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Large N_c, the $I_t = J_t$ selection rule, and meson-baryon reactions

J. T. Donohue
Laboratoire de Physique Théorique, Université de Bordeaux I, 33170 Gradignan, France
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Mattis and Braaten have derived a linear relation among partial-wave amplitudes for meson-baryon reactions which is valid in the limit $N_c \to \infty$, where N_c is the number of quark colors. From this they establish the selection rule $I_t = J_t$ found previously by Mattis and Mukerjee. Using the new relation, I present a simplified derivation of earlier results for the spin-projection amplitudes in meson-baryon reactions. Consequently, the $I_t = J_t$ selection rule may be confronted with experimental data even when a partial-wave analysis is unavailable.

In a recent paper, Mattis and Braaten[1] have derived a linear relation among the partial-wave amplitudes for two-flavor meson-baryon scattering, which is valid in the large-N_c limit, where N_c is the number of quark colors. The reaction considered is

$$\phi + B \to \psi + B' ,$$

where ϕ and ψ represent arbitrary, nonstrange mesons, and where B and B' are nonstrange baryons whose spins and isospins are identical and are denoted by R and R', respectively. The relation holds among the partial-wave amplitudes T of fixed orbital angular moment L (initial state) and L' (final state), and Mattis and Braaten write it as

$$T_{I_sJ_sSS'LL'RR'} = \sum_{\tilde I_s \tilde J_s \tilde S \tilde S' \tilde R \tilde R'} \Pi(L,L')^{\tilde I_s \tilde J_s \tilde S \tilde S' \tilde R \tilde R'}_{I_s J_s S S' R R'} T_{\tilde I_s \tilde J_s \tilde S \tilde S' L L' \tilde R \tilde R'} , \tag{1}$$

where S and S' are the initial-state and final-state spins, I_s and J_s are the (conserved) s-channel isospin and total angular momentum, respectively, and the matrix Π is, to within numerical factors, essentially an 18-j symbol of the second kind. The sum over the tilde variables is unrestricted, except for requiring that the matrix elements be nonvanishing. Following the notation of Yutsis, Levinson, and Vanagas,[2] Mattis and Braaten express the matrix Π in the form

$$\Pi(L,L')^{\tilde I_s \tilde J_s \tilde S \tilde S' \tilde R \tilde R'}_{I_s J_s S S' R R'} = k^{-1}[\tilde I_s][\tilde J_s]([R][R'][S][S'][\tilde R][\tilde R'][\tilde S][\tilde S'])^{1/2}(-1)^{\xi} \begin{bmatrix} \tilde R & & \tilde S & & \tilde J_s & & \tilde S' & & \tilde R' & & \tilde I_s & \\ & S_\phi & & L & & L' & & S_\psi & & I_\psi & & I_\phi \\ R & & S & & J_s & & S' & & R' & & I_s & \end{bmatrix} , \tag{2}$$

where the quantity ξ is defined by

$$\xi = I_s + J_s + S + S' - \tilde I_s - \tilde J_s - \tilde S - \tilde S' . \tag{3}$$

In Eq. (2), S_ϕ and I_ϕ are the spin and isospin of the meson ϕ, S_ψ and I_ψ are those of the meson ψ, and the notation $[R]$ means $(2R+1)$. The quantity in large brackets is an 18-j symbol of the second kind, which is symmetric under exchange of top and bottom rows, as well as cyclic advance of the columns. The coupling schemes correspond to triangles whose base is in the bottom row, such as $(RS_\phi S)$, and similar, but inverted, triangles for the top row. The quantity k is determined by the condition that the matrix Π be a projection operator, i.e., $\Pi^2 = \Pi$, and it is represented by the infinite sum

$$k = \sum_R [R]^2 . \tag{4}$$

The meaning of Eq. (1) is clear. The projection operator Π has eigenvalues zero and unity, and the equation simply says that the partial-wave amplitude T must be an arbitrary linear combination of eigenvectors whose eigenvalue is unity. Mattis and Braaten then proceed to solve the equation by making a change of variables. Introducing I_t, the t-channel isospin, and three new quantities which they call J_t, J_ϕ, and J_ψ, they define

$$T_{I_t J_t J_\phi J_\psi LL'RR'} = \sum_{I_s J_s SS'} [I_s][J_s]([J_\phi][J_\psi][S][S'])^{1/2}(-1)^{I_s+J_s+I_t+J_t+S_\phi+S_\psi+S+S'+J_\phi+I_\psi} \times \begin{Bmatrix} R & I_\phi & I_s \\ I_\psi & R' & I_t \end{Bmatrix} \begin{Bmatrix} R & J_\phi & J_s \\ J_\psi & R' & J_t \end{Bmatrix} \begin{Bmatrix} J_s & J_\phi & R \\ S_\phi & S & L \end{Bmatrix} \begin{Bmatrix} J_s & J_\psi & R' \\ S_\psi & S' & L' \end{Bmatrix} T_{I_s J_s SS'LL'RR'} , \tag{5}$$

and find, upon substitution into Eq. (1), that their new partial-wave amplitudes $T_{I_t J_t J_\phi J_\psi LL'RR'}$ vanish unless the variable $J_t = I_t$, the t-channel isospin. Furthermore the R and R' dependence of these amplitudes is entirely contained in a factor $([R][R'])^{1/2}$. The $J_t = I_t$ rule was first obtained by Mattis and Mukerjee,[3] using an expression for the partial-wave amplitudes derived earlier by Mattis[4] in the context of the Skyrmion model of nonstrange baryons.

In a previous work,[5] I had shown, after considerable algebra, that the Skyrmion-based formula of Mattis for the partial-wave amplitudes permits one to express the spin-projection amplitudes, for fixed t-channel isospin, in terms of a set of unknown reduced amplitudes. The advantage gained thereby is that the number of independent reduced amplitudes is generally less than the number of independent spin-projection amplitudes, and one obtains a number of predictions which may be compared with experimental results even when a partial-wave analysis is not feasible. The aim of this paper is to show that the same result follows immediately from the linear relation derived by Mattis and Braaten. Once again, the necessary formulas are to be found in the extraordinarily rich monograph of Yutsis, Levinson, and Vanagas, in particular Eqs. (A.6.45) and (A.6.46). Using these expressions one may rewrite the projection operator Π as a sum of products of 6-j and 9-j symbols, such that each term factors into one piece which depends on the variables I_s, J_s, S, S', R and R', and another which depends on the tilde counterparts. The result is

$$\Pi(L,L')_{I_s J_s SS'RR'}^{\tilde I_s \tilde J_s \tilde S \tilde S' \tilde R \tilde R'} = k^{-1}[\tilde I_s][\tilde J_s]([R][R'][S][S'][\tilde R][\tilde R'][\tilde S][\tilde S'])^{1/2} \times \sum_{A,B,C} [A][B][C](-1)^{\Omega-\tilde\Omega} \begin{Bmatrix} R & I_\phi & I_s \\ I_\psi & R' & A \end{Bmatrix} \begin{Bmatrix} S' & L' & J_s \\ L & S & B \end{Bmatrix} \begin{Bmatrix} B & A & C \\ S' & R' & S_\psi \\ S & R & S_\phi \end{Bmatrix} \times \begin{Bmatrix} \tilde R & I_\phi & \tilde I_s \\ I_\psi & \tilde R' & A \end{Bmatrix} \begin{Bmatrix} \tilde S' & L' & \tilde J_s \\ L & \tilde S & B \end{Bmatrix} \begin{Bmatrix} B & A & C \\ \tilde S' & \tilde R' & S_\psi \\ \tilde S & \tilde R & S_\phi \end{Bmatrix}, \qquad (6)$$

where the summation variables A, B, and C are limited only by the triangle inequalities occurring in the 6-j and 9-j symbols. The substitutions

$$\Omega = I_s + J_s + S' - R', \quad \tilde\Omega = \tilde I_s + \tilde J_s + \tilde S' - \tilde R' \qquad (7)$$

have been used in writing this expression. If in Eq. (1) one then replaces Π by Eq. (6), one finds the result

$$T_{I_s J_s SS'LL'RR'} = ([R][R'][S][S'])^{1/2}(-1)^{\Omega} \times \sum_{A,B,C} [A][B][C] \begin{Bmatrix} R & I_\phi & I_s \\ I_\psi & R' & A \end{Bmatrix} \begin{Bmatrix} S' & L' & J_s \\ L & S & B \end{Bmatrix} \begin{Bmatrix} B & A & C \\ S' & R' & S_\psi \\ S & R & S_\phi \end{Bmatrix} \hat T_{ABCLL'}, \qquad (8)$$

where the reduced amplitudes $\hat T_{ABCLL'}$ are defined by

$$\hat T_{ABCLL'} = k^{-1} \sum_{\substack{\tilde I_s, \tilde J_s, \tilde S \\ \tilde S', \tilde R, \tilde R'}} [\tilde I_s][\tilde J_s]([\tilde R][\tilde R'][\tilde S][\tilde S'])^{1/2}(-1)^{-\tilde\Omega} \times \begin{Bmatrix} \tilde R & I_\phi & \tilde I_s \\ I_\psi & \tilde R' & A \end{Bmatrix} \begin{Bmatrix} \tilde S' & L' & \tilde J_s \\ L & \tilde S & B \end{Bmatrix} \begin{Bmatrix} B & A & C \\ \tilde S' & \tilde R' & S_\psi \\ \tilde S & \tilde R & S_\phi \end{Bmatrix} T_{\tilde I_s \tilde J_s \tilde S \tilde S' LL' \tilde R \tilde R'} . \qquad (9)$$

The essential result is Eq. (8), which shows that the partial-wave amplitudes may be written as sums of products of known coefficients with unknown reduced partial-wave amplitudes. The entire dependence on the six variables I_s, J_s, S, S', R, and R' is carried by the numerical factors and the 6-j and 9-j symbols appearing therein.

In order to show clearly what sort of reduction in the number of partial-wave amplitudes is entailed by Eq. (8), I introduce a fully general way of writing the amplitudes for fixed values of L, L', R, and R'. The starting point is the resolution of the identity

$$\delta_{J_s \tilde J_s}\delta_{S\tilde S}\delta_{S'\tilde S'} = ([S][S'][\tilde S][\tilde S'])^{1/2}[\tilde J_s](-1)^{J_s - \tilde J_s + S' - \tilde S'} \times \sum_{A,B,C} [A][B][C] \begin{Bmatrix} S' & L' & J_s \\ L & S & B \end{Bmatrix} \begin{Bmatrix} B & A & C \\ S' & R' & S_\psi \\ S & R & S_\phi \end{Bmatrix} \begin{Bmatrix} \tilde S' & L' & \tilde J_s \\ L & \tilde S & B \end{Bmatrix} \begin{Bmatrix} B & A & C \\ \tilde S' & R' & S_\psi \\ \tilde S & R & S_\phi \end{Bmatrix}, \qquad (10)$$

which is a consequence of the standard orthogonality properties of the 6-j and 9-j symbols.[2,6] Using this identity one may then write

$$T_{I_sJ_sSS'LL'RR'}=([S][S'])^{1/2}(-1)^{J_s+S'}\sum_{A,B,C}[A][B][C]\begin{Bmatrix} S' & L' & J_s \\ L & S & B \end{Bmatrix}\begin{Bmatrix} B & A & C \\ S' & R' & S_\psi \\ S & R & S_\phi \end{Bmatrix} t_{I_s ABCLL'RR'} , \qquad (11)$$

where the quantities $t_{I_s ABCLL'RR'}$ are defined by

$$t_{I_s ABCLL'RR'} = \sum_{J_s,S,S'} [J_s]([S][S'])^{1/2}(-1)^{J_s+S'} \times \begin{Bmatrix} S' & L' & J_s \\ L & S & B \end{Bmatrix}\begin{Bmatrix} B & A & C \\ S' & R' & S_\psi \\ S & R & S_\phi \end{Bmatrix} T_{I_sJ_sSS'LL'RR'} . \qquad (12)$$

They may be considered as just a convenient way of coding the information contained in the usual partial-wave amplitudes. It should be noted that in contrast with Eq. (8), the sum over the variable A in Eq. (11) is not constrained by the meson isospins. Following Rebbi and Slansky,[7] I then form the linear combinations which correspond to definite t-channel isospin I_t:

$$t_{I_t ABCLL'RR'} = \sum_{I_s} [I_s](-1)^{R+I_s+I_t}\begin{Bmatrix} R & I_\phi & I_s \\ I_\psi & R' & I_t \end{Bmatrix} t_{I_s ABCLL'RR'} . \qquad (13)$$

If one then evaluates this expression, using Eqs. (12) and (8), one finds the result

$$t_{I_t ABCLL'RR'}=\frac{\delta_{I_t A}}{[A]}(-1)^{R+R'+I_t}([R][R'])^{1/2}\hat{T}_{ABCLL'} , \qquad (14)$$

which succinctly summarizes the result of Mattis and Braaten. One sees that if the quantity A is not equal to the t-channel isospin I_t, then the amplitude $t_{I_t ABCLL'RR'}$ is zero. Furthermore, the dependence on the baryon spins is explicit.

As an illustration I examine in detail the predictions of the model for the reaction originally considered by Mattis,[4] $\pi N \rightarrow \rho N$. Here one has $S_\phi=0$, hence $C=S_\psi=1$, and $|L-L'|=0$ or 2. If $I_t=0$, then $A=0$, which implies $B=1$ [through the 9-j symbol in Eq. (12)], and the unique reduced amplitude is $\hat{T}_{011LL}$. Those amplitudes with $|L-L'|=2$ are zero because of the 6-j symbol in Eq. (12). In contrast, if $I_t=1$, then one has $B=0,1$ (both of which imply $|L-L'|=0$), and 2 (which permits $|L-L'|=0$ or 2). There are thus five allowed reduced amplitudes: $\hat{T}_{101LL}$, $\hat{T}_{111LL}$, $\hat{T}_{121LL}$, $\hat{T}_{121LL+2}$, and $\hat{T}_{121LL-2}$. In this way the results originally obtained by Mattis and cast into t-channel isospin language by me[8] are easily reproduced.

The relation between the partial-wave amplitudes introduced by Mattis and Bratten and my partial-wave amplitudes $t_{I_t ABCLL'RR'}$ may be obtained by substituting Eq. (11) on the right-hand side of Eq. (5), and using some identities among the 12-j symbols: namely, Eqs. (19.3) and (A.6.47) of Ref. 2. One finds that the variable which I call A is identical to the variable J_t of Mattis and Braaten, and the relation between the two sets of partial-wave amplitudes is

$$T_{I_tJ_tJ_\phi J_\psi LL'RR'} =([J_\phi][J_\psi])^{1/2}\sum_{B,C}[B][C](-1)^{B-\eta}\begin{Bmatrix} J_t & B & C \\ J_\phi & L & S_\phi \\ J_\psi & L' & S_\psi \end{Bmatrix} \times t_{I_tJ_tBCLL'RR'} , \qquad (15)$$

where the quantity η is defined by

$$\eta=R-R'+J_\phi+J_\psi-L+I_\psi . \qquad (16)$$

One sees from Eq. (15) that my partial-wave amplitudes and those of Mattis and Braaten are essentially related by a 9-j symbol, which, for fixed values of J_t, L, L', S_ϕ, and S_ψ, may be considered as forming an orthogonal matrix between the pairs of variables (J_ϕ,J_ψ) and (B,C). Thus my Eq. (14) simply confirms the two major results of Mattis and Braaten, the $J_t=I_t$ and $([R][R'])^{1/2}$ proportionality rules. If one is interested in testing these predictions of the model at the level of partial-wave amplitudes, there is no reason to prefer one set to the other. However, there is a second aspect of Eq. (8) which does not have its equivalent in the approach of Mattis and Braaten. The J_s dependence of my partial-wave amplitudes $t_{I_t ABCLL'RR'}$ is such that the J_s sum which relates the partial-wave amplitudes to the spin-projection amplitudes can be carried out explicitly. Therefore my relations may be extended from the partial-wave amplitudes to a particular form of the spin-projection amplitudes.

Let the spin-projection amplitudes of definite I_t for the reaction be denoted by $\mathcal{T}^{I_t}_{m_cm_dm_am_b}(\theta,\varphi,\theta',\varphi')$, where c and d (a and b) denote final (initial) meson and baryon, respectively, and where the choice of axes in the center-of-momentum frame is such that (θ,φ) are the angles of the incident meson and (θ',φ') are those of the final meson. The spin projections m refer to axes in the rest frames of the various particles obtained by pure Lorentz transformations (boosts) along the particle directions from the axis system in the c.m. frame. Following the derivation given in Ref. 5, and using Eq. (10), one may write this amplitude as

$$\mathcal{T}^{I_t}_{m_c m_d m_a m_b}(\theta,\varphi,\theta',\varphi') = \sum_{A,B,C,N} \tilde{C}^{ACBN}_{m_c m_d m_a m_d} \mathcal{F}^{I_t}_{ACBNRR'}(\theta,\varphi,\theta',\varphi') , \quad (17)$$

where the numerical coefficients $\tilde{C}$ are defined in terms of 3-j and 9-j symbols by

$$\tilde{C}^{ACBN}_{m_c m_d m_a m_d} = ([C][B][R][R'])^{1/2}(-1)^{R+R'} \times \sum_{s,\mu,S',\mu'} [S][S'](-1)^{S'-\mu'} \begin{pmatrix} S & S' & B \\ -\mu & \mu' & N \end{pmatrix} \begin{pmatrix} S_\phi & R & S \\ m_a & m_b & -\mu \end{pmatrix} \begin{pmatrix} S_\psi & R' & S' \\ m_c & m_d & -\mu' \end{pmatrix} \begin{Bmatrix} A & B & C \\ R' & S' & S_\psi \\ R & S & S_\phi \end{Bmatrix} \quad (18)$$

and the functions $\mathcal{F}$ may be expressed in terms of my partial-wave amplitudes as

$$\mathcal{F}^{I_t}_{ACBNRR'}(\theta,\varphi,\theta',\varphi') = [A] \left[\frac{[C][B]}{[R][R']} \right]^{1/2} (-1)^{A+C+R'-R} \times \sum_{\substack{L,M \\ L',M'}} (-1)^{L-L'-M'} \begin{pmatrix} L' & L & B \\ M' & -M & -N \end{pmatrix} Y^{M'}_{L'}(\theta',\varphi') Y^{M*}_{L}(\theta,\varphi) t_{I_t\,ABCLL'RR'} , \quad (19)$$

where the Y^M_L denote the usual spherical harmonics. The quantity N is equal to the net amount of spin flip, $m_a+m_b-m_c-m_d$, the sum upon N in Eq. (17) being formal. The sum over the variables A, B, and C is then constrained by $B \geq |N|$, as well as by the triangle inequalities implicit in the 9-j symbol. At this point I have merely written the spin-projection amplitudes in an unconventional way, since there are just as many ($ABCN$) combinations as there are $(m_c m_d m_a m_b)$. If one introduces Eq. (14), one derives the consequences at the amplitude level of Mattis and Braaten's result:

$$\mathcal{F}^{I_t}_{ACBNRR'}(\theta,\varphi,\theta',\varphi') = \delta_{I_t A}([C][B])^{1/2}(-1)^{C+2R} \times \sum_{\substack{L,M \\ L',M'}} (-1)^{L-L'-M'} \begin{pmatrix} L' & L & B \\ M' & -M & -N \end{pmatrix} Y^{M'}_{L'}(\theta',\varphi') Y^{M*}_{L}(\theta,\varphi) \hat{T}_{ABCLL'} , \quad (20)$$

which implies that the functions $\mathcal{F}$ vanish unless $A=I_t$, and that they are independent of the baryon spins R and R'. Except for some obvious changes of notation, this result is identical to Eq. (29) of Ref. 5, which was derived from the formula found by Mattis in the Skyrmion approach. The relative simplicity of the present derivation is evident.

The $I_t=J_t$ selection rule of Mattis and Mukerjee has been extended to the level of spin-projection amplitudes. All of the fairly successful phenomenological consequences discussed in Ref. 5 may then be seen in retrospect as evidence in favor of this rule.

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