covariantly defined gauge-invariant quantities in a FLRW universe? We can easily determine them by writing down a list of all the simple covariantly defined quantities in a general fluid flow, and then seeing which ones vanish in a FLRW universe model (the other two options in the Stewart and Walker lemma are not useful in our context, as the only invariantly defined constant in the FLRW universes is the cosmological constant, and no tensors that are constant products of Kronecker deltas occur naturally).

To carry this out, it is convenient to use the general formalism developed by Schücking, Ehlers, and Trümper. We turn to this in the next section.

III. COVARIANTLY DEFINED GAUGE-INVARIANT VARIABLES

A. The covariant approach

We consider now a completely general perfect-fluid flow in a curved space-time. The exact covariant fluid equations we utilize are based on the classic paper by Ehlers[19] plus unpublished work by Trümper.[20] They are presented in the papers of Hawking[4] and Ellis.[21,22] We will repeat here only those equations important for our present derivations.

In the context of cosmology, there will always be a preferred family of world lines (the *fundamental world lines*) representing the motion of typical observers in the universe ("fundamental observers"). We will often refer to the flow lines as "fluid flow lines", as we will use the standard fluid approximation.

Let the four-velocity vector tangent to these world lines be

$$u^a = dx^a/d\tau \quad \Rightarrow \quad u^a u_a = -1\,, \tag{3}$$

where τ is proper time along the fluid flow lines. The projection tensor into the tangent three-spaces orthogonal to u^a (the rest-space of an observer moving with four-velocity u^a) is

$$h_{ab} \equiv g_{ab} + u_a u_b \quad \Rightarrow \quad h^a{}_b h^b{}_c = h^a{}_c, \quad h_a{}^b u_b = 0. \tag{4}$$

The time derivative of any tensor $T^{ab}{}_{cd}$ along the fluid flow lines is[23]

$$\dot{T}^{ab}{}_{cd} \equiv T^{ab}{}_{cd;e} u^e. \tag{5}$$

It is important to note that, because of (3), this is the derivative relative to *proper time* defined along these lines.

The first covariant derivative of the four-velocity vector is

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} - \dot{u}_a u_b\,, \tag{6}$$

where $\Theta \equiv u^a{}_{;a}$ is the expansion, $\omega_{ab} = \omega_{[ab]}$ is the vorticity tensor ($\omega_{ab}u^b = 0$), and $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab}u^b = 0$, $\sigma^a{}_a = 0$). It is convenient to define a representative *length scale* $S(\tau)$ by the relation

$$\dot{S}/S = \tfrac{1}{3}\Theta, \tag{7}$$

determining S up to a constant factor along each world line; then the volume of any fluid element varies as S^3 along the flow lines (this quantity is the generalization to arbitrary anisotropic flows of the Robertson-Walker scale parameter), so S represents the *average* distance behavior of the fluid. The vorticity and shear magnitudes are defined by $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$, $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$.

The space-time curvature tensor R_{abcd} is made up of Ricci-tensor and Weyl-tensor components. The Einstein equations

$$R_{ab} - \tfrac{1}{2}(R^c{}_c)g_{ab} + \Lambda g_{ab} = \kappa T_{ab} \tag{8}$$

determine the Ricci tensor R_{ab} directly from the matter energy-momentum-stress tensor T_{ab} at each point; κ is the gravitational constant, and we include a cosmological constant Λ in the field equations for generality. The "free gravitational field," determined nonlocally by matter and suitable boundary conditions, is represented by the Weyl tensor C_{abcd}, related to the Ricci tensor through the Bianchi identities

$$R_{ab[cd;e]} = 0. \tag{9}$$

Together (8) and (9) imply the energy-momentum-conservation equations

$$T^{ab}{}_{;b} = 0. \tag{10}$$

In this paper we will assume a one-component "perfect fluid" unless otherwise specified: that is, T_{ab} takes the form

$$T_{ab} = \mu u_a u_b + p h_{ab} = (\mu + p)u_a u_b + p g_{ab}. \tag{11}$$

The physics of the fluid is expressed by suitable equations of state relating the energy density μ and pressure p.

These will be discussed in the next section.

When the fluid vorticity vanishes (and only then) there exists a family of three-surfaces $\Sigma_\perp$ everywhere orthogonal to the fluid flow vector u^a; these are instantaneous surfaces of simultaneity for all the fundamental observers. In a general fluid flow, we can define the quantity

$$\mathcal{K} = 2(-\tfrac{1}{3}\Theta^2 + \sigma^2 + \kappa\mu + \Lambda). \tag{12}$$

When $\omega = 0$, this quantity acquires a special significance: it is the Ricci scalar ${}^3R$ of the three-dimensional spaces $\Sigma_\perp$; that is, $\omega = 0 \Rightarrow {}^3R = \mathcal{K}$. The meaning of this variable when $\omega \neq 0$ is discussed in Appendix B.

B. Gauge-invariant quantities

A FLRW universe model is a perfect-fluid space-time characterized by the conditions[19,21]

$$\sigma_{ab} = \omega_{ab} = \dot{u}^a = 0 \tag{13}$$

which imply

$$\mu = \mu(t),\ p = p(t),\ \Theta = \Theta(t), \tag{14}$$

where t is the cosmic time defined (up to a constant) by the FLRW fluid flow vector: $u_a = -t_{,a}$; and

$$E_{ab} = 0,\ H_{ab} = 0; \tag{15}$$

i.e., the Weyl tensor vanishes (so these space-times are conformally flat).

From the above characterization plus the Stewart and Walker lemma quoted in the previous section,[18] the basic gauge-invariant quantities for an almost-FLRW universe are as follows.

(1) The vorticity, shear, and acceleration:

$$\begin{aligned} &\omega_{ab} \equiv h_a{}^c h_b{}^d u_{[c;d]}, \\ &\sigma_{ab} \equiv h_a{}^c h_b{}^d u_{(c;d)} - \tfrac{1}{3} u^c{}_{;c} h_{ab}, \quad \dot{u}^a \equiv u^a{}_{;b} u^b. \end{aligned} \tag{16}$$

(2) The electric and magnetic parts E_{ab}, H_{ab} of the Weyl tensor C_{abcd}:

$$E_{ab} \equiv C_{acbd} u^c u^d, \quad H_{ab} \equiv \tfrac{1}{2} C_{acst} u^c \eta^{st}{}_{bd} u^d. \tag{17}$$

(3) The matter tensor components:

$$q_a \equiv -h_a{}^c T_{cd} u^d, \quad \pi_{ab} \equiv h_a{}^c h_b{}^d T_{cd} - \tfrac{1}{3}(h^{cd} T_{cd}) h_{ab} \tag{18}$$

(which vanish identically in the perfect-fluid case; however, we may need to consider nonzero components of these tensors in some physically significant situations; see the next section).

These are the simplest covariantly defined quantities which vanish in FLRW models, and so are gauge invariant. The problem is that the list so far does not contain quantities characterizing the variation of the zero-order variables (the energy density μ, pressure p, and fluid expansion Θ), which are in general nonzero in expanding FLRW models, and so are not gauge invariant. However, we can find associated gauge-invariant quantities: namely, the *orthogonal spatial gradients* of these variables. We define

$$X_a \equiv \kappa h_a{}^b \mu_{,b}, \quad Y_a \equiv \kappa h_a{}^b p_{,b}, \quad Z_a \equiv h_a{}^b \Theta_{,b} \tag{19}$$

(we include the gravitational constant κ in these definitions for later convenience); each quantity is gauge invariant, as they all vanish in the FLRW universes [because of (4) and (14)].

We can easily find many further gauge-invariant quantities by finding more complex invariantly defined quantities that vanish in the FLRW universe models, for example the gradients of the squared magnitudes of the shear and vorticity, $(\omega^2)_{,a}$ and $(\sigma^2)_{,a}$; the scalar products of the shear with the Weyl tensor components, $\sigma^{ab}E_{ab}$ and $\sigma^{ab}H_{ab}$; and so on. These will not be significant to us in considering linearization around the FLRW universes, for they will be of second or higher order. However, there are two other gauge-invariant quantities that will be important subsequently, namely, the divergence of the acceleration, and its spatial gradient:

$$A \equiv \dot{u}^c{}_{;c}\ , \quad A_a \equiv h_a{}^b A_{,b}. \tag{20}$$

In the case of vanishing vorticity, the Ricci scalar ${}^3R$ of the orthogonal three-spaces is gauge invariant if and only if the homogeneous space sections in $\bar{S}$ are flat, i.e., if that idealized universe is at the critical density. However, its spatial gradient is always gauge invariant. Thus for a general fluid flow, it is interesting to define from $\mathcal{K}$ [see (12)] the gauge-invariant quantity

$$\mathcal{K}_a \equiv h_a{}^b \mathcal{K}_{,b} = -\tfrac{4}{3}\Theta Z_a + 2X_a + 2(\sigma^2)_{,c} h^c{}_a, \tag{21}$$

the equivalence following from (19). Then *isocurvature fluctuations* can be defined as the zero-vorticity perturbations for which $\mathcal{K}_a = 0$.

C. The key variables

The point of this discussion is that instead of concentrating on $\delta\mu$, with the arbitrariness that implies, we can find three simple gauge-invariant quantities that will give us the information we need to discuss the time evolution of density perturbations, without the complexity of the Bardeen[3] analysis.

The first is *the spatial projection of the energy density gradient*, i.e., the vector $X_a \equiv h_a{}^b \mu_{,b}$. This vanishes in the FLRW universes, and so is a gauge-invariant quantity; it is covariantly defined in the real Universe. It is measurable in the sense that (a) it can be determined from virial theorem estimates (indeed, dynamical mass estimates determine precisely spatial density gradients), and (b) the contribution to it from luminous matter can be found by observing gradients in the numbers of observed sources and estimating the mass-to-light ratio [Kristian and Sachs,[20] Eq. (39)]. It describes the density inhomogeneities which we wish to investigate, for if there is an overdensity which is a viable protogalaxy, this will be evidenced by a nonzero value of X_a (the magnitude of X_a directly indicating how rapid the spatial variation of density is). Thus X_a seems to encapsulate much of the information we want.

However, we normally will wish to compare the density gradient with the existing density, to characterize its significance. Thus we can define the second quantity, the *fractional density gradient*

$$\mathcal{X}_a \equiv \frac{X_a}{\kappa\mu} = h_a{}^b \left(\frac{\mu_{,a}}{\mu}\right) \tag{22}$$

which is also gauge invariant, and represents the relative importance of the density gradient. While both are observable in principle, it is a moot point whether X_a or $\mathcal{X}_a$ is more easily observable in practice.

Both these vectors can be used to determine the spatial variation of the energy density μ. One important point should be noticed. In the case where $\omega = 0$, they will characterize the distribution of the density μ in the three-spaces $\Sigma_\perp$ orthogonal to the fluid flow (which might naturally be chosen as the surfaces $\{t = \text{const}\}$). However, when $\omega \neq 0$, no such orthogonal three-surfaces exist. These vectors still characterize the gradient of μ orthogonal to u^a, but cannot be immediately integrated to give the distribution of density in the surfaces $\{t = \text{const}\}$ for a suitable set of coordinates[24] because these surfaces cannot be everywhere orthogonal to the fluid flow lines. Even if $\omega = 0$, the time t such that the surfaces $\{t = \text{const}\}$ are orthogonal to the fluid flow will not measure proper time τ along the fluid flow lines unless the acceleration is zero also, that is, unless there are no pressure gradients.

There remains a problem with $\mathcal{X}_a$: it is not dimensionless. This is related to the fact that in essentials, when we consider the time evolution of the fluid, both X_a and $\mathcal{X}_a$ represent the change in density to a *fixed distance*, whereas in the context of considering the growth of protogalaxy fluctuations we want to consider density variations at a *fixed comoving scale*. Thus the third quantity of interest is the *comoving fractional density gradient* obtained by multiplying (22) by the scale factor $S(\tau)$:

$$\mathcal{D}_a \equiv S\mathcal{X}_a, \tag{23}$$

which is gauge invariant and dimensionless. We must remember here that S is defined only up to a constant by (7), so $\mathcal{D}_a$ is similarly defined up to a constant along each flow line; this reflects the fact that it represents the density variation to *any* neighboring comoving region. The time variation of this quantity precisely reflects the relative growth of density in neighboring fluid comoving volumes,[25] and this is what we wish to investigate.

The vector $\mathcal{D}_a$ can be separated into a direction e_a and magnitude $\mathcal{D}$ where

$$\mathcal{D}_a = \mathcal{D}e_a,\ e_a e^a = 1,\ e_a u^a = 0 \ \Rightarrow\ \mathcal{D} = (\mathcal{D}^a\mathcal{D}_a)^{1/2}. \tag{24}$$

The magnitude $\mathcal{D}$ is the gauge-invariant variable that most closely corresponds to the intention of the usual $(\delta\mu/\mu)$ in representing the fractional density increase in a comoving density fluctuation. The crucial difference from the usual definition is that $\mathcal{D}$ represents a (real) spatial fluctuation, rather than a (fictitious) time fluctuation.

The vectors X_a, $\mathcal{X}_a$, and $\mathcal{D}_a$ are closely related to the vectors Y_a and Z_a defined above; indeed they are dynamically dependent on each other, as shown in the following sections. All are gauge invariant, and directly determinable (at any desired scale) from a description of the real (lumpy) Universe at that scale. Thus our further analysis will concentrate on these quantities.

These definitions are only useful if we can determine useful equations for these quantities. We turn to this in the next section.

IV. DYNAMIC EQUATIONS

A. Exact equations

We can determine propagation equations along the fluid flow lines in an arbitrary fluid flow for the quantities defined in the previous section. In particular we can do so for the zero-order quantities μ, p, and Θ on the one hand, and the first-order quantities (16)–(24), that are gauge invariant in the almost-FLRW context, on the other.

1. Zero-order quantities

The energy- and momentum-conservation equations [the time and space components of the four-dimensional equation (10)] for perfect fluids take the forms

$$\dot{\mu} + (\mu + p)\Theta = 0 \tag{25}$$

and

$$\kappa(\mu + p)\dot{u}_a + Y_a = 0, \tag{26}$$

respectively. The time evolution of p is determined by (25) plus the equation of state determining p from μ.

The *Raychaudhuri equation*[26] is the fundamental equation of gravitational attraction, giving the evolution of Θ along the fluid flow lines:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - A + \tfrac{1}{2}\kappa(\mu + 3p) - \Lambda = 0, \tag{27}$$

where A is defined by (30). It is this equation that establishes that $\mu + 3p$ is the active gravitational mass of the fluid.

Finally, the time derivative of $\mathcal{K}$ [see (12)] along the fluid flow lines obeys the equation

$$(\mathcal{K} - 2\sigma^2)^{\cdot} = \tfrac{2}{3}\Theta(6\sigma^2 - \mathcal{K} - 4\omega^2 - 2A). \tag{28}$$

When the vorticity vanishes, this is an equation for ${}^3R$.

2. Gauge-invariant quantities

Propagation equations for ω_{ab}, σ_{ab}, E_{ab}, and H_{ab} are given in the literature cited above, and will not concern us directly here. By the perfect-fluid assumption, q_a and

π_{ab} vanish in S, so their "propagation equations" are $q_a = 0$, $\pi_{ab} = 0$.

The propagation equation for the acceleration follows from (25) and (26); it is

$$h_a{}^c(\dot{u}_c)^{\cdot} = \dot{u}_a\Theta\left(\frac{dp}{d\mu} - \frac{1}{3}\right) + h_a{}^b\left(\frac{dp}{d\mu}\Theta\right)_{,b} - \dot{u}_c(\omega^c{}_a + \sigma^c{}_a). \quad (29)$$

where $dp/d\mu$ is taken along the fluid flow lines. The propagation equations for the spatial gradients of the energy density and expansion [defined in (19)] are[27]

$$S^{-4}h_c{}^a(S^4X_a)^{\cdot} = -\kappa(\mu+p)Z_c - (\omega^a{}_c + \sigma^a{}_c)X_a \quad (30)$$

with source term Z_a, and

$$S^{-3}h_c{}^a(S^3Z_a)^{\cdot} = \dot{u}_c\mathcal{R} + h_c{}^a\left(-\tfrac{1}{2}X_a - 2(\sigma^2)_{,a} + 2(\omega^2)_{,a} + A_a\right) - Z_b(\sigma^b{}_c + \omega^b{}_c) \quad (31)$$

with source terms $\mathcal{R}\dot{u}_c$, X_c, A_c, $h_a{}^c(\sigma^2)_{,c}$, and $h_a{}^c(\omega^2)_{,c}$, where

$$\mathcal{R} \equiv -\tfrac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + A + \kappa\mu + \Lambda = \tfrac{1}{2}\mathcal{K} + A - 3\sigma^2 + 2\omega^2, \quad (32)$$

where $\mathcal{K}$ [defined by (12)] is the Ricci curvature ${}^3R$ of the surfaces orthogonal to the fluid flow when $\omega = 0$. In effect these equations are the spatial gradients of (25) and (27).

The equation for the evolution of Y_a will follow from that for X_a if the equation of state of the fluid is known (see Sec. V A); those for $\mathcal{X}_a$ and $\mathcal{D}_a$ follow from (30) and (25). They are

$$h_c{}^a(\mathcal{X}_a)^{\cdot} = \mathcal{X}_c\Theta\left(\frac{p}{\mu} - \frac{1}{3}\right) - Z_c\left(1 + \frac{p}{\mu}\right) - \mathcal{X}_a(\omega^a{}_c + \sigma^a{}_c) \quad (33)$$

and

$$h_c{}^a(\mathcal{D}_a)^{\cdot} = \frac{p}{\mu}\Theta\mathcal{D}_c - \left(1 + \frac{p}{\mu}\right)\mathcal{Z}_a - \mathcal{D}_a(\omega^a{}_c + \sigma^a{}_c) \quad (34)$$

where we have defined $\mathcal{Z}_a \equiv SZ_a$.

It should be emphasized that, given the perfect-fluid assumption, these are *exact propagation equations* for these quantities, valid in any fluid flow whatever. With suitable choice of the equation of state, the propagation equations close to give a higher-order equation for X_a only (see the next section). As well as these propagation equations, these quantities obey various constraint equations, given in the references cited above.

A significant feature follows immediately from (30): provided $(\mu + p) \neq 0$, $Z_a \neq 0 \Rightarrow X_a \neq 0$. The converse result ($X_a \neq 0 \Rightarrow Z_a \neq 0$) will hold in general as well [if $X_a \neq 0$, $Z_a = 0$ then the right-hand side of (31) must be zero; this is unlikely to remain true even if it is true at some initial time]. Indeed (30) and (31) show that, in general, nonzero acceleration, or spatial gradients in either the shear or the vorticity, will generate spatial gradients in both the expansion and the energy density.

B. Linearization about Robertson-Walker universes

We now specialize the above equations to the situation where the universe is almost FLRW. We do so by treating the quantities μ, p, and Θ as zero order, the quantities in (16), (17), (19)–(24), and their derivatives, as first order, and assuming the quantities in (18) vanish (this is the perfect-fluid assumption, with u^a the fluid velocity vector). Then in each equation we drop the higher-order terms relative to the lower-order ones, keeping only the lowest two orders. Note this does not mean we can always drop the second-order terms (in some equations the largest term is first order); and also that although we treat the pressure p as zeroth order, it may vanish; but we must allow for those cases where it is large. On carrying out this procedure, we will have linearized the covariant equations about an as yet unspecified FLRW universe model; as the linearized equations hold for all choices of background FLRW models, they are gauge invariant.

The basic equations resulting from this process are given in Hawking's pioneering paper[4] [see his Eqs. (13)–(19)]; we add to them the propagation equations for the gauge-invariant spatial gradients defined above.

1. Propagation equations

The relevant resulting equations for determining the density fluctuation behavior along the flow lines are the energy- and momentum-conservation equations (25) and (26), which are unaffected by the linearization procedure; the linearized Raychaudhuri equation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - A + \tfrac{1}{2}\kappa(\mu + 3p) - \Lambda = 0; \quad (35)$$

and the linearized equations for propagation of X_a and Z_a:

$$S^{-4}h_c{}^a(S^4X_a)^{\cdot} = -\kappa(\mu+p)Z_c, \quad (36)$$

$$S^{-3}h_c{}^a(S^3Z_a)^{\cdot} = \dot{u}_c\mathcal{R} - \tfrac{1}{2}X_c + A_c, \quad (37)$$

where now[28]

$$\mathcal{R} = -\tfrac{1}{3}\Theta^2 + \kappa\mu + \Lambda = \tfrac{1}{2}\mathcal{K}, \quad \dot{\mathcal{K}} = -\tfrac{2}{3}\Theta(\mathcal{K} + 2A). \quad (38)$$

These are the linearized equations determining the propagation of the gradients along the fluid flow lines. Their development depends on the equation of state of the fluid, as discussed in the next section. The equation for $\mathcal{X}_a$ follows directly [or can be obtained by linearizing (33), by dropping the last term]. Similarly, the linearized equation for $\mathcal{D}_a$ follows directly from (34); in terms of $\mathcal{Z}_a \equiv SZ_a$, the basic perturbation equations are

$$h_a{}^c(\mathcal{D}_c)^{\cdot} = \frac{p}{\mu}\Theta\mathcal{D}_a - \left(\frac{p}{\mu}+1\right)\mathcal{Z}_a, \tag{39}$$

$$h_c{}^a(\mathcal{Z}_a)^{\cdot} = -\tfrac{2}{3}\Theta\mathcal{Z}_c - \tfrac{1}{2}\kappa\mu\mathcal{D}_c + S(\dot{u}_c\mathcal{R} + A_c). \tag{40}$$

2. Constraint equations

While the constraint equations are not needed to determine the propagation of interesting quantities along the flow lines, they must of course be satisfied at some initial time on each world line. This gives interesting information about what is and is not possible.

Specifically, the linearized momentum constraint equations [(10) in Hawking[4]] are

$$h_a{}^b(\omega_b{}^c{}_{;c} - \sigma_b{}^c{}_{;c}) = -\tfrac{2}{3}Z_a. \tag{41}$$

This shows that if Θ varies spatially, i.e., $Z_a \neq 0$, then either the shear or the vorticity must also be nonzero. Conversely only restricted shear and vorticity perturbations will be compatible with Z_a remaining zero.

Similarly, the linearized "div$E$" Bianchi identity [(13) in Hawking[4]] is

$$E^{ab}{}_{;b} = \tfrac{1}{3}h^{ab}\kappa\mu_{,b} = \tfrac{1}{3}X^a \tag{42}$$

showing that the electric part E_{ab} of the Weyl tensor must be nonzero if there is a nonzero density gradient (i.e., if $X_a \neq 0$).

These results give a warning that consistent solutions to the field equations may demand inclusion of nonzero gauge-independent variables not initially anticipated.

C. The implied "gauge"

Before turning to specific equations of state, we briefly consider the gauge issue relative to the formulation here. Our equations are gauge invariant, so we can choose any map Φ we like from $\bar{S}$ to S when using this formalism (just as we can use any coordinates we like in $\bar{S}$ and in S, because the formalism is covariant). However, there is a natural map Φ from an idealized FLRW model $\bar{S}$ to the realistic model S associated with our formalism, which is the obvious one to choose unless there is some good reason to use a different correspondence. We consider here this map, naturally implied by the analysis (see Fig. 5).

(A) A very specific choice of timelike lines has been made in S, namely to examine the propagation of each quantity along the fluid flow lines. Because of the perfect fluid form (11), these are uniquely defined provided $(\mu + p) \neq 0$, which we almost always assume. The naturally associated map Φ from $\bar{S}$ to S maps fluid flow lines to fluid flow lines. This means we compare observations made by fundamental observers in the two universes.

(B) Because of the spatial homogeneity of the FLRW models it does not matter which specific flow line in $\bar{S}$ is mapped into which one in S.

(C) The implied time coordinate t in S is proper time along the fluid flow lines; it is the time coordinate of normalized comoving coordinates (t, y^ν) (see, e.g., Ehlers,[24] Ellis,[21] and Treciokas and Ellis[24]), characterized by $\dot{t} \equiv t_{,a}u^a = 1$, $\dot{y}^\nu \equiv (y^\nu)_{,a}u^a = 0$. It is arbitrary by choice of some initial surface Σ_0; i.e., the freedom in t is

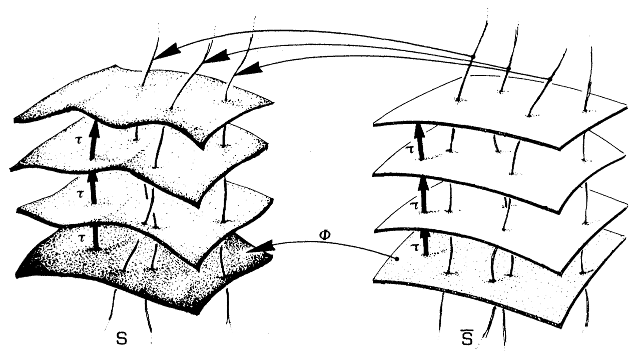

FIG. 5. In the *proper time gauge* (the time coordinate denotes proper time along the fluid flow lines), we have freedom to choose an initial time surface Σ_0: $\{t = t_0\}$ arbitrarily; then the other time surfaces are determined by measuring proper time from it along the fluid flow lines. This has the advantage of corresponding to time measurements made by fundamental observers.

$$t \rightarrow t' = t + f(y^\nu), \tag{43}$$

where f is an arbitrary function of the "spatial coordinates" y^ν. Thus we compare evolution in the universes $\bar{S}$ and S with respect to proper time measured by the fundamental observers in each model [the standard time $\bar{t}$ in $\bar{S}$ is proper time along the fluid flow lines, without the freedom (43) because we take $\bar{t} = 0$ at the big bang $\bar{S} = 0$]. The objections raised to this choice by Bardeen[3] do not apply here, for the variables X_a, Y_a, and Z_a will be small in any space-time region where the universe S is "near" some FLRW universe $\bar{S}$, irrespective of how the time coordinate t is chosen, and the *definition* of our variables is independent of the time choice; thus the "nonlocality" issue discussed previously (Sec. II B 1) does not affect the physical interpretation of our variables.

(D) The specific map of times from the idealized model $\bar{S}$ to the realistic model S will be represented by a choice of constants of integration in the solutions to the zero-order propagation equations (25) and (35), which then determine the solutions to the propagation equations (36), (37) or (39), (40) for the gauge-independent variables; in effect, the zero-order solutions are arbitrary by independent constants along each world line. This choice corresponds to the gauge freedom above, and may be thought of as choosing specific initial conditions for the perturbed universe at an initial time t_0.

In the present approach, the definition of the perturbation quantities is independent of the gauge chosen; however, we have to choose a specific gauge to obtain detailed specific solutions of the equations (just as we have to choose specific coordinates to write down a specific detailed solution to covariant equations). This freedom should be left to the end (being represented by the integration constants that naturally arise). Variation of these constants then corresponds to variation of the gauge, and also enables us to explore the effects of different initial conditions on the evolution of the gauge-independent variables (or equivalently, to explore their evolution in families of differing FLRW models instead of only one model). The essential problem in the non-gauge-invariant approach—that the *definition* of $\delta\mu$ depends on this choice—does not arise with the variables proposed here.

V. SPECIFIC MATTER DESCRIPTIONS

The implications of the linearized equations (25), (26), (35)–(40) depend on the equations of state of the matter, which describe the physics of the situation. The present paper will consider only the two simplest cases, after briefly commenting on more complex possibilities.

A. Fluid equations of state

In general we may wish to study perturbations with a scalar field, fermionic matter, or other matter sources; or using a kinetic theory description. We here concern ourselves with situations where a simple or multifluid description is appropriate. Three rather different cases arise.

(1) *Imperfect fluids* will have nonzero energy flux vectors q_a and/or anisotropic pressures π_{ab}. These could occur due to dissipative processes, when suitable equations of state will determine the quantities q_a and π_{ab} (see, e.g., Ehlers[19]); but this description is also appropriate for *multicomponent perfect fluids with different four-velocities.*

In the latter case it would be natural to describe the situation relative to the four-velocity of the dominant component; the effective stress tensor of any other perfect fluid, moving relative to this four-velocity, will be that of an imperfect fluid.[29] This would be the situation for example in *isothermal perturbations* where the surfaces of constant matter density are different from the surfaces of constant radiation density, for in general their four-velocities will differ, leading to such phenomena as radiation drag.

The methods used in this paper can be adapted to this case, but the resulting equations are rather more complex than those presented here.

(2) *Nonbarotropic perfect fluids* occur when there are two essential thermodynamic variables, so that (11) and (25)–(42) hold but $p \neq p(\mu)$. The importance of this is that then in general $\dot{u}_{[a}X_{b]} \neq 0$, so that $\dot{u}_a$ and X_c are not parallel in (36), (37), implying Z_a (and so X_a) will not be Fermi propagated along the fluid flow lines (they will rotate relative to a local inertial rest frame).[30]

A particular case of interest is that of *multicomponent perfect fluids with the same four-velocity*, e.g., baryonic matter plus radiation that is isotropic about that matter. This might be expected to be the case in *isentropic perturbations*, where both the matter and radiation are significant but the surfaces of constant matter density are the same as the surfaces of constant radiation density (the baryon-to-photon ratio is constant), then in general their four-velocities must coincide, else this condition will not be maintained. We can then represent the equation of state in terms of the simple relativistic *γ-law equation of state*

$$p = (\gamma - 1)\mu, \tag{44}$$

where $\gamma = \gamma(S)$ takes a simple form when the fluid components are noninteracting (cf. Madsen and Ellis[31]).

(3) *Barotropic perfect fluids* are perfect fluids where p and μ are functionally dependent: $p = p(\mu)$. Then there will be a well-defined speed of sound $v_s = (dp/d\mu)^{1/2}$ limiting communication by fluid processes, and from (26) $\dot{u}_a$ and X_b are necessarily functionally dependent and parallel. Equations (35)–(40) apply.

The simplest situation is when v_s is constant (cf. Olson[5]); then the relativistic γ-law description (44) may be used where now γ is constant. The important cases are $\gamma = 1$ (dust), $\frac{4}{3}$ (radiation), or 0 (false vacuum). The first and third cases are considered below. The other cases will be discussed in a further paper[32]; we merely comment here that a crucial issue is whether A_a is significant or not in Eqs. (37) and (40).[33,9]

B. A false vacuum

The "false vacuum" equation of state occurs if the stress tensor is Lorentz invariant, i.e., if $T_{ab} \propto g_{ab}$. This will be a good representation of the stress tensor of a scalar field ϕ when $\dot{\phi}$ is nearly zero (e.g., it underlies the concept of exponential inflation in the early Universe.)

The false vacuum is equivalent to a perfect fluid for which $\mu + p = 0$; we see directly from (36) that then $S^4 X_a$ is constant along the fluid flow lines (which are not uniquely defined, in this case). Thus spatial density gradients die away as S^{-4}, independent of their wavelength; relative gradients $\mathcal{X}_a$ also die away as S^{-4}; but comoving fractional density gradients $\mathcal{D}_a$ die away as S^{-3}. This is the density-gradient aspect of the "no-hair" theorem for inflationary universes.

C. Pressure-free matter

This is the case of "pure gravity", often called "dust"; $p = 0$, $\mu > 0$ so no kinetic or pressure effects are taken into account. Thus it is not very physical, but enables us to see how gravity alone functions. It may be a reasonable approximation to the equation of state of the Universe at late times.

1. The zero-order equations

Pressure-free matter must move geodesically: from the momentum equation (26),

$$p = 0 \Rightarrow \dot{u}^a = 0 \Rightarrow A = A_a = 0 \tag{45}$$

(there are no pressure gradients to deviate the motion from freefall). This enables us to omit the projection tensors in the perturbation evolution equations. The energy equation (25) shows

$$\mu = \frac{M_1}{S^3}, \quad \dot{M}_1 = 0, \tag{46}$$

and the Raychaudhuri equation (27) becomes

$$3\ddot{S}/S = -\tfrac{1}{2}\kappa\mu + \Lambda. \tag{47}$$

Provided $\dot{S} \neq 0$ we can multiply by $S\dot{S}$ and integrate to find

$$3(\dot{S})^2 - (\kappa\mu + \Lambda)S^2 = -3k, \quad \dot{k} = 0\,, \tag{48}$$

which is just the Friedmann equation which governs the time evolution of FLRW universe models; it is the same as Eq. (38) with $\mathcal{K} = 6k/S^2$. When $\omega = 0$, $\mathcal{K} = {}^3R$ and k, constant on each world line γ, characterizes the three-space curvature of the three-surfaces $\Sigma_\perp$ where they intersect γ (when $\omega \neq 0$ this is approximately true; see Appendix B). Thus *there is a separate FLRW evolution along each world line*;[4] these evolutions will differ only in their energies and starting times.[34]

2. The perturbation equations

The equations for propagation of X_a and Z_a are now[35]

$$S^{-4}(S^4 X_a)^{\cdot} = -\kappa\mu Z_a, \tag{49}$$

$$S^{-3}(S^3 Z_b)^{\cdot} = -\tfrac{1}{2}X_b. \tag{50}$$

From these, we obtain a second-order equation for X_a:

$$[S^2(S^4X_a)^{\cdot}]^{\cdot} = \tfrac{1}{2}\kappa\mu(S^6X_a)\,, \tag{51}$$

where we can substitute for μ from the Friedmann equation on each world line to obtain

$$(X_a)^{\cdot\cdot} + \tfrac{10}{3}\Theta(X_a)^{\cdot} + \frac{1}{2}\left(\frac{11}{3}\Theta^2 - 7\frac{k}{S^2} + 5\Lambda\right)X_a = 0 \tag{52}$$

[if we use expresion (46) for μ directly, we must ensure the correct relation between the constant M_1 and k].

The corresponding equations for the relative density gradient $\mathcal{X}_a$ can be written

$$(\mathcal{X}_a)^{\cdot} = -\tfrac{1}{3}\Theta\mathcal{X}_a - Z_a, \tag{53}$$

$$(Z_a)^{\cdot} = -\Theta Z_a - \tfrac{1}{2}\kappa\mu\mathcal{X}_a, \tag{54}$$

$$(\mathcal{X}_a)^{\cdot\cdot} + \tfrac{4}{3}\Theta(\mathcal{X}_a)^{\cdot} - \left(\frac{2k}{S^2} - \Lambda\right)\mathcal{X}_a = 0. \tag{55}$$

Similarly, the equations for $\mathcal{D}_a$, in terms of $\mathcal{Z}_a \equiv SZ_a$, are

$$\dot{\mathcal{D}}_a = -\mathcal{Z}_a, \tag{56}$$

$$(\mathcal{Z}_a)^{\cdot} = -\tfrac{2}{3}\Theta\mathcal{Z}_a - \tfrac{1}{2}\kappa\mu\mathcal{D}_a, \tag{57}$$

leading to the second-order equation

$$\ddot{\mathcal{D}}_a + \tfrac{2}{3}\Theta\dot{\mathcal{D}}_a - \tfrac{1}{2}\kappa\mu\mathcal{D}_a = 0. \tag{58}$$

To determine the solutions explicitly, we have to substitute for $S(t)$ from the zero-order equations (or change to the conformal time variable $\eta = \int dt/S(t)$ and give S in terms of that time along each world line). Before looking at two simple cases, we comment on some general properties of these equations.

(1) Inhomogeneity on a world line γ is indicated by at least one of X_a, Z_a being nonzero. Because the equations governing its evolution are homogeneous, inhomogeneity cannot arise spontaneously: if both X_a and Z_a are zero at any event p on γ, then they are both zero at all events on γ; if either is nonzero at any event on γ, they are both nonzero at almost all events on γ (one or the other may be zero at exceptional events).

(2) In general, X_a and Z_a are not parallel. However, if they are parallel at one event p on γ, they are parallel at all events on γ; and if either vanishes at any event q on γ, they are parallel at all events on γ where they are nonzero. In these cases, the vector equations reduce to scalar equations, giving the rate of change of the relevant magnitude along γ; for example, (58) implies

$$\ddot{\mathcal{D}} + \tfrac{2}{3}\Theta\dot{\mathcal{D}} - \tfrac{1}{2}\kappa\mu\mathcal{D} = 0 \tag{59}$$

for $\mathcal{D}$ [defined by (24)]. We can always find such "scalar" solutions (take initial data at p on γ with $\dot{X}_a$ parallel to X_a), and they will indicate the extreme behavior of the vector solutions (the magnitudes of those solutions should not be able to grow larger than those of the scalar solutions). Thus we may use the scalar equations to investigate how fast density inhomogeneities can grow.

(3) In these scalar equations, the sign of the gravitational term is positive, expressing the feature of gravitational instability of inhomogeneities. Thus, for example, the scalar equation from (51) is

$$[S^2(S^4X)^{\cdot}]^{\cdot} = \tfrac{1}{2}\kappa\mu(S^6X), \quad X \equiv (X^aX_a)^{1/2}\,, \tag{60}$$

where the source term is positive (assuming, as usual, $\mu > 0$). However, of course, the expansion of the universe, expressed in the factor S^6, works against this instability.

(4) In the case considered here (vanishing pressure), because the evolution along each world line is independent, the evolution of each of X_a, $\mathcal{X}_a$, and $\mathcal{D}_a$ is unaffected by the wavelength of the density fluctuations [(58) and (59) hold independent of wavelength]. Furthermore, the evolution is unaffected by particles horizons; they are irrelevant to this evolution, whether we consider large or small scale inhomogeneity, because the individual world lines evolve independently. This is in accord with the analysis of Ehlers *et al.*,[36] showing that timelike world lines are characteristics of perturbations of pressure-free matter.

Equation (59) is the standard equation for zero-pressure density perturbation growth relative to proper time along the flow lines in an expanding universe, obtained by Lifshitz[37] in his pioneering study of the instability of FLRW models. It can also be obtained from Newtonian theory.[38] We have here obtained it as an equation describing scalar modes of the vector equation (58). An alternative way of deriving it as an equation governing the relative magnitudes of the density in neighboring comoving volumes is given in Appendix A.

3. The Einstein static universe

As a first example, we consider a universe that is static at some event p on a world line γ: $\ddot{S} = \dot{S} = 0$ at p. Then, from (47), and (48), $k = +1$, $\frac{1}{2}\kappa\mu = S^{-2} = \Lambda$ at p. Equation (55) becomes

$$(\mathcal{X}_a)^{\cdot\cdot} = \tfrac{1}{2}\kappa\mu\mathcal{X}_a \tag{61}$$

at p, independent of the wavelength of the fluctuation, showing the gravitational instability to inhomogeneity; any nonzero initial inhomogeneity in a static situation will grow. This supplements the usual proof of instability to homogeneous (FLRW) modes, which follows direct from the Raychaudhuri equation (47).

1. The Einstein–de Sitter universe

For comparison with the standard case, we consider the simplest expanding solution, the Einstein–de Sitter universe with $k = 0 = \Lambda$. Then the zero-order solution is

$$S(t) = a(t - t_*)^{2/3},\ a = (3\kappa M_1/4)^{1/3},\ \Theta = 2/(t - t_*)\ , \tag{62}$$

where t is proper time along the world lines; a, M_1, and t_* are constants. From equations (53)–(60), we find as follows: in a parallel propagated orthonormal frame along a world line, the spatial density gradients X_a have power-law solutions

$$X_a = a_{+a}(t - t_*)^{-2} + a_{-a}(t - t_*)^{-11/3}\ , \tag{63}$$

where the a_{ia} are constant along each world line; that is, there are only decaying modes. Correspondingly, the fractional spatial density gradients $\mathcal{X}_a$ have power-law solutions

$$\mathcal{X}_a = b_{+a} + b_{-a}(t - t_*)^{-5/3}\ , \tag{64}$$

where the b_{ia} are constant on each world line. Again there is no growing mode. Finally the comoving fractional density gradients $\mathcal{D}_a$ have power-law solutions

$$\mathcal{D}_a = c_{+a}(t - t_*)^{2/3} + c_{-a}(t - t_*)^{-1}\ , \tag{65}$$

where the c_{ia} are constant on each world line, giving the expected modes with powers of $\frac{2}{3}$ and -1. From (24) it follows that the magnitude $\mathcal{D}$ goes as

$$\mathcal{D} = (\mathcal{D}_a\mathcal{D}^a)^{1/2} = [\ c_{+a}c_+{}^a(t - t_*)^{4/3} + 2c_{+a}c_-{}^a(t - t_*)^{-1/3} + c_{-a}c_-{}^a(t - t_*)^{-2}]^{1/2}$$

showing that there is also an extra mode in this magnitude in the general case, that is, if c_{+a} and c_{-a} are not parallel. Note, however, this is the magnitude of the scaled energy density gradient $\mathcal{D}_a$, which does not necessarily directly relate the density change between neighboring world lines (it gives the density variation in the instantaneous direction of maximum density change, but particles in that direction at one time will not necessarily remain in that direction at other times). The relative density change Δ between two comoving fluid elements will not show this extra mode, because it will be governed by Eq. (A11), identical to Eq. (59) for the scalar modes of the vector equation. The growth of this quantity will thus show only the $\frac{2}{3}$ and -1 modes, agreeing with the standard results for growth of $\delta\mu/\mu$ in terms of proper time along the flow lines.[37,4,5]

It is quite clear in our analysis that these are physically well-defined modes of growth and decay of a density inhomogeneity; whereas, because of the remaining gauge freedom (choice of the initial surface from which to measure proper time), the situation is much more ambivalent if we use the usual variables. Because the evolution along each world line γ_i individually is like a FLRW model F_i, it is clear that the "best fit" FLRW model along γ_i *is* F_i (irrespective of the world model $\bar{S}$ we first thought of). If we define the map Φ to assign the reference density $\bar{\mu}$ correspondingly, we will have chosen the zero-density perturbation gauge (see Sec. II A). Suppose we more conventionally choose a time coordinate which measures proper time along the world lines in S. Then Olson shows (see p. 329 of his paper[5]) that the decaying mode of $\delta\mu/\mu$ can be eliminated by the remaining gauge freedom, while this is not true for the growing mode. However, when the decaying mode has been eliminated from $\delta\mu/\mu$, it will still be evident in other quantities. The gauge-invariant approach avoids this kind of problem.

As Eq. (59) is a standard, we will not discuss its properties further here; the solutions for $k = +1$ and $k = -1$ may be found, for example, in Weinberg's book:[39] see p.573ff.

D. Isocurvature perturbations

As a last example, we look at the implications of imposing geometrical restrictions on the fluctuations. Specifically, suppose the isocurvature condition following from (21) holds; the linearized version of this condition is

$$\mathcal{K}_a = 0 \quad\Leftrightarrow\quad X_a = \mu\mathcal{X}_a = \tfrac{2}{3}\Theta Z_a. \tag{66}$$

The issue is as follows: suppose this condition is satisfied initially; under what circumstances will it remain satisfied? We examine this by taking the time-derivative of this equation, substituting from the propagation equations (36), (37), and simplifying again by use of the isocurvature condition (66). We find

$$\mathcal{K}Z_a + 2\Theta(\mathcal{R}\dot{u}_a + A_a) = 0 \tag{67}$$

as a consistency condition. When $p = 0$ this becomes

$$\mathcal{K}Z_a = 0 \tag{68}$$

showing that in this case isocurvature perturbations with nonzero density gradients are only possible if $\mathcal{K} = 0$, i.e., if the unperturbed universe has flat spatial sections. When (66) holds, substituting into (50) or (54) gives a *first-order* equation for Z_a [compatible with the second-order equations (51), (55) provided the consistency condition (68) is satisfied]. We find that when $\Lambda = 0$, $p = 0$ the isocurvature condition holds if and only if $k = 0$ (the Einstein–de Sitter case, Sec. V C 4) with $Z_a \propto S^{-4}$ and $\mathcal{D} \propto t^{-1}$; that is, the isocurvature condition is precisely equivalent to existence of a decaying mode alone.

When $p \neq 0$, we have to examine the time-evolution of the constraint equation (67) to see under what circumstances such fluctations can exist at all times.

VI. CONCLUSION

We have found a set of covariantly defined gauge-invariant quantities that characterize spatial density variation in almost-Robertson-Walker universes. In particular, we have identified the quantity $\mathcal{D}_a$, the *comoving fractional density gradient* and its magnitude $\mathcal{D}$, defined by Eqs. (22)–(24), as the covariant and gauge-invariant quantities that embody most closely the intention of the usual (gauge-dependent) definition $\delta\mu/\mu$.

We have obtained exact (fully nonlinear) and linearized propagation equations for these quantities, and examined their solutions in the simplest case (pressure-free matter). Comparison with the usual approach shows we can obtain the same results as usual but in a much more transparent way, because in the standard approach the *definition* of the density fluctuation $\delta\mu$ depends on the gauge chosen. In our case we need a specific gauge to write down the solutions to the equations, but the definitions of the fundamental quantities are gauge invariant. The key difference is that the standard approach compares two evolutions (the actual one, and a fictitious comparison one) along a world line, whereas our variables specifically reflect the spatial density variation in the fluid (they compare evolutions along neighbouring world lines in the actual Universe).

The following paper[32] will use this formalism to examine the solutions in the case of nonvanishing pressure. Because we have obtained fully nonlinear equations (30)–(34) for the quantities considered, we can hope to extend our analysis to looking at nonlinear effects, such as when a protogalaxy separates out from the Hubble expansion, or perhaps even aspects of the effects of averaging on the effective field equations (cf. Ellis[14], Futamase[16]).

An implicit gauge choice is made in our equations: we always examine the evolution along the uniquely defined fluid flow lines in the universes, and take derivatives with respect to proper time along these flow lines; thus we compare the real situation S with a FLRW universe model $\bar{S}$ where the same choices have been made. This leaves arbitrary the choice of an initial surface $\{t = t_0\}$ in S to correspond to a chosen surface $\{\bar{t} = \bar{t}_0\}$ in $\bar{S}$. We advocate use of appropriate "fitting conditions"[6,7,13] to make this choice; that is, choice of some explicit procedure to optimally fit the FLRW model to the real Universe model at a chosen initial time. This will completely fix the gauge choice [(A)–(D) in Sec. II]. A major issue then is how well this optimal fitting will be preserved at later times in the universe models.

How do our variables relate to the gauge invariant variables of Bardeen?[3] While the quantities he uses as variables are gauge invariant, they do not directly represent the density contrast on neighboring world lines, unless supplemented by extra conditions fixing the gauge by an appropriate fitting procedure (e.g., taking a spatial average across world lines). Without such a condition we can, for example, first determine his variables ϵ_m or ϵ_g for some specific fluid flow, and then choose a zero-density perturbation gauge (see Sec. II A); we will find then that

$$\epsilon_m = 3(1+w)\frac{\dot{S}}{kS}(v^{(0)} - B^{(0)})$$

in Bardeen's notation, giving us heavily disguised information on the density variation.

If his gauge-invariant variables do not directly represent the density contrast, what are they? They should correspond to one or other of the quantities listed in Sec. III B. Nel[40] has calculated these quantities using the Newman-Penrose formalism, and identified them as Weyl tensor components E_{ab} [see also Goode,[17] Eq. (2.7.3)]. At first this seems highly mysterious, for how could they then relate to density perturbations? The key is the constraint Eq. (42), showing that E^{ab} is a potential for X_a. The nature of the central Bardeen relations [Eq. (4.3) in his paper] is obscured by the Fourier analysis undertaken at the beginning of that paper, but our contention is that *they are essentially the same as the relations* (42) *in this paper*, relating the divergence of the Weyl tensor to the gradient of the energy density (which is what we want to characterize). Our approach is to deal directly with the quantities X_a, instead of their potentials E_{ab}.

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APPENDIX A: EVOLUTION ON NEIGHBORING WORLD LINES

In considering galaxy formation, what we really wish to examine is the relative density growth in two neighboring comoving volumes. Suppose we tackle this directly. Let the relative position vector $\eta^a_\perp = h^a{}_b\eta^b$ link world lines O, G where the connecting vector η^a obeys the Lie derivative relation[19,21] $\eta^a{}_{;b}u^b = u^a{}_{;b}\eta^b$. Then the density $\tilde{\mu}$ on G is related to the density μ on O by

$$\tilde{\mu} = \mu[1 + \Delta + O(\Delta^2)], \qquad (A1)$$

where [using (22)]

$$\Delta \equiv \frac{\mu_{,a}}{\mu}\eta^a_\perp = \mathcal{X}_a \eta^a_\perp. \qquad \text{(A2)}$$

Now the time variation of $\eta^a_\perp$ is given by[19,21]

$$h_{ab}(\eta^b_\perp)^{\cdot} = \frac{\dot{S}}{S}\eta_{\perp a} + (\sigma_{ab} + \omega_{ab})\eta^b_\perp \qquad \text{(A3)}$$

showing that in the almost-FLRW case, $\eta^a_\perp$ varies with $S(t)$ to first order; that is, $\Delta \simeq \mathcal{X}_a S(t)\eta^a_{\perp 0}$. Hence if we define $\mathcal{D}_a = S(t)\mathcal{X}_a$ [as in (23)], we see that

$$\Delta \simeq \mathcal{D}_a \eta^a_{\perp 0}; \qquad \text{(A4)}$$

that is, the time variation of the density difference between two neighboring comoving volumes is determined by $\mathcal{D}_a$.

1. *Exact propagation equations*

One can obtain an exact first-order propagation equation for Δ, defined by (A2), from (33) and (A3):

$$\dot{\Delta} = \frac{p}{\mu}\Theta\Delta - \left(1 + \frac{p}{\mu}\right)\Xi, \qquad \text{(A5)}$$

where

$$\Xi \equiv \Theta_{,a}\eta^a_\perp = Z_a \eta^a_\perp \qquad \text{(A6)}$$

[so the expansion $\tilde{\Theta}$ on G is related to the expansion Θ on O by $\tilde{\Theta} = \Theta + \Xi + O(\Xi^2)$]. From (31) and (A3), the exact first-order propagation equation for Ξ is

$$\dot{\Xi} = -\tfrac{2}{3}\Theta\Xi - \tfrac{1}{2}\kappa\mu\Delta + (\mathcal{R}\dot{u}_c + A_c - 2\sigma^2{}_{,c} + 2\omega^2{}_{,c})\eta^c_\perp. \qquad \text{(A7)}$$

2. *The case of zero pressure*

In the case of *dust*, we find from (A5) and (A7) the simple exact relations

$$\dot{\Delta} = -\Xi, \qquad \text{(A8)}$$

$$\dot{\Xi} = -\tfrac{2}{3}\Theta\Xi - \tfrac{1}{2}\kappa\mu\Delta + (-2\sigma^2{}_{,c} + 2\omega^2{}_{,c})\eta^c_\perp \qquad \text{(A9)}$$

leading to the completely general *exact second-order equation*

$$\ddot{\Delta} = -\tfrac{2}{3}\Theta\dot{\Delta} + \tfrac{1}{2}\kappa\mu\Delta + (2\sigma^2{}_{,c} - 2\omega^2{}_{,c})\eta^c_\perp. \qquad \text{(A10)}$$

The last equation is linearized in the almost-FLRW context by dropping the last term, to give

$$\ddot{\Delta} + \tfrac{2}{3}\Theta\dot{\Delta} - \tfrac{1}{2}\kappa\mu\Delta = 0, \qquad \text{(A11)}$$

which is essentially the well-known equation (59) for zero-pressure density perturbations (obtained in the standard literature by other means).

APPENDIX B: THE MEANING OF $\mathcal{K}$ WHEN $\omega \neq 0$

When $\omega \neq 0$, there are no surfaces orthogonal to the family of fluid flow lines, but we can find normalized comoving coordinates $\{t, y^\nu\}$ as in Sec. IV C (see Ehlers,[24] Treciokas and Ellis[24]). Using such coordinates, the surfaces $\{t = \text{const}\}$ can be set orthogonal to a particular chosen world line γ and almost orthogonal to neighboring world lines by the remaining gauge freedom (43) (e.g., if we choose an initial surface $\{t = t_0\}$ to be generated by orthogonal geodesics emanating from γ). Then $\mathcal{K}$, given by (12), will be nearly the Ricci-scalar of these three-spaces on and near γ. Note, however, these surfaces do not directly correspond to the FLRW surfaces $\{t = \text{const}\}$ when there are spatial density gradients, because if $X_a \neq 0$ the surfaces $\{\mu = \text{const}\}$ do not lie orthogonal to the world lines; similarly if $Z_a \neq 0$ the surfaces $\{\Theta = \text{const}\}$ do not lie orthogonal to the world lines.

More generally, if u^a is not too different from the normals n^a to a family of surfaces, then $\mathcal{K}$ will be not too different from the Ricci scalar of those three-spaces. The meaning of "not too different" can be made precise by either using (a) a formalism equivalent to the Arnowitt-Deser-Misner lapse and shift formalism (cf. Bardeen,[3] Sec. VI), (b) the tilted flow vector formalism of King and Ellis,[29] or (c) adapted comoving coordinates mentioned above.

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*Permanent address.

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