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Analytical evaluation of the anomalous fermion-number nonconservation at high temperatures in the (1+1)-dimensional Abelian Higgs model

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The Abelian Higgs model with a fermionic current nonconserved due to an anomaly is considered in 1+1 dimensions. The one-loop expression for the rate of the fermionic-number nonconservation at high temperatures is obtained analytically for arbitrary values of the Lagrangian parameters.

The non-Abelian nature of the standard electroweak theory remains a subject of intense interest. The existence of the θ vacuum in this $SU(2)\times U(1)$ gauge theory leads to the nonconservation of leptonic and baryonic numbers which is, however, negligibly small at zero temperature.[1] Nevertheless, as was pointed out in Ref. 2 in matter at high temperatures which took place in the early Universe anomalous nonconservation of fermionic numbers is not suppressed. The relevant considerations are usually performed in the $A_0=0$ gauge. There is a static energy barrier (SEB) between the classical vacua with different values of the Chern-Simons number.[3] At high temperatures the system has enough energy to pass through the SEB (Ref. 4) via classical thermodynamical fluctuations.[5] At temperatures smaller than the height E_s of the SEB the probability Γ of the transitions over the barrier is small and may be evaluated in the semiclassical approximation $\Gamma = A\exp(-E_s/T)$. Here the preexponential factor A is important.[2] Exact analytical evaluation of the preexponential factor A in 3+1 dimensions is a serious problem.[6,7] So the semiclassical calculations in various toy models are valuable.[8–10] In Ref. 10 the γ_5 version of the Abelian Higgs model in 1+1 dimensions was shown to reproduce many essential features of the real case. It was solved analytically in the limit $g^2/\lambda \gg 1$ (where g is gauge and λ is scalar self-coupling constants) for integer values of the ratio $g/\sqrt{\lambda/2}$. In this paper we give an analytical solution for arbitrary values of the coupling constants g and λ.

The theory under consideration is defined by the Lagrangian of the form

$$L = -\tfrac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 + i\bar\psi\gamma_\mu(\partial_\mu - ig\gamma_5 A_\mu)\psi - V(\phi)\,,$$
$$V(\phi) = \lambda(|\phi|^2 - c^2/2)^2\,, \tag{1}$$

where ϕ, ψ, and A_μ are scalar, spinor, and vector gauge fields, respectively. The particle spectrum contains vector and Higgs bosons with masses $m_W^2 = g^2c^2$ and $m_H^2 = 2\lambda c^2$.

The gauge-invariant fermionic current $J_\mu = \bar\psi\gamma_\mu\psi$ is not conserved due to an anomaly:

$$\partial_\mu J_\mu = -\frac{g}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}\,. \tag{2}$$

Nonconservation of the fermionic number is associated with the fluctuations of gauge fields which in the $A_0=0$ gauge change the value of the Chern-Simons number. The theory has a θ-vacuum structure.[11] The classical vacua with different values of the Chern-Simons number are separated by SEB, the minimum height of which E_s is nonzero. A statistical system built in the vicinity of one such vacua is slightly unstable with respect to penetration through the SEB. The decay rate Γ of such a state coincides with the rate of anomalous fermionic-number nonconservation in hot plasma.[12] In the one-loop approximation it is related to the imaginary part of the free energy $\mathcal{F}$:[5,12]

$$\Gamma = \kappa\, \mathrm{Im}\mathcal{F}\,, \tag{3}$$

the coefficient κ is to be defined later. The relation (3) is useful because there is a regular representation for the free energy in terms of the Matsubara functional integral. The functional integral for the imaginary part of the free energy is saturated by the fluctuations around a stationary point called a "sphaleron." In the continuum limit this unstable static solution coincides with the kink:[10,13]

$$A_\mu^{\rm sph} = 0\,,\quad \phi^{\rm sph} = \frac{c}{\sqrt{2}}\tanh(m_H x/2)\,. \tag{4}$$

The coefficient κ in (3) is determined by the magnitude of the negative eigenmode ω_- in the sphaleron background: $\kappa = \omega_-/(2\pi T)$. The sphaleron energy $E_{\rm sph}$ is just the height of SEB.

For the preexponential factor one has

$$\frac{Z_0}{T}\mathrm{Im}\mathcal{F}\exp(E_{\rm sph}/T) = \int DA_1\, DA_0\, D\phi^*\, D\phi\, D\chi\, D\bar\chi \exp\left[-\int_0^{1/T} d^2x\, L_{\rm eff}\right]\,, \tag{5}$$
$$L_{\rm eff} = L_{\rm gauge} + L_{\rm Higgs} + L_{\rm mix} + L_{\rm ghost} + L_{\rm gauge\ fixing}\,.$$

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$L_{\rm eff}$ is the quadratic part of the Lagrangian in the sphaleron background, χ and $\bar\chi$ are ghost fields, and Z_0 is the perturbative partition function. Integration in (5) is performed over the fields periodic on the interval $[0,1/T]$. From now on we use dimensionless variables $z=xm/2$, $\beta=m_H/(2T)$, and substitution $\phi=(\phi_1+i\phi_2)/\sqrt{2}$. It is essential to use the $R_{\xi=1}$ background gauge:

$$L_{\text{gauge fixing}}=\frac{1}{2}\left[\partial_\mu A_\mu-\frac{2g}{m_H}\phi_{\rm sph}\phi_2\right]^2 . \tag{6}$$

The corresponding ghost term is

$$L_{\rm ghost}=\partial_\mu\bar\chi\partial_\mu\chi+\frac{4g^2}{m_H^2}\phi_{\rm sph}^2\bar\chi\chi . \tag{7}$$

At this point $L_{\rm eff}$ has been diagonalized in Ref. 10 by means of the rotation $\sqrt{2}A_1=\rho_1+\rho_2$ and $\sqrt{2}\phi_2=\rho_1=\rho_2$ in the limit $g^2\gg\lambda$. However, the specific form of $\phi_{\rm sph}$ allows one to diagonalize $L_{\rm eff}$ exactly for any values of g and λ by the orthogonal rotation in (A_1,ϕ_2) plane with angle ω:

$$\tan(2\omega)=-4m_W/m_H\equiv-2\alpha . \tag{8}$$

Then for the spatial density of the imaginary part of the free energy Im$\mathcal{F}$ one obtains

$$\text{Im}\mathcal{F}=TN_{\rm tr}\,\text{Det}'^{1/2}\left[\frac{M_1^{\rm vac}M_2^{\rm vac}M_3^{\rm vac}M_4^{\rm sph}}{M_1^{\rm sph}M_2^{\rm sph}M_3^{\rm sph}M_4^{\rm vac}}\right]\times\exp(-\beta E_{\rm sph}) , \tag{9}$$

where

$$\begin{aligned}&M_i^{\rm sph}=-\partial^2-s_i(s_i+1)/\cosh^2(z)+\alpha_i^2 ,\quad i=1,2,3,4;\\ &s_1=2 ,\quad s_2+1=s_3-1=s_4=s;\\ &\alpha_1\equiv 2 ,\quad \alpha_2^2=\alpha_3^2=\alpha_4^2=\alpha^2=s(s+1) .\end{aligned} \tag{10}$$

$M_i^{\rm vac}$ are the same operators as (10) but with $s_1=0$, $N_{\rm tr}$ is a normalization factor of the translational zero-mode contribution which comes from $M_1^{\rm sph}$

$$N_{\rm tr}=[E_{\rm sph}/(2\pi T)]^{1/2} . \tag{11}$$

The prime in (9) indicates that the zero mode is omitted.

As the sphaleron solution is static the eigenvalues of the operators (10) have a general form

$$E_{n,k}^2=(2\pi n/\beta)^2+\omega_k^2 ,\quad n=0,\pm1,\pm2,\dots , \tag{12}$$

where ω_k^2 are the eigenvalues of the corresponding one-dimensional quantum-mechanical operators. It allows one to perform the summation over Matsubara frequencies explicitly in the following expression for the determinants:

$$\begin{aligned}&(\text{Det}M_i^{\rm sph}/\text{Det}M_i^{\rm vac})^{-1/2}=\prod_{n,k}(E_{n,k}^{\rm vac}/E_{n,k}^{\rm sph})\equiv\exp(J) ,\\ &J=-\sum_k[\beta\omega_k^{\rm sph}/2-\beta\omega_k^{\rm vac}/2+\Phi(\omega_k^{\rm sph})-\Phi(\omega_k^{\rm vac})] ,\\ &\Phi(\omega_k)\equiv\ln[1-\exp(-\beta\omega_k)] .\end{aligned} \tag{13}$$

After the zero-temperature renormalization of the sphaleron mass we are left with

$$J(T)=-\sum_k[\Phi(\omega_k^{\rm sph})-\Phi(\omega_k^{\rm vac})] . \tag{14}$$

Both discrete eigenvalues and continuum spectrum contribute to the sum in (14). The discrete spectrum of the operators (10) is given by

$$\omega_i^{\rm sph}=\sqrt{\alpha_i^2-(s_i-n)^2} ,\quad 0\le n<s_i . \tag{15}$$

Operators $M_{2,3}^{\rm sph}$ have only positive eigenvalues, $M_1^{\rm sph}$ has one zero mode (corresponding to the translation of the sphaleron), and operator $M_3^{\rm sph}$ has one negative eigenvalue $\omega_-^2=s+1$ or (in terms of dimensional variables) $\omega_-^2=-m_H^2/4+m_Hm_W/2$. One can see that the negative eigenmode does not vanish when gauge interactions are switched off ($g\to 0$). The presence of the imaginary part of the scalar field accounts for the instability of the sphaleron solution.

For the continuum one has

$$\omega_i^{\rm sph}(k)=\omega_i^{\rm vac}(k)=\sqrt{\alpha_i^2+k^2}\equiv\omega_i^2(k) . \tag{16}$$

To do the sum over the continuum in (14) one should impose periodic boundary conditions on the eigenfunctions of the operators (10) at the finite interval $x\in[-L/2,L/2]$. The general solution of the corresponding Schrödinger equation may be constructed from $\psi_k(z)$ and $\psi_k^*(z)$ with eigenfunctions $\psi(z)$ satisfying the relations

$$\begin{aligned}&\psi_k(z)\underset{z\to+\infty}{\sim}\exp(ikz) ,\\ &\psi_k(z)\underset{z\to-\infty}{\sim}A(k)\exp(-ikz)+B(k)\exp(ikz) ,\end{aligned} \tag{17}$$

For the potentials (10),

$$\begin{aligned}&A=\frac{\Gamma(1-ik)\Gamma(ik)}{\Gamma(1+s)\Gamma(-s)} ,\quad B=\frac{\Gamma(1-ik)\Gamma(-ik)}{\Gamma(-ik-s)\Gamma(-ik+s+1)} ,\\ &\arg B(k)=2\sum_{n=1}^{s_{0i}}\arctan[k/(n+\epsilon)]+\arctan(k/\epsilon)+\frac{\pi}{2}\text{sgn}(k)\\ &\qquad+\sum_{n=1}^{\infty}\{2\arctan(k/n)-\arctan[k/(n+\epsilon)]-\arctan[k/(n-\epsilon)]\} ,\end{aligned} \tag{18}$$

where s_{0i} is an integer part of s_i while ϵ is a fractional one which is unique for s_2, s_3, s_4.

There are two branches in the spectrum $k_{1,2}(n)$ defined as

$$2\pi n=Lk_{1(2)}-\delta_{1(2)}(k_{1(2)}) ,\qquad \delta_{1(2)}(k)=\arg B+(-)\arcsin[\,|A/B|\sin(\arg A)] . \tag{19}$$

So for the continuum contribution to (14) one obtains

$$J^{\rm cont}_{(T)} = \int_0^\infty dk\, \delta(k) \frac{d}{dk}\Phi(\omega_k)\,, \tag{20}$$

with

$$\delta(k) \equiv \delta_1(k) + \delta_2(k) = 2\arg B\,.$$

In the high-temperature limit one may use $\Phi \approx \ln(\beta\omega_k)$ and the integral

$$\frac{a}{2\pi}\int_{-\infty}^{\infty} dk \ln(k^2+\alpha^2)/(k^2+a^2) = \ln(\alpha+a)\,, \qquad a>0\,,\ \alpha>0 \tag{21}$$

to obtain

$$\mathrm{Det}'^{-1/2}(M_1^{\rm sph}/M_1^{\rm vac}) = 4\sqrt{3}\,,$$

$$\mathrm{Det}^{-1/2}(M_i^{\rm sph}/M_i^{\rm vac}) = \left[\frac{\Gamma(\alpha_i+s_i+1)\Gamma(\alpha_i-s_i)}{\Gamma(\alpha_i+1)\Gamma(\alpha_i)}\right]^{1/2}. \tag{22}$$

One may obtain the final expression for the spatial density of rate Γ of the anomalous fermion-number nonconservation at high temperatures using (3), (9), (10), and (22). In dimensionful units it reads

$$\Gamma = \frac{\sqrt{3}m_H^2}{2\pi}\left[\frac{E_{\rm sph}}{2\pi T}\right]^{1/2}\left[(s+1)\frac{\Gamma(\alpha+s+1)\Gamma(\alpha-s)}{\Gamma(\alpha+1)\Gamma(\alpha)}\right]^{1/2} \times \exp(-E_{\rm sph}/T)\,. \tag{23}$$

In the Coleman-Weinberg limit[10] we get

$$\Gamma(\alpha\to\infty) = \frac{\sqrt{3}m_H^2}{2\pi}\sqrt{\alpha}\,2^{\alpha-1/4}\left[\frac{E_{\rm sph}}{2\pi T}\right]^{1/2}\exp(-E_{\rm sph}/T)\,. \tag{24}$$

In the limit $\alpha\to 0$ (23) yields

$$\Gamma(\alpha\to 0) = \frac{\sqrt{3}m_H^2}{2\pi}\left[\frac{E_{\rm sph}}{2\pi T}\right]^{1/2}\exp(-E_{\rm sph}/T)\,. \tag{25}$$

One may compare the analytical result (23) with the corresponding exact numerical evaluations on the lattice[14] (see Fig. 1). One can see that the results coincide within the error bars which imply that both high-temperature expansion and the semiclassical approximation are efficient.

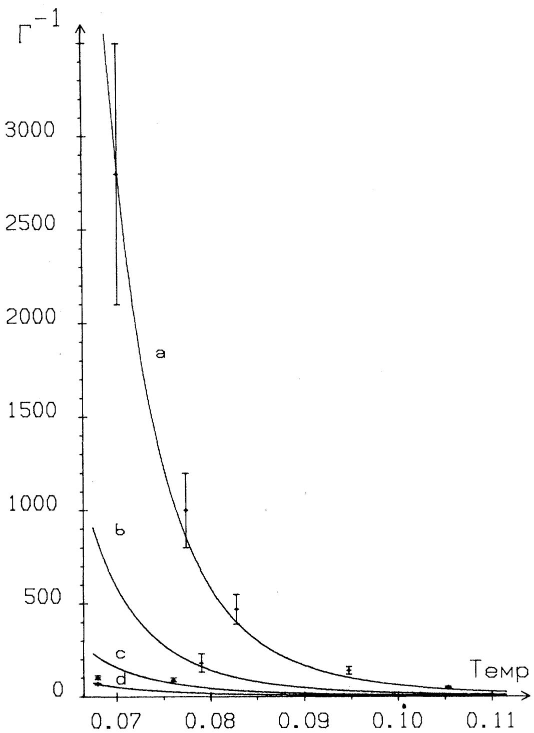

FIG. 1. Analytical vs numerical results (Ref. 14) for the rate of the anomalous fermion-number nonconservation as a function of temperature: curve *a*, $m_H/m_W = 0.5$; curve *b*, $m_H/m_W = 0.395$; curve *c*, $m_H/m_W = 0.32$; curve *d*, $m_H/m_W = 0.264$.

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